CrasyDSE: A framework for solving Dyson-Schwinger equations

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Abstract

Dyson-Schwinger equations are important tools for non-perturbative analyses of quantum field theories. For example, they are very useful for investigations in quantum chromodynamics and related theories. However, sometimes progress is impeded by the complexity of the equations. Thus automatizing parts of the calculations will certainly be helpful in future investigations. In this article we present a framework for such an automatization based on a $C++$ code that can deal with a large number of Green functions. Since also the creation of the expressions for the integrals of the Dyson-Schwinger equations needs to be automatized, we defer this task to a Mathematica notebook. We illustrate the complete workflow with an example from Yang-Mills theory coupled to a fundamental scalar field that has been investigated recently. As a second example we calculate the propagators of pure Yang-Mills theory. Our code can serve as a basis for many further investigations where the equations are too complicated to tackle by hand. It also can easily be combined with DoFun, a program for the derivation of Dyson-Schwinger equations.

Keywords: Dyson-Schwinger equations, correlation functions, quantum field theory

PROGRAM SUMMARY

Program Title: CrasyDSE
Version: 1.0.0
Licensing provisions: CPC non-profit use license
Programming language: Mathematica 8 and higher, $C++$
Operating system: all on which Mathematica and $C++$ are available (Windows, Unix, MacOS)
PACS: 11.10.-z,03.70.+k,11.15.Tk
CPC Classification: 11.1 General, High Energy Physics and Computing
11.4 Quantum Electrodynamics
11.5 Quantum Chromodynamics, Lattice Gauge Theory
11.6 Phenomenological and Empirical Models and Theories
Nature of problem: Solve large systems of Dyson-Schwinger equations numerically.
Solution method: Create $C++$ functions in Mathematica to be used for the numeric code in $C++$. This code uses structures to handle large numbers of Green functions.

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1. Introduction

Strongly coupled field theories play an essential role in the physical description of nature. Both established theories like quantum chromodynamics and conjectured ones like technicolor theories cannot be fully understood without non-perturbative methods. Typical approaches include Monte-Carlo simulations on a discretized space-time or functional equations. Functional renormalization group equations, see, e. g., [1–4], Dyson-Schwinger equations, see, e. g., [5–7], and the n-PI formalism, see, e. g., [8], belong to the second group. Their advantages are well appreciated and they provided many new insights.

In this article we will focus on Dyson-Schwinger equations (DSEs) and propose a concrete way how to handle them when they become too complex to be treated by hand alone. DSEs consist of a system of coupled integral equations which relate different Green functions with each other. Since there are infinitely many Green functions there are also infinitely many DSEs. Unfortunately no subset of these equations forms a closed system so that we have to deal with an infinitely large system of equations. Naturally one hopes that only a (small) finite number of Green functions is relevant and looks for truncations capturing the most important features of the theory. Of course, in order to check the validity of such an approach one should test the influence of neglected Green functions, which, however, often is very tedious. On the other hand there are also theories where it is known that current truncation and approximation schemes and available methods are insufficient and need to be extended. For example, standard truncations restrict the DSEs to one-loop diagrams [3, 7, 9–11] but Yang-Mills theories in the maximally Abelian gauge require the inclusion of two-loop diagrams in order to be consistent in the non-perturbative regime [12]. Consequently the present technical methods have to be improved.

The reason that more sophisticated truncations or more complicated theories require so much more effort is mainly that the length and complexity of the explicit expressions increase considerably with the number of interactions and the number of external legs. Also the numbers of dressing functions for higher Green functions grow. We will illustrate this below explicitly with the example of Yang-Mills theory coupled to a scalar: We will see that extending a simple truncation beyond the propagators by dynamically including the vertex between the gauge field and the scalar triples the number of dressing functions to be calculated and requires five times as many kernels. Furthermore, the new kernels are substantially more complicated than the first one. Seeing such complexity arise from such a simple extension we felt it was time to think about automatizing this process. This seems even more necessary since computing time is no longer as restrictive as it was ten years ago. For example, fourteen years ago the first solution of the DSE system of Yang-Mills theory that was complete at the propagator level [13, 14] relied on an angle approximation and took several hours. Nowadays it is possible to do it with the full momentum integration in a few minutes. Thus more complicated truncations and theories are
definitely doable. However, right now one has to invest much time in deriving DSEs and implementing them. In a sense we fell behind the possibilities today’s computers offer and we think we should try to change this and find means that allow to focus more on the physical rather than the technical problems.

The technical part of investigating a theory numerically with DSEs consists of two main steps: First, the equations have to be derived. Second, one has to implement them in a numeric code. A tool that assists in the first part is already available with the Mathematica [15] application DoFun [16]. Here we present a generic numeric code that can serve as basis for the second step. CrasyDSE (Computation of Rather lArge SYstems of DSEs) is capable of dealing with a high number of Green functions and their dressing functions[1]. Furthermore it provides several predefined integration routines and numerical approximation techniques. It also can be extended to multicore environments and finite temperature without much effort. Part of CrasyDSE is the Mathematica package CrasyDSE.m to generate C++ expressions for the kernels. This first of all alleviates the generation of the code tremendously and reduces human errors and secondly is the easiest way to transform the notation of the user into the notation of CrasyDSE. Note that the functions of the package can deal with all regular Mathematica input and do not rely on DoFun. In order to use the package, the file CrasyDSE.m has to be copied from main Mathematica to a place where Mathematica can find it. We suggest to copy it to the Mathematica subdirectory Applications. Now the package can be loaded with <<CrasyDSE'.

In the following we will describe the general procedure to solve DSEs in section 2. The numerical problem is formulated in section 3 and section 4 contains details on the provided routines to solve DSEs. Secs. 5 and 6 explain the application of CrasyDSE using as examples the calculation of two DSEs of a scalar field coupled to Yang-Mills theory and the calculation of the propagators of pure Yang-Mills theory. Finally, we give a summary and an outlook in section 7.

2. Solving Dyson-Schwinger equations

Our approach to solving DSEs can be separated into three parts as illustrated in Fig. 1:

1. Derive the equations from the given action by the method of choice. If not done in Mathematica, enter them into Mathematica.
2. Use the Mathematica package CrasyDSE.m to generate the C++ files with the kernels. Alternatively, in simple cases one can write the kernel files manually.
3. Use the kernel files with the C++ code of CrasyDSE to solve the DSE numerically.

We illustrate the last steps with two examples in sections 5 and 6.

For the first step, the derivation of the DSEs, we recommend the Mathematica application DoFun (Derivation Of FUNctional equations) [16]. Its predecessor is the Mathematica package DoDSE (Derivation Of DSEs) [6] which was of great help in the investigation of big systems of DSEs like that of the maximally Abelian gauge [12] and the Gribov-Zwanziger action [17, 18]. The calculation of some infrared properties in the maximally Abelian gauge would even have been impossible

without automatization due to the huge number of terms \cite{19, 20}. Later on DoDSE was considerably extended and the derivation of functional renormalization group equations was included \cite{16}, thus it was renamed to DoFun (Derivation of functional equations). However, note that CrasyDSE works completely independent of DoFun.

