Dualities and Identities for Maximal-Entanglement Quantum Codes

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Abstract—The dual of an entanglement-assisted quantum error-correcting (EAQEC) code is the code resulting from exchanging the original code’s information qubits for its ebits and vice versa. As an introduction to this notion, we show how entanglement-assisted repetition codes and entanglement-assisted accumulator codes are dual to each other, much like their classical counterparts, and we give an explicit, general quantum shift-register circuit that encodes both classes of codes. We later show that our constructions are optimal, and this result completes our understanding of these dual classes of codes. We obtain linear programming bounds for non-degenerate EAQEC codes, by exploiting general forms of these dualities and corresponding MacWilliams identities. We establish the Gilbert-Varshamov bound and the Plotkin bound for EAQEC codes, and all of these bounds allow us to formulate a table of upper and lower bounds on the minimum distance of any maximal-entanglement EAQEC code with length up to 15 channel qubits. Finally, we provide an upper bound on the block error probability when transmitting maximal-entanglement EAQEC codes through the depolarizing channel.

Index Terms—quantum dual code, entanglement-assisted quantum error correction, MacWilliams identity, linear programming bound, entanglement-assisted repetition codes, entanglement-assisted accumulator codes.

I. INTRODUCTION

The theory of quantum error correction is the theory underpinning the practical realization of quantum computation and quantum communication [1], [2], [3], [4], [5], [6]. Quantum stabilizer codes are an extensively analyzed class of quantum error-correcting codes because their encoding, decoding, and recovery are straightforward to describe with algebraic theory [7], [8], [9], [10]. In particular, a quantum code designer can produce quantum stabilizer codes from classical binary and quaternary self-orthogonal codes by means of the CSS and CRSS code constructions, respectively [11], [12], [7], [8].

Entanglement-assisted quantum error correction (EAQEC) is a paradigm for quantum error correction in which the sender and receiver share entanglement before quantum communication begins [13]. An $[[n,k,d]]$ EAQEC code encodes $k$ information qubits into $n$ channel qubits with the help of $c$ pairs of maximally-entangled Bell states. The code can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors acting on the $n$ channel qubits, where $d$ is the minimum distance of the code. Standard stabilizer codes are a special case of EAQEC codes with $c = 0$, and we use the notation $[[n,k,d]]$ for such codes.

Bowen proposed the first EAQEC code [14], by replacing the two ancilla qubits of the three-qubit bit-flip code with two pairs of ebits. He then applied a unitary transformation on the code space to make the EAQEC code equivalent to the well-known five-qubit code [15]. Fattal et al. established a technique for handling entanglement in the stabilizer formalism [16]. Brun, Devetak, and Hsieh then devised the entanglement-assisted stabilizer formalism and showed how to transform any $[[n,k,d]]$ classical quaternary code [14] into an $[[n,2k-n+c,d;e]]$ EAQEC code, where $c$ depends on the properties of the classical code [17]. Lai and Brun further explored the properties of EAQEC codes and proposed an optimization method to find optimal EAQEC codes that cannot be obtained by the aforementioned construction [18]. An $[[n,k,d;c]]$ EAQEC code is optimal in the sense that $d$ is the highest achievable minimum distance for given parameters $n$, $k$, and $c$.

In this paper, we define the dual of an EAQEC code as the code obtained by exchanging the information qubits and the ebits of the original EAQEC code. We establish the EAQecc version of the MacWilliams identity, which we then exploit to construct a linear programming bound on the minimum distance of a non-degenerate EAQEC code. In the special case of maximal-entanglement EAQEC codes [19] they have no degeneracy, and the linear programming bound provides a good upper bound on their minimum distance. We can now show that several code parameters proposed in Ref. [18] are optimal, by exploiting the linear programming bound developed here.