The second step consists in making the Mathematica expressions accessible for C++. We chose to write our own functions that generate complete C++ files. Thus we maintain as much control as possible over the process and make it more transparent for the user. However, in principle it would also be possible to let Mathematica and C++ interact in a more direct way via MathLink. All the necessary functions to create the C++ files are included in the package CrasyDSE.m, but the notebook from which it is created is also provided so that the user has direct access and can most easily adapt code if required.

Finally, after all kernels have been written into C++ files, one uses the provided C++ modules to solve the equations. Typical initial work includes defining model parameters, defining the required Green functions and their dressing functions, choosing integration routines and defining the renormalization procedures. The way dressing functions are defined is thereby quite arbitrary: One can use closed expressions, for example, from fits to lattice data, interpolations or expansions in sets of polynomials.

In order to help with starting calculations with CrasyDSE we provide several examples with the main code. Furthermore we want to stress that CrasyDSE can not be considered a black box. In order to successfully use it, the user has to understand many of the employed routines and adapt them if required.

3. General formulation of the numerical problem

As already mentioned DSEs form an infinitely large set of equations. For numeric calculations we take a subset of these equations, but they will always depend on Green functions whose DSEs are not part of the truncation. Depending on the details of our truncation scheme we can either provide expressions for them as external input or drop diagrams containing such Green functions. Furthermore, every Green function can consist of several dressing functions and before we can do any calculation we have to project the DSEs such that we deal with scalar integrals. Sometimes it is not possible or not feasible to project directly onto the dressing functions and additionally a linear system of equations has to be solved to get results for them.

But let us for now assume for the sake of the argument that it is possible to project directly onto the dressing functions. They are called $A^{\sigma,i}(x_g)$ here where $g$ denotes the Green function to which it belongs and $i$ labels the dressing functions of
a Green function. $x_g$ denotes the external momenta. We have to solve the following integral equations:

$$A^{g,i}(x_g) = A^{g,i}_{\text{bare}} + \sum_l Z^{g,i,l} \int_{R^d} d^d y F_l(y, x_g, \{A^{h,j}\}, \{A_{\text{model}}\}),$$

(1)

$$A^{g,i} : \Omega_g \subseteq \mathbb{R}^{d_g} \to \mathbb{R},$$

where $A^{g,i}_{\text{bare}}$ are the bare dressing functions (if non-zero and possibly including a renormalization constant). The sum over $l$ denotes contributions from different graphs and $Z^{g,i,l}$ are the renormalization constants of the bare $n$-point function of a given graph. The integration is over the loop momenta $y$. The dimensions of external and internal momenta are indicated by $d_g$ and $d_l$, respectively. The $F_l(y, x_g, \{A^{h,j}\}, \{A_{\text{model}}\})$ denote the kernels of the integrals. They depend on the internal and external momenta explicitly and via several Green functions also implicitly. Some of them, the $A^{h,j}$, are a dynamic part of the truncation, whereas others, the $A_{\text{model}}$, are given by external input.

Before solving this system of equations numerically the following steps are required:

1. The dressing functions $A^{g,i}$ have to be approximated, e. g. by discretization of the argument or expansion in an orthogonal set of functions (see section 4.1).
2. If the integrals are divergent, one needs a regularization prescription, e. g. a sharp cutoff in $|y|$ or BPHZ [21–23] (see section 5.1).
3. Expressions for the dressings $A_{\text{model}}$ need to be provided (see sections 4.3 and 5.1).
4. A renormalization procedure needs to be defined (see sections 4.3 and 5.1).

After this is settled, the integrals can be evaluated with the provided quadrature (see section 4.2). A solution can be found, for example, by fixed point iteration or Newton’s method, see, e. g., [24, 25]. Both these solution methods are implemented in CrasyDSE, see section 4.3.

### 4. Implementation in C++

Simple solutions of the different problems stated in section 3 are implemented in C++ in three modules

- dressing.cpp/hpp,
- quadrature.cpp/hpp,
- DSE.cpp/hpp,

where dressing.cpp/hpp uses the structure dse from DSE.cpp/hpp.

Additionally some simple general functions are stored in function.cpp/hpp. All these files can be found in the directory main. The provided examples are located in separate directories in examples. They are initialized and called in the files sphere_main.cpp, interp_main.cpp, scalar_main.cpp and YM4d_main.cpp. A very basic Makefile using the g++ GNU compiler is provided with each of the examples.
The provided modules are as self-contained as possible and can be arbitrarily extended, e. g. by including further interpolations in dressing.cpp/hpp, adaptive integration algorithms in quadrature.cpp/hpp or additional solving strategies in DSE.cpp/hpp.

4.1. Approximation (dressing.cpp/hpp)

All functions and parameters are referenced to or stored in the structure dse defined in DSE.hpp where default values can be set via init_dse_default and allocation/deallocation works via init_A_xA/dealloc_A_xA.

A Green function represented by a structure dse contains dim_A functions of dim_x - dim_mat continuous arguments x and dim_mat discrete variables z ∈ Ω ⊂ Zdim_mat that are interpolated in the first dim_x - dim_mat continuous variables. All functions have the same domain Ω which is assumed to be a cube in the continuous variables which can in general depend on the discrete arguments i. e.

\[
\Omega = \{(z_0, \ldots, z_{\text{dim}_\text{mat}-1}, x_0, \ldots, x_{\text{dim}_\text{mat}-\text{dim}_\text{x}}) | z \in \Omega, x_i \in [a_i(z), b_i(z)]\}. \tag{2}
\]

In other words we interpolate functions of dim_x - dim_mat variables defined on the direct product \(\prod_{i=0}^{\text{dim}_x-\text{dim}_\text{mat}-1} [a_i(z), b_i(z)]\) for every discrete variable z. If the interpolated function is called outside its domain Ω checked via def_domain a user defined function offdomain will be called where additional information on which side the interval \([a_i(z), b_i(z)]\) has been left is stored in the argument pos with \(\text{pos}[i]=1\) or 2 if \(x_i < a_i(z)\) or \(x_i > b_i(z)\), respectively.

The \(\prod_{i=0}^{\text{dim}_x-1} n_A[i]\) expansion coefficients of the functions are stored in the array A where two methods are provided to express the dressing functions:

- linear interpolation: Lin_gen_dress_interp,
- expansion in Chebyshev polynomials: Cheb_gen_dress_interp.

In case of a linear interpolation A contains the function values on a rectilinear grid stored (together with the discrete arguments) in x_A whereas in case of an expansion in Chebyshev polynomials the expansion coefficients are stored in A and only the discrete arguments need to be stored in x_A. For a Chebyshev expansion it is possible to either transform the standard interval \([-1,1]\] linearly or logarithmically to \([a_i(z), b_i(z)]\) via cheb_trafo[i]. Furthermore, it is possible to expand the logarithm of a function instead of the function itself using cheb_func_trafo[i]. For an example on this see 6.2.

To summarize the user has to provide the following functions:

- def_domain: Checks whether the domain of interpolation has been left.
- offdomain: Determines which function value is returned if the domain of interpolation is left depending on which side the intervals defining the domain Ω have been left.
The correct initialization and definition of the required parameters and functions is illustrated by the example `interp_main.cpp` interpolating three functions linearly and with Chebyshev polynomials. These functions have two discrete and three continuous variables where the domains in the continuous variables depend on the values of the discrete variables.

4.2. Integration (quadrature.cpp/hpp)

All functions and parameters are referenced to or stored in the structure `quad` defined in `quadrature.hpp` where allocation and deallocation works via `init_quad` and `dealloc_quad`.

This module provides the means to integrate the function

\[
\text{integrand}: \mathbb{R}^{\text{dim}} \to \mathbb{R}^{\text{nint}},
\]

multiplied with some Jacobian `jacob` and a constant `coeff` over

\[
\int_{a_0}^{b_0} \int_{a_1(x_0)}^{b_1(x_0)} \cdots \int_{a_{\text{dim}-1}(x_0,\ldots,x_{\text{dim}-2})}^{b_{\text{dim}-1}(x_0,\ldots,x_{\text{dim}-2})} dx_{\text{dim}-1}.
\]

The boundaries have to be defined in `boundary`. One also has to specify if the boundaries depend on the external momenta via `bound_type`, because boundaries independent of the external momenta allow a faster integration routine. Furthermore one can define `nint_para` different sets of parameters `int_param` for which the `integrand` is integrated by once calling the integration routine. They are initialized in `init_para` and are here mainly used for storing the external momenta.

The integration of every integration variable \(x_i\) can be split into several parts. The number of integration regions is given by `n_part[i]`. For each region, labeled here by \(j\), a quadrature rule can be chosen with `type[j]` and the number of integration points is set by `nodes_part[j]`. Currently the following quadrature rules are available:

- Gauss-Legendre
- Gauss-Chebyshev type two
- Fejers second rule
- double exponential (needs additional parameter `param[0]`)

They are defined on the interval \([-1, 1]\), which can be transformed linearly or logarithmically [26] to the integration interval defined in `boundary` via setting `traf[j]` appropriately.

To summarize the user has to provide the following functions:

- `init_para`: Initializes parameters `int_param` for every one of the `nint_para` different sets of parameters.
- `boundary`: Initializes the boundaries \(a_0, b_0, \ldots, a_{\text{dim}-1}(x_0,\ldots,x_{\text{dim}-2})\) and \(b_{\text{dim}-1}(x_0,\ldots,x_{\text{dim}-2})\), where each boundary can depend on previous integration variables.
• **integrand**: Defines the integrand of the integral. Usually this will be the kernel function created by the *Mathematica* notebook.

• **const**: Constant factor of integrands (independent of `nint_para` different values for `int.param`).

• **jacob**: Defines the Jacobian of the integral measure.

The correct initialization and definition of the needed parameters and functions is illustrated by a simple example. In *sphere_main.cpp* the function

\[
sphere\_integrand : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (x, y, z) \mapsto (1, x^2 + y^2, (x^2 + y^2)^2),
\]

times the Jacobian `sphere_jacob= r` is integrated over

\[
\int_{-R}^{R} dz \int_{0}^{2\pi} d\phi \int_{0}^{\sqrt{R^2 - z^2}} dr.
\]

Furthermore, we exemplify here the use of external parameters. They are initialized in *sphere_init_para* and are used as multiplicative factors for the integrals.

**4.3. Solving DSEs (DSE.cpp/hpp)**

Assuming that together with the quadrature a proper regularization has been chosen all the integrals in Eq. (1) are known and we are left with the task of solving the given integral equations including a proper renormalization.

Every Green function present in our truncated set of DSEs is represented by its own structure `dse` which contains all the necessary functions and parameters in order to evaluate its dressing functions via the function `dress`. This is already all one needs to initialize the modeled Green functions, whereas those we are going to solve for need additional information in their structures. The fact that the equations are coupled and the iteration of one dressing function needs information about dressing functions of other Green functions is handled by `otherGF` which contains a copy of the needed structures which are then called when evaluating the integration kernels. Therefore it is necessary to allocate the arrays in all other Green functions before copying them to `otherGF` such that the correct pointer addresses are available. Only variables which are not supposed to be changed during the iteration procedure will be copied by value. Additionally some model parameters might be needed by the (modeled) dressing functions which are pointed to by every structure `dse` via `mod` which has to be defined by the user. For an example see Fig. 4, where also the `quad` structures are indicated which contain specifics on the integration routines needed to evaluate the loop integrals in the graphs.

Focusing now on one specific Green function these graphs are grouped into `n_looporder` contributions where all `n_loop[i]` members of one of the groups can be evaluated using the same quadrature `Q[i]`. The evaluation of the self energy for the `dim_A` dressing functions and `ntot_A` different array points `x_g` is then performed by calling `selfenergy` and stored in `self_A`.

Translated to the language of the structure `quad Q[i]` the correspondences are then `nint ↔ dim_A*n_loop[i]`, `int_param ↔ dse`, `nint_para ↔ ntot_A` and
4.3.1. Iteration

We implemented a fixed point iteration solution technique, i.e. the system is solved by calculating

\[ A_{g,i}^{(k+1)}(x_g) = A_{g,i,bare}^{(k)} + \sum_l Z_{g,i,l}^{(k)} \int_{\mathbb{R}^d} dy_F l \left( y, x_g, \{ A_{h,j}^{(k)} \}, \{ A_{\text{model}} \} \right), \tag{7} \]

from the previous dressing functions \( A_{g,i}^{(k)} \). The calculation of the new dressing functions \( A_{g,i}^{(k+1)} \) is performed in several steps, where it might be necessary that a subset of Green functions is iterated till convergence for every “meta-iteration” of the full set of equations.

Before starting the iterations the initial dressings \( A_{g,i}^{(0)} \) of every Green function have to be set via \texttt{init.dress}. The last step in every iteration is then to renormalize such that the \( A_{g,i}^{(k+1)} \) have the correct values/derivatives at the renormalization scale(s) \( \mu \) via \texttt{renorm.param} parameters which may need some initialization in \texttt{init.renorm.param}. In the example of section 5 this will be done by fixing the \texttt{renorm.Z} renormalization constants \( Z_{g,i,l}^{(k)} \) accordingly in the function \texttt{renorm}.

Note that renormalization constants also appear in the bare Green functions \( A_{g,i,bare}^{(k)} \), but one could employ subtracted equations to drop them.

The iterations of a single Green function are then performed in \texttt{solve.iter} where different stopping criteria (absolute difference \texttt{epsabs}, relative difference \texttt{epsrel} and maximum number of iterations \texttt{maxiter}) are available. The same stopping criteria are available for the full set of implemented Green functions in \texttt{meta.solve.iter} where for every meta-iteration \texttt{solve.iter} is called once for every Green function. This allows different relative iterations, e.g. all Green functions are iterated once for every meta-iteration or a subset of Green functions is iterated till convergence while another subset is iterated only once.