We establish the Gilbert-Varshamov bound for EAQECs, proving the existence of EAQEC codes with certain parameter values. We then apply the encoding optimization algorithm from Ref. [18] to find good EAQEC codes with maximal entanglement for $n \leq 15$. All of these EAQEC codes have minimum distance greater than or equal to that given by the Gilbert-Varshamov bound. Lai and Brun recently proposed a family of $[[n,1,n; n-1]]$ entanglement-assisted repetition codes for $n$ odd [18]; in this paper, we propose a family of

1An $[[n,k,d]]$ classical linear code over a certain field encodes $k$ information digits into $n$ digits, where $d$ is its minimum distance.

2One might wonder why we are considering EAQEC codes that exploit the maximum amount of entanglement possible, given that noiseless entanglement could be expensive in practice. But there is good reason for doing so. The one-shot father protocol is a random entanglement-assisted quantum code [19], [20], and it achieves the entanglement-assisted quantum capacity of a depolarizing channel (the entanglement-assisted hashing bound [21], [14]) by exploiting maximal entanglement. Furthermore, there is numerical evidence that maximal-entanglement turbo codes come within a few dB of achieving the entanglement-assisted hashing bound [22].
entanglement-assisted repetition codes [n, 1, n − 1; n − 1] for n even, which we prove here to be optimal, and thus complete the family of entanglement-assisted repetition codes for any n. We also derive the quantum version of the Plotkin bound [23], which is tight for codes with small k and maximal entanglement. Combining the results of linear programming bounds and the existence of EAQECC codes, we give a table of upper and lower bounds on the highest achievable minimum distance of any maximal-entanglement EAQEC code for n ≤ 15.

The weight enumerator of a classical code gives an upper bound on the block error probability when transmitting coded bits through a binary symmetric channel [24, 25]. Since maximal-entanglement EAQEC codes have many similarities with classical codes [18], [22], we can find an upper bound on the block error probability of transmitting coded quantum information through the depolarizing channel, and this derivation is similar to the classical derivation [24]. We also exploit this result to find an upper bound on the expected block error probability when decoding a random maximal-entanglement EAQEC code.

We organize this paper as follows. We first review the notion of a dual code in classical coding theory and recall a MacWilliams identity for quantum codes. We then give the definition of the dual of an EAQEC code, and we explain this notion with the example of the dual repetition and accumulator EAQEC codes. We follow the terminology and notations of EAQECCs used in [18]. For details, we point the reader to Refs. [13], [18]. The MacWilliams identity for EAQECCs and the linear programming bound for nondegenerate EAQEC codes are derived in Section IV. In Section V, we begin with the Gilbert-Varshamov bound for EAQECCs. Then we describe the construction of [[n, 1, n − 1; n − 1]] entanglement-assisted repetition codes for n even and prove other results about the existence of EAQECCs. We finish this section with a table of upper and lower bounds on the minimum distance of any EAQEC codes with maximal entanglement for n ≤ 15. Section VI details an upper bound on the block error probability under maximum-likelihood decoding, and the final section concludes with a summary and future questions.

II. PRELIMINARIES

In classical coding theory, a well-established notion is that of a dual code. Suppose that C is an [n, k] linear code over an arbitrary field GF(q) with a k × n generator matrix G and a corresponding (n − k) × n parity check matrix H such that \( H G^T = 0_{(n-k) \times k} \). The transpose of the above matrix equation holds, i.e., \( G H^T = 0_{k \times (n-k)} \), revealing that H can play the role of a generator matrix and G can play the role of a parity check matrix. The dual code of C is the [[n, n − k]] linear code \( C^\perp \) with H as a generator matrix and G as a parity check matrix. Suppose the code C has a weight enumerator

\[
W(C; x, y) = \sum_{w=0}^{n} A_w x^w y^{n-w},
\]

where \( A_w \) is the number of codewords in C of weight w. The generalized MacWilliams identity gives a relation between the weight enumerators of C and its dual code \( C^\perp \) [23]:

\[
W(C^\perp; x, y) = \frac{1}{|C|} W(C; x + (q-1)y, x - y).
\]

Consequently, we can determine the minimum distance of the dual code \( C^\perp \) from the weight enumerator of C.