To get an idea of the efficiency of our code we compared the calculation from section 5 with an independently created code that was optimized for this problem [27]. In general the difference in time depends on how much optimization is possible in the given problem. In case of the scalar-gluon vertex the time difference for one iteration was less than a factor of 2.

4.3.2. Newton method

Another method for solving DSEs is based on Newton’s method to solve a nonlinear system of equations. It was already used in many DSE calculations, see, for instance, [24, 25, 28]. For this method the system of DSEs is rewritten into the following form:

\[ E^{(g,i)} = -A^{g,i}(x_g) + A_{g,i,bare}^{(k)} + \sum_l Z_{g,i,l}^{(k)} \int_{\mathbb{R}^d} dy_F l \left( y, x_g, \{ A_{h,j}^{(k)} \}, \{ A_{\text{model}} \} \right). \tag{8} \]
We assume here that the dressing functions $A^{g,i}(x_g)$ are expanded in a set of polynomials. The corresponding expansion coefficients are the unknown variables $c^{(g,i,j)}$, where $j$ labels the polynomials. The goal is to find those values for $c^{(g,i,j)}$ that make all $E^{(g,i)}$ vanish. Newton’s method yields new coefficients by the following formula:

$$c^{(g,i,j)} = c^{(g,i,j)} - \lambda \sum_{(g',i')} \left( J^{(g,i,j)}_{(g',i')} \right)^{-1} E^{(g',i')} ,$$

where the Jacobian $J$ is given by

$$J^{(g,i,j)}_{(g',i')} := \frac{\partial E^{(g',i')}}{\partial c^{(g,i,j)}} .$$

The backtracking parameter $\lambda$ can be used to optimize this step by choosing an appropriate value between zero and one. The determination of $\lambda$ can be subject of sophisticated algorithms, see, for example, [25]. Here, however, we simply cut $\lambda$ in half if the norm of the new $E^{(g,i)}$ is not smaller than that of the old one. If the starting functions are well chosen this is sufficient for the example of section 6.

The stopping criteria for the iteration are the norm of the vector of the $E^{(g,i)}$ and a maximal number of iterations. A single iteration step takes here much longer than for the fixed point iteration described in section 4.3.1 because the calculation of the Jacobian is rather expensive. In principle the derivatives required to get $J$ can be done directly, but here Broyden’s method is used which defines an approximate Jacobian by a simple forward differentiation with small step size $h$:

$$J^{(g,i,j)}_{(g',i')}^{\text{approx}} := \frac{E^{(g',i')}(c^{(g,i,j)}) - E^{(g',i')}(c^{(g,i,j)} + h)}{h} .$$

This prescription proved very reliable for the example treated in section 6.

5. Solving the gap and vertex equations of Yang-Mills theory coupled to a scalar field

In this section we describe how to solve a truncated set of DSEs of Yang-Mills theory coupled to a scalar field. We will first give a short overview of the employed truncation and then explain how to solve it with CrasyDSE.

The derivation of the DSEs is not discussed here, but we provide details in the Mathematica notebook DoFun_YM+Scalar.nb. The results of this notebook form the basis on which we create the functions for the C++ code. The corresponding steps are contained in a second notebook, CrasyDSE_YM+scalar.nb. It describes how the expressions of the DSEs have to be modified so they can be used as input for CrasyDSE. In a second part all required definitions are provided and the kernels are created. We will explain only the latter here, since the first steps consist only of standard Mathematica transformations and are not special to CrasyDSE. Finally we explain some details for this specific example in the C++ code. The provided files allow to follow the complete procedure, from the derivation of the DSE to their numeric solution, in detail.
5.1. Yang-Mills theory coupled to a scalar field

In nature elementary matter fields are fermions. In quantum chromodynamics, for example, these are the quarks which interact via gluons. However, since their spin is \(1/2\), quarks are Dirac fields and consequently represented by spinors. An advantage of functional methods is that they do not suffer from fundamental problems when dealing with anti-commuting fields. However, calculations are complicated by the Dirac structure, since it allows more dressing functions than for a simple scalar, see, e. g., [29]. While at the level of propagators this is still doable, see, e. g., [30–33], three-point functions become already quite tedious [34]. Since some non-perturbative phenomena like confinement may not depend on the fields being spinors or scalars, one can alleviate calculations by replacing the quarks by scalar fields. In order to mimic quarks such fields also have to be in the fundamental representation. The calculation used as an example for the presentation of \textit{CrasyDSE} in this section is motivated by investigations along these lines [27, 35–38].

The action of this theory in Landau gauge reads in momentum space

\[
S[A, \bar{\varphi}, \varphi] = \int \frac{d^d q}{(2\pi)^d} \left( \frac{1}{2} Z_3 A^a_\mu(q) \left( q^2 g_{\mu\nu} - q_\mu q_\nu \right) A^a_\nu(-q) \right.
+ \hat{Z}_3 \varphi^i(q^2 + Z_m m^2) \varphi^i \bigg) \\
+ \int \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \left( i Z_1 g f^{abc} q_1^\mu A^a_\mu(q_1) A^b_\nu(q_2) A^c_\nu(-q_1 - q_2) \right.
+ \hat{Z}_1 g T^a_\mu(q_2 \mu + q_1 \mu) A^a_\mu(q_1) \bar{\varphi}^i(q_2) \varphi^i(-q_1 - q_2) \bigg) + \ldots \tag{12}
\]

The dots correspond to four-point vertices and terms with Faddeev-Popov ghosts. The former are dropped in our truncation and the latter do not appear in the DSEs considered here.

In the following we will focus on the scalar sector of the theory, viz. the scalar propagator and the scalar-gauge field vertex. As can be seen from the truncated DSEs of the scalar two-point function and the scalar-gauge field vertex in Fig. 3 we have the following four quantities left in our truncation: the propagators of the scalar and the gauge fields, the three-gauge field vertex and the scalar-gauge field vertex.

The scalar propagator and the scalar-gluon vertex are parametrized as

\[
(D_s)_{mn}(p) = \frac{A_s(p^2)}{p^2} \delta_{mn} \tag{13},
\]

\[
\Gamma^s_{mn,\mu} (p, q) = g(T^a)_{mn} \left( A^a(p^2, q^2, z) p_\mu + A_a(p^2, q^2, z) q_\mu \right),
\]

where in case of the vertex \(p\) and \(q\) are the momenta of the scalar particles and \(z = p \cdot q/(|p||q|)\). Introducing a sharp momentum cutoff \(\Lambda\) to regularize the self-energy contributions we need to approximate the three dressing functions
\[ A_s : [0, \Lambda^2] \to \mathbb{R} , \]
\[ A_{p,q} : [0, (\Lambda/2)^2] \times [-1, 1] \to \mathbb{R} , \]

which will be done linearly for \( A_{p,q} \) and linearly as well as with Chebyshev polynomials in case of \( A_s \). Choosing the cutoff in the vertex to be smaller by a factor of two has the effect that only the dressing functions \( A_{p,q} \) will be called at large momenta outside their domain when evaluating the self-energy integrals. We will approximate them with their bare value \( A_{p,q} = \hat{Z}_1 \) where necessary.