A good example of this duality occurs with a repetition code and an accumulator code. The [n, 1, n] binary repetition code C has the 1 × n generator matrix

\[
G = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},
\]

and an \((n-1) \times n\) parity check matrix

\[
H = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.
\]

Its weight enumerator is \( W(C; x, y) = y^n + x^n \). Flipping the roles of these matrices gives an [n, n − 1, 2] binary accumulator code with the following weight enumerator:

\[
W(C; x, y) = \frac{1}{2} ((x+y)y^n + (x-y)y^n) = \sum_{w=0; w\text{ even}}^{n} \left( \binom{n}{w} x^w y^{n-w} \right).
\]

Prior to this paper, the notion of a dual code did not exist in quantum coding theory. However, the quantum analogue of the MacWilliams identity does exist [26, 9]. Suppose \( S \) is a stabilizer subgroup of the n-qubit Pauli group \( G^n \), and \( \mathcal{N}(S) \) is its normalizer group in \( G^n \). Let

\[
A(z) = \sum_{w=0}^{n} A_w z^w, \quad B(z) = \sum_{w=0}^{n} B_w z^w,
\]

where \( A_w \) and \( B_w \) are the number of elements of the stabilizer \( S \) and the normalizer \( \mathcal{N}(S) \) of weight w, respectively. Then \( A(z) \) is the weight enumerator of the stabilizer \( S \), and \( B(z) \) is the weight enumerator of the normalizer \( \mathcal{N}(S) \). Shor and Laflamme proposed the quantum MacWilliams identity for general quantum codes [26], and Gottesman derived the same identity for stabilizer codes [9]. This identity states that

\[
B(z) = \frac{1}{2^n-n} (1 + 3z)^n A \left( \frac{1 - z}{1 + 3z} \right),
\]

or

\[
B_w = \frac{1}{2^{n-k}} \sum_{w'=0}^{n} \sum_{u=0}^{w} (-1)^{u} 3^{w-u} \binom{w'}{u} (n - w') A_{w'},
\]

for \( w \in \{0, \cdots, n\} \). Note that \(|S| = 2^{n-k}\). This quantum MacWilliams identity is exactly the same as the MacWilliams identity in [1] for a classical linear quaternary code with q = 4 after a transformation of variables.

III. THE DUAL OF AN EAQEC CODE

We begin our exposition of the dual of an EAQEC code by considering an entanglement-assisted quantum code with logical qubits and ebits alone (the code has no ancillas). We call such a code a maximal-entanglement EAQEC code.
Suppose the simplified stabilizer group of an $[[n, k, d; c]]$ EAQEC code with maximal entanglement ($c = r = n - k$) is $S' = \{g_1, \ldots, g_r, h_1, \ldots, h_r\}$, and its simplified logical group is $L = \{x_1, \ldots, x_k, z_1, \ldots, z_k\}$, where $g_i$ and $h_i$ are symplectic partners, and $x_j$ and $z_j$ are symplectic partners. In what follows, we refer to simplified stabilizer generators as symplectic partners, and $S_r$ is the symplectic group.

In other words, if the simplified check matrix and logical are in some sense similar to classical codes because they have no degeneracy [18], [22]. Observe that the normalizer of the simplified stabilizer $S'$ is equivalent to the logical group $L$ and vice versa:

$$N(S') = L, \quad N(L) = S'. $$

In other words, if the simplified check matrix and logical matrix are $H$ and $L$, respectively, we have

$$H \Lambda_2n L^T = 0_{(2r \times 2k)},$$

where

$$\Lambda_2n = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ I_{n \times n} & 0_{n \times n} \end{bmatrix},$$

$0_{i \times j}$ is an $i \times j$ zero matrix, and $I_{n \times n}$ is an $n$-dimensional identity matrix. Thus $S'$ and $L$ are dual to each other with respect to the symplectic product. If we interchange the roles of $S'$ and $L$, they define another EAQEC code, which is the dual of the original EAQEC code.

**Definition 1.** The dual of an $[[n, k, d; n - k]]$ EAQEC code, defined by a stabilizer group $S'$ and a logical group $L$, is an $[[n, n - k, d'; k]]$ EAQEC code, defined by the stabilizer group $S' \subseteq S$ and the logical group $L$ for some minimum distance $d'$.