Additional information is required for the gauge field propagator and the three-gauge field vertex. For the latter we use for simplicity

\[ \Gamma^{AAA,abc}_{\mu\nu\rho}(p_1, p_2, p_3) = Z_3 \hat{Z}_3 \Gamma^{AAA,abc,(0)}_{\mu\nu\rho}(p_1, p_2, p_3) , \]

where finiteness of the ghost-gauge field vertex in Landau gauge \( \hat{Z}_1 = 1 \) and the Slavnov-Taylor identity \( Z_1/\hat{Z}_1 = Z_3/\hat{Z}_3 \) have been used. The bare vertex \( \Gamma^{AAA,abc,(0)}_{\mu\nu\rho}(p_1, p_2, p_3) \) is given in Eq. (23). For the dressing function \( Z(p^2) \) of the Landau gauge field propagator we employ a fit to the solution of the ghost-gluon system within the DSE framework provided in ref. [10] (see also section 6):

\[ Z_g(x) = \left( \frac{\alpha(x)}{\alpha(\mu^2)} \right)^{-\gamma} R^2(x) , \]

\[ R(x) = \frac{c \left( \frac{x}{\Lambda_{QCD}^2} \right)^{\kappa} + d \left( \frac{x}{\Lambda_{QCD}^2} \right)^{2\kappa}}{1 + c \left( \frac{x}{\Lambda_{QCD}^2} \right)^{\kappa} + d \left( \frac{x}{\Lambda_{QCD}^2} \right)^{2\kappa}} , \]

where \( c = 1.269, d = 2.105, \gamma = -13/22, \kappa = 0.5953, \Lambda_{QCD} = 0.714 \text{ GeV} \) and

\[ \alpha(x) = \frac{\alpha(0)}{\ln \left[ e + a_1 \left( \frac{x}{\Lambda_{QCD}^2} \right)^{a_2} + b_1 \left( \frac{x}{\Lambda_{QCD}^2} \right)^{b_2} \right]} , \]

with \( a_1 = 1.106, a_2 = 2.324, b_1 = 0.004, b_2 = 3.169 \) and \( \alpha(0) = 8.915/N_c, N_c \) being the number of colors which we take to be 3. Note that the renormalization constants \( Z_3, \hat{Z}_3 \) can be calculated from the running coupling \( \alpha \) as described in [10].

For the present system the iteration procedure is very stable and we can start from a massless bare scalar propagator \( A_s \equiv 1 \) and a bare scalar-gauge field vertex. As previously mentioned the integrals are regularized via an ultraviolet cutoff. To determine the renormalization constants of the scalar propagator we fix the value \( A_s(\mu^2) \) and \( Z_m m^2 \). Furthermore the vertex will be renormalized by forcing the Slavnov-Taylor identity \( \hat{Z}_1/\hat{Z}_3 = \hat{Z}_3/\hat{Z}_3 \) to be true. With this prescription the system is then multiplicatively renormalizable, i. e. the expressions \( \hat{Z}_3 A_s \) and \( (\hat{Z}_1)^{-1} A_{p,q} \) are independent of the renormalization point \( \mu^2 \). Results confirming this for the propagator are shown in fig. 2.
Figure 2: Dressing function of the propagator times its renormalization constant. Multiplicative renormalizability of the system propagator and vertex is shown by plotting the results for two different choices of $\mu^2$. Independence of the results from the approximation method is shown by showing additionally the results obtained via the Chebyshev expansion method. For comparison the result with a bare scalar-gluon vertex is included.

We can here also illustrate how extending truncations complicates the system of equations: A simple truncation takes into account only the scalar propagator. In this case we need the gluon and the scalar-gauge field vertex as input and we only have to calculate one integral. Its integrand is comparatively simple. Going only one step further by including also the scalar-gauge field vertex requires solving in total for three dressing functions (the vertex has two and the propagator one) by calculating five integrals, whereby the complexity of the integrands has also increased considerably.

We should mention that the employed truncation is not state-of-the-art but it is sufficient to illustrate many features of CrasyDSE. More elaborate results for Yang-Mills theory coupled to a fundamental scalar will be presented elsewhere [27].

5.2. Generating the C++ files with the integrands

We turn now to the creation of the kernels with Mathematica. The following explanations follow the notebook CrasyDSE.nb. To initialize the required functions and the expressions for the DSEs we evaluate the initialization cells with

FrontEndTokenExecute["EvaluateInitialization"]
or via the menu *Evaluation*. Now the variables containing the expressions of the integrals are defined: \texttt{gapAlgProjLoopIntegrand} is the integral of the gap equation and \texttt{vertexAbelianProj1Final}, \texttt{vertexNonAbelianProj1Final}, \texttt{vertexAbelianProj2Final} and \texttt{vertexNonAbelianProj2Final} correspond to the integrals of the Abelian and non-Abelian diagrams projected onto the external momenta \( p_1 \) and \( p_2 \).

Basically for the generation of the C++ files only one function is needed. However, before we can use it we need to define several expressions. They are split into lists containing parameters, momentum variables and dressing function names. In the following the order of the elements in lists is very important. Thus one has to be very careful when one changes something later on, because this could lead to inconsistencies with the C++ code.

In the gap equation two parameters appear: the number of colors \( N_c \) and the coupling constant \( g \). We put them both into a list:

\[
\texttt{parasGap} = \{ N_c, g \};
\]

Furthermore we specify the external (\( \texttt{ps} := p^2 \)) and internal (\( \texttt{qs} := q^2 \), \( \texttt{ct} := \cos(\varphi) = p \cdot q / |p||q| \)) variable names:

\[
\texttt{extVarsGap} = \{ \texttt{ps} \};
\]
\[
\texttt{intVarsGap} = \{ \texttt{qs}, \texttt{ct} \};
\]

Although we did not introduce abbreviations of momentum combinations for the gap equation, we need to define a variable for this, because we will need it later:

\[
\texttt{extraVarsCListGap} = \{ \};
\]

Finally we have to specify which dressings appear in each equation. They are separated into two lists, one for the dressings of the Green function we are calculating and one for all other dressings belonging to other Green functions. The propagator has only one dressing function \( D_s \), so the first list contains one item only:

\[
\texttt{dressingsGap} = \{ \texttt{Ds} \};
\]
But the gap equation also depends on other dressing functions, namely on the one of the gauge field, $D_A$, and on the two of the scalar-gauge field vertex, $D_{As}^{(1)}$ and $D_{As}^{(1)}$. All dressings belonging to the same Green function have to be grouped together in one sublist:

otherGreenFuncsGap = {{DA}, {DAssb1, DAssb2}};

These are the lists required for the gap equation as input for generating the C++ files.

The lists for the vertex equation have the same structure and we only list them here:

parasVertex = {Nc, g};

extVarsVertex = {p1s, p2s, ca};

intVarsVertex = {qs, ct1, ct2};

extraVarsCListVertex = {{p1p2, ca Sqrt[p1s p2s]}, {p1q, Sqrt[p1s qs] Cos[ct2]}, {p2q, (ca ct2 + Sqrt[1 - ca^2] ct1 Sqrt[1 - ct2^2]) Sqrt[p2s qs]}, {p1mq, p1s - 2 p1q + qs}, {p2mq, p2s - 2 p2q + qs}, {p1mp2s, p1s + p2s - 2 Sqrt[p1s p2s] ca}};

dressingsVertex = {DAssb1, DAssb2};

otherGreenFuncsVertex = {{DA}, {Ds}, {DAAA}};

Note that the vertex has two dressings by itself and depends on three other Green functions. Furthermore we provided with the list extraVarsCListVertex the definitions of employed abbreviations, e. g. $p1p2 := p_1 \cdot p_2 = \cos(\alpha)|p_1||p_2|$

Before we generate the C++ files we split off numeric coefficients of the integrands. We call the resulting expressions kernels and coefficients. Instead of doing this by hand, we use the function splitIntegrand. It takes as arguments an expression and a list of variables. Everything in the overall factor that is not a variable will be put into the coefficient:

{coeffGap, kernelGap} = splitIntegrand[gapAlgProjLoopIntegrand, 
Join[extVarsGap, extraVarsCListGap[[All, 1]], intVarsGap]] /. 
Z1h :> 1 // Simplify

--> {(g^2 (-1 + Nc^2))/(
8 Nc [Pi]^3), (1/pplusqs)(1 - ct^2)^^(3/2)

DA[qs] (DAssb1[ps, pplusqs, -((ps + ct Sqrt[ps qs])/Sqrt[pplusqs ps])] - 
DAssb2[ps, pplusqs, -((ps + ct Sqrt[ps qs])/Sqrt[pplusqs ps])]) Ds[pplusqs]}

We discarded the renormalization function Z1h since for the demonstration purposes in this article an unrenormalized vertex is sufficient.

For the vertex we do the same but bear in mind the following structure: Both for coefficients and kernels every loop integral is treated as a single expression, and for every equation all loops are grouped into lists. The syntax of the list of coefficients or kernels is thus
{loop 1 of eq. 1, loop 2 of eq. 1, ...},
{loop 1 of eq. 2, loop 2 of eq. 2, ...}, ...

For the first and second projections we split the integrands as follows:

\[
\{\text{coeffsVertexProjp1}, \text{kernelsVertexProjp1}\} = \text{Transpose[}
\text{splitIntegrand[#,}
\text{Join[extVarsVertex, extraVarsCListVertex[All, 1],
\text{intVarsVertex]} & /@ \{\text{vertexAbelianProjp1Final,}
\text{vertexNonAbelianProjp1Final}\} /. \{Z1h \to 1, Z1 \to 1\} // \text{Simplify}];
\{\text{coeffsVertexProjp2, kernelsVertexProjp2}\} = 
\text{Transpose[}
\text{splitIntegrand[#,}
\text{Join[extVarsVertex, extraVarsCListVertex[All, 1],
\text{intVarsVertex]} & /@ \{\text{vertexAbelianProjp2Final,}
\text{vertexNonAbelianProjp2Final}\} /. \{Z1h \to 1, Z1 \to 1\} // \text{Simplify}]
\]

Again we have discarded the renormalization functions. The final lists of kernels
and coefficients are

\[
\text{kernelsVertex} = \{\text{kernelsVertexProjp1, kernelsVertexProjp2}\};
\text{coeffsVertex} = \{\text{coeffsVertexProjp1, coeffsVertexProjp2}\}
\]

\[
\rightarrow \{\{g/\left(16 \ Nc \ Pi^{-3}\right), -(g \ Nc)/(32 \ Pi^{-3})\}, \{g/\left(16 \ Nc \ Pi^{-3}\right), -(g \ Nc)/(32 \ Pi^{-3})\}\}
\]

We show the coefficients explicitly. One can easily spot the $1/N_c$ and $N_c$ dependences
of the Abelian and non-Abelian diagrams, respectively.

Finally we have everything to generate the C++ code. We do so with the function

\[
\text{exportKernels[\{FileNameJoin[\{NotebookDirectory[], ",\..\}\],}
\text{"scalar\_QCD.hpp"},
\{\{"coeffGap", "kernelGap"},
\{coeffGap},
\{kernelGap},
dressingsGap,
otherGreenFuncsGap,
parasGap,
extVarsGap,
intVarsGap,
extraVarsCListGap},
\{\"coeffsVertex", \"kernelsVertex"},
coeffsVertex,
kernelsVertex,
dressingsVertex,
otherGreenFuncsVertex,
parasVertex,
extVarsVertex,
intVarsVertex,
intVarsVertex,
\]
It will create two files kernelsAll.hpp and kernelsAll.cpp. The filenames are determined by the first argument where we also indicate that the files should be exported to the parent directory. The second argument here is the name of an additional header file which contains functions specific to this example. The third argument contains all the information we gathered above: It is a list where every item corresponds to one DSE. For every DSE we have the following entries:

- The C++ names of the functions containing the coefficients and the kernels.
- A list with the expression(s) for the coefficient(s).
- A list with the expression(s) for the kernel(s).
- The list of dressings for this Green function.
- The list of dressings from other Green functions.
- The list of parameters.
- The list of external variables.
- The list of internal variables.
- The list of extra variables.

Note that without specifying a path in the first argument of exportKernels the files will be created in the directory of the notebook.

We want to mention here the function functionToString, which is used by createFileString but can also be used directly by the user. It creates a string of the expression given as its argument similar to the Mathematica function CForm, but it replaces some common functions like Power or Sin by its C++ counterparts pow or sin. If a function is not included, it can be added by hand, for example:

```
functionToString[a^b + Sin[b]/10 - 5 Sinh[a], {Sinh -> sinh}]
```

--> 0.1*(sin(b)) + -5.*(sinh(a)) + (pow(a, b))

This finishes our work in Mathematica and we proceed with the C++ code.

### 5.3. Numerical code

The C++ code, contained in scalar_main.cpp, is extensively commented and every variable that appears is described directly in the file. Here we only give a rough overview of the required initializations.

In the file scalar_main.cpp first the model and then all Green functions are initialized. For the former the definitions are as simple as providing numeric values for some parameters, e. g. $N_c = 3$. All Green functions are defined as a dse structure, which contains a pointer to mod which is reserved for hosting model parameters, see also Fig. 1. Since the gauge field propagator and the three-gauge field vertex are given by ansätze the only additional information these structures need are the corresponding definitions. The dynamically calculated Green functions also contain
an array of quad structures, namely one for each different integration. Consequently variables like the numbers of integration points and the quadrature types have to be initialized. We also have to provide starting expressions for the dressings and information on the other Green functions contained in a DSE - handled via the array otherGF. For example, for the gap equation these are the gauge field propagator and the scalar-gauge field vertex. In the C++ code dressing functions do not have a specific name, but are just collected in the function dress where it is important at all times to maintain the same assignment of the dressing functions as in the notebook. The integrands of the self-energy are defined in the kernels file created with CrasyDSE.nb. Also a renormalization procedure has to be defined. Finally, the iteration is done with the function meta solve iter.

6. Landau gauge Yang-Mills theory

As a second example we use pure Yang-Mills theory which requires some different methods. The most obvious change is that we use a Newton method instead of a direct fixed point iteration. Again we provide the complete Mathematica and C++ code together with the program. However, as the creation of the kernel files is rather similar to the case of the previous section we refrain from showing here any details on this.

6.1. Truncation and ansätze

As in the last section we will employ the Landau gauge. The DSE system truncated at the level of propagators has been investigated with DSEs for some time now, see, for example, [9, 11, 24, 28, 30, 41]. We will here reproduce the solution of ref. [28]. Besides employing a Newton procedure to solve this set of equations another difference to the previous section is in the renormalization procedure: Here we work with subtracted DSEs.

The system we investigate consists of the ghost and gluon two-point DSEs. The former is used without change, while the gluon DSE is truncated [9, 40]: We neglect all diagrams involving a bare four-gluon vertex. Besides the tadpole diagram this

---

Figure 4: Schematic illustration of structures defined in the code and their dependencies.
Figure 5: The truncated two-point DSEs of pure Yang-Mills theory. Gluons have red, continuous lines and ghost fields green, dashed lines. Thick blobs denote dressed vertices. All internal lines are dressed propagators.

are the two-loop diagrams. They are sub-leading in the UV and it was shown analytically for the scaling solution that they are also subleading in the IR \[42\]. The remaining unknown quantities are the ghost-gluon and the three-gluon vertices for which we use suitable ansätze. The truncated set of DSEs is depicted in Fig. 5.

The ghost and gluon propagators are given by

\[
D_{ab}^{\mu\nu}(p) = \delta_{ab} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{Z(p^2)}{p^2},
\]

\[
G^{ab}(p) = -\delta^{ab} \frac{G(p^2)}{p^2}.\]

For the ghost-gluon vertex \(\Gamma^{A\bar{c},abc}_\mu(p_1, p_2, p_3)\) the bare version, \n
\[
\Gamma^{A\bar{c},abc}_\mu(0)(p_1, p_2, p_3) = ig f^{abc} p_\mu,
\]

is used as motivated originally by an argument of Taylor. However, several studies both on the lattice \[43\] and in the continuum \[44, 45\] confirmed this to be a very reliable ansatz. For the full three-gluon vertex \(\Gamma^{AAA,abc}_\mu(p_1, p_2, p_3)\) we use the tensor structure of the bare vertex \(\Gamma^{AAA,abc}_\mu(0)(p_1, p_2, p_3)\) amended by a dressing \(D^{AAA}(p^2_1, p^2_2, p^2_3)\) that guarantees the correct UV behavior of the gluon dressing function \[28\]:

\[
\Gamma^{AAA,abc}_\mu(0)(p_1, p_2, p_3) = ig f^{abc} (g_{\mu\nu}(p_2 - p_1)_\rho + g_{\nu\rho}(p_3 - p_2)_\mu + g_{\rho\mu}(p_1 - p_3)_\nu),
\]

\[
\Gamma^{AAA,abc}_\mu(p_1, p_2, p_3) = \Gamma^{AAA,abc}_\mu(0)(p_1, p_2, p_3) D^{AAA}(p^2_1, p^2_2, p^2_3),\]

\[
D^{AAA}(p^2_1, p^2_2, p^2_3) = \frac{1}{Z_1} \frac{(G(p^2_1)G(p^2_2))^{1-a/\delta-2a}}{(Z(p^2_1)Z(p^2_3))^{1+a}}.
\]

\(a\) is a parameter chosen as \(3\delta\), where \(\delta\) is the anomalous dimension of the ghost propagator. \(Z_1\) is the renormalization constant of the three-gluon vertex. The dependencies of all Green functions on each other are shown in Fig. 6.

An important issue of the gluon DSE are spurious quadratic divergences. They appear because we employ a numerical cutoff as UV regularization which breaks
gauge invariance. There are several ways to deal with them, see, for example, [11, 28, 46]. Here we subtract them by an additional term in the kernel of the gluon loop in the gluon DSE [28]. The derivation of the DSEs with DoFun and the projection to scalar quantities are described in the notebook DoFun_YM_4d.nb and the creation of the kernel files in CrasyDSE_YM_4d.nb. For details we refer to them.

6.2. Renormalization and solution

For the present system we will use subtracted DSEs, i. e., we subtract from a DSE at external momentum \( p \) the DSE at a fixed external momentum \( p_0 \):

\[
D^{-1}(p^2) = Z^{-1} + \Pi(p^2) \quad \Rightarrow \quad D^{-1}(p^2) = D^{-1}(p_0^2) + \Pi(p^2) - \Pi(p_0^2).
\]  

Thus we can trade the renormalization constant \( Z \) for specifying the value of the dressing function at the subtraction point \( p_0 \). For the gluon propagator we choose the subtraction point \( p_0 \) at sufficiently high momenta since we expect the two-point function to become divergent at low momenta. For the ghost, however, it is most advantageous to specify the dressing at zero momentum. These conditions are boundary conditions for the integral equations. As it turns out two different types of solutions can be found depending on the value of the ghost dressing at zero momentum: Choosing finite values a solution of the family of decoupling solutions emerges [11, 47], while with an infinite zero-momentum dressing we get the scaling solution [9, 11]. The former has a finite gluon propagator and a finite ghost dressing function at zero momentum and the latter an IR vanishing gluon propagator and an IR divergent ghost dressing function. Thereby the divergence of the ghost dressing and the vanishing of the gluon dressing can be described by power laws whose exponents \( \delta_{gh} \) and \( \delta_{gl} \), respectively, are related by \( \delta_{gl} + 2\delta_{gh} = 0 \), whereby \( \delta_{gh} := \kappa = 0.595353 \) can be calculated analytically [44, 48].

There are several choices at which point of the calculation the subtraction of a DSE can be performed. For illustration purposes we employ a different one for each propagator: For the ghost we use the subtracted expression in the kernel file. This is advantageous because the limit of vanishing external momentum can be done analytically but is problematic numerically. For the gluon propagator, on the
other hand, we only create the unsubtracted expressions. The subtraction is then performed with a function in C++. Here the subtraction is numerically no problem.

The specific renormalization procedure has to be taken into account in the renormalization function in the C++ code. Furthermore, it is important to note that this function does not calculate the right-hand side of a DSE as in the example of the scalar system but the difference between the right- and left-hand side of a (subtracted) DSE:

\[ E(p^2) := -D^{-1}(p^2) + D^{-1}(p^0_0^2) + \Pi(p^2) - \Pi(p^0_0^2). \]  

(27)

The reason is the employed Newton procedure as described in section 4.3.2 which attempts to bring \( E(p^2) \) to zero. \( E(p^2) \) is calculated for every external momentum and saved as an array of the DSE structure.

The behavior of the dressing functions in the IR and UV is known analytically. In the intermediate regime, between two given momenta \( \epsilon \) and \( \Lambda \), above and below the IR and UV cutoffs, respectively, they are expressed by an expansion in \( N \) Chebyshev polynomials:

\[ G_{IM}(p^2) = \exp \sum_{i=0}^{N-1} c_i^{(gh)} T_i(M(p^2)), \]  

(28)

\[ Z_{IM}(p^2) = \exp \sum_{i=0}^{N-1} c_i^{(gl)} T_i(M(p^2)), \]  

(29)

where \( M(p^2) \) maps the regime \([\epsilon, \Lambda]\) to \([-1,1]\). For momenta below \( \epsilon \) we employ a power law with the given exponent and the coefficient calculated from the lowest known point in the Chebyshev expansion:

\[ G_{IR}(p^2) = A^{(gh)}(p^2)^{\delta_{gh}}, \]  

(30)

\[ Z_{IR}(p^2) = A^{(gl)}(p^2)^{\delta_{gl}}. \]  

(31)

For momenta higher than \( \Lambda \) an extrapolation in agreement with the UV behavior is chosen:

\[ G_{UV}(p^2) = G(s^2)(w \log(p^2/s^2) + 1)^\delta, \]  

(32)

\[ Z_{UV}(p^2) = Z(s^2)(w \log(p^2/s^2) + 1)^\gamma. \]  

(33)

\( s \) is the highest momentum at which the Chebyshev expansion is known, \( \delta \) or \( \gamma \) are the anomalous dimensions of the ghost and gluon, respectively, and \( w = 11 N_c \alpha(s) G(s)^2 Z(s)/12 \pi \). \( \alpha(p^2) \) is a possible non-perturbative definition of the running coupling \[49, 50\]:

\[ \alpha(p^2) := \alpha(\mu^2) G(p^2)^2 Z(p^2). \]  

(34)

The value of \( \alpha(\mu) \) is an input parameter and sets the scale: At \( \mu \) we have \( G(\mu^2)^2 Z(\mu^2) = 1 \).

2The exponential is chosen due to better convergence properties. For such an expansion see also, for example, ref. [25].
Figure 7: Ghost (left) and gluon (right) dressing functions. The continuous line corresponds to
the scaling solution, the dashed lines are solutions of the decoupling type.

As starting functions for the propagator dressings in the intermediate regime we use

\begin{align}
Z_{\text{ans}}(p^2) &= f_{\text{IR}}(p^2)^2(p^2)^{\delta_{\text{gl}}} + 1, \\
G_{\text{ans}}(p^2) &= f_{\text{IR}}(p^2)(p^2)^{\delta_{\text{gl}}} + c_{\text{UV}} f_{\text{UV}}(p^2),
\end{align}

with the IR and UV damping factors given by

\begin{align}
f_{\text{IR}}(p^2) &= \frac{L_{\text{IR}}}{L_{\text{IR}} + p^2}, \\
f_{\text{UV}}(p^2) &= \left(\frac{p^2}{L_{\text{UV}} + p^2}\right)^2.
\end{align}

The parameter $c_{\text{UV}}$ is used to adjust the starting function to the starting conditions
in order to speed up convergence. These ansätze respect the qualitative IR behavior
which lead to a faster convergence than starting with constant functions. In general
the Newton procedure becomes more stable when using starting functions close to
the solution. The starting functions being not differentiable at the UV matching
poses no problem.

For the integration we used Gauss-Legendre quadratures. Furthermore it was
advantageous to split the integration over the momentum at the value of the external
momentum by setting bound_type of all quadratures to 1. This allows a higher
precision of the final result. However, this comes at the prize of slowing down the
integration as the integration boundaries have to be calculated for every external
momentum.

In Fig. 7 we show the results of the calculations for different boundary conditions
of the ghost. It is clearly visible that all solutions coincide in the UV and only show
their distinct behavior in the IR. In table 1 we provide the input parameters used
for our calculations. The reached precision can be seen from how well the results
fulfill the DSEs, i.e., how close $E(p^2)$ approaches zero. Its norm goes with double
precision down to about $10^{-6}$. Using long double variables instead this value can
be made even lower.

We also tried a second choice for $\alpha(\mu^2)$ to test the code, namely $\alpha(\mu^2) = 0.5$. As
expected the propagators change by a constant factor as expected by multiplicative
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_c$</td>
<td>3</td>
<td>$\kappa$</td>
<td>0.595353</td>
</tr>
<tr>
<td>$\alpha(\mu^2)$</td>
<td>1, 0.5</td>
<td>$h$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>UV cutoff</td>
<td>$10^3$</td>
<td>$\epsilon$</td>
<td>$2 \times 10^{-8}$</td>
</tr>
<tr>
<td>IR cutoff</td>
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<td>$\Lambda$</td>
<td>$0.99 \times 10^4$</td>
</tr>
<tr>
<td>$\delta$</td>
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<td>gluon subtraction point $p_0$</td>
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<tr>
<td>$\gamma$</td>
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<td>value of ghost dressing at $p = 0$</td>
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</tr>
<tr>
<td>$L_{UV}$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Parameters for the calculation. Where several values are given see text for details.

Figure 8: Left: Ghost and gluon propagators. The continuous (red) lines corresponds to the gluon, the dashed (green) lines to the ghost. Thick and thin lines corresponds to different choices of $\alpha(\mu^2)$. Right: Coupling for the two choices of $\alpha(\mu^2)$. The two lines (black straight and gray dashed) are almost indistinguishable.

renormalizability, whereas the running coupling $\alpha(p^2)$ is independent of $\mu^2$, see, for example, [11, 51]. A comparison between $\alpha(\mu^2) = 1$ and 0.5 is depicted in Fig. 8.

Finally we want to make some technical remarks: With these examples we only want to illustrate the basic use of CrasyDSE so the code is not optimized and we expect that the runtime can be improved considerably. Another point is that we also tried a simple fixed point iteration but did not get a solution. This may indicate that this method is not suited for this problem.

7. Summary and outlook

The strength of the framework provided by CrasyDSE lies in its ability to handle large numbers of dressing functions. Thus one of its fields of application is the extension of current truncation schemes by including higher vertices and/or enlarging the tensor bases of Green functions. For example, it is expected that going beyond the current truncation schemes in the Landau gauge makes DSE solutions more competitive to lattice solutions in the mid-momentum regime. Since the number of dressing functions increases at non-zero temperature and/or non-zero density corresponding calculations can profit from CrasyDSE too. Finally there are also interesting cases for which no numerical calculations have been done successfully yet because their systems of DSEs are very complex.
interpolations: linear
expansions: Chebyshev
solution methods: fixed point iteration, Newton
quadratures: Gauss-Legendre, Chebyshev, Fejer, double exponential

Table 2: Currently implemented numerical methods.

Combining CrasyDSE with DoFun there exists a sound framework for the treatment of all such systems of DSEs, from their derivations to their numeric solutions. Furthermore, we plan to extend CrasyDSE by adding further approximation methods or new solving algorithms to the ones given in table 2 as required by individual cases.

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References