Equivalently, consider a Clifford unitary encoding operator acting only on ebits and logical qubits is completely specified by the logical operators and the stabilizer operators. To obtain the dual of an EAQEC code, we can just substitute all of the information qubits with halves of ebits and vice versa. Note that the minimum distance $d'$ of the dual code can be determined by the MacWilliams identity for EAQECs, which will be derived in the next section.

As an example, consider the class of $[[n, 1; n - 1]]$ entanglement-assisted repetition codes for $n$ odd [18]. The two logical operators for the one logical qubit of this code are as follows:

$$X X X \cdots X X$$

$$Z Z Z \cdots Z Z$$

These logical operators are directly imported from the generator matrix in (2). The stabilizer generators are as follows:

$$Z Z I \cdots I I$$

$$I Z Z \cdots I I$$

$$\vdots \quad \vdots \quad \vdots$$

$$I I I \cdots Z Z$$

$$X X I \cdots I I$$

$$I X X \cdots I I$$

$$\vdots \quad \vdots \quad \vdots$$

$$I I I \cdots X X$$

One can determine the symplectic pairs by performing a symplectic Gram-Schmidt orthogonalization of the above operators [28]. If we interchange the roles of the stabilizer subgroup and the logical operator subgroup, we obtain an $[[n, n - 1, 2; 1]]$ accumulator code.

To make this more precise, consider the encoding circuits in Figure 1. The circuit in Figure 1(a) is the encoder of a $[[9, 1, 9:8]]$ entanglement-assisted repetition code. Swapping the information qubit for an ebit and all of the ebits for information qubits gives the encoder of Figure 1(b), which encodes a $[[9, 8, 2; 1]]$ entanglement-assisted accumulator code. To illustrate that the circuit is working as expected, let us consider it acting on the first five qubits only (just to simplify the analysis). Inputting the following two operators at the information qubit slot

$$X I I I I$$

$$Z I I I I$$

gives the following two logical operators:

$$X X X X X$$

$$Z Z Z Z Z$$

and these operators match the form of the logical operators for the repetition code in (4). Inputting the following operators at the ebit slots

$$I X I I I$$

$$I Z I I I$$

$$I I X I I$$

$$I I I Z I$$

$$I I I I X$$

$$I I I I Z$$

gives the following operators

$$X X I I I$$

$$I Z Z Z Z$$

$$I X X X X$$

$$Z Z I I I$$

$$I I X I I$$

$$I I I Z Z$$

$$I I I I X$$

$$I I Z Z I$$

which we can transform by row operations to be the same as the operators in (5).

In general, consider an $[[n, k, d; c]]$ EAQEC code with a stabilizer group $S' = \{g_1, \ldots, g_r, h_1, \ldots, h_r\}$ and a logical group $L = \{x_1, \ldots, x_k, z_1, \ldots, z_k\}$, where $g_i$ and $h_i$ are symplectic partners and $x_j$ and $z_j$ are symplectic partners. Let $S_I = \{g_{c+1}, \ldots, g_r\}$ be the isotropic subgroup and $S_S = \{g_1, \ldots, g_c, h_1, \ldots, h_c\}$ be the symplectic subgroup. To extend the notion of a dual code to arbitrary entanglement-assisted codes, the approach is to exchange all of the information qubits with ebits and vice versa, but do nothing with the ancillas. For convenience, define

$$L' = L \cup S_I = \{x_1, \ldots, x_k, z_1, \ldots, z_k, g_{c+1}, \ldots, g_r\},$$

and $S' = S_S \cup S_I$. Thus, $|L'| = 2^{k+r+c}$ and $|S'| = 2^{r+c}$. [14]
Observe that $N(S_S \cup S_I) = L \cup S_I$ and $N(L \cup S_I) = S_S \cup S_I$. Thus $S'$ and $L'$ are dual to each other. The ancillas seem to have little to do with duality in the sense above. That is, the isotropic subgroup $S_I$ only plays a role in error correcting power.

**Definition 2.** The dual of an $[[n, k, d; c]]$ EAQEC code, defined by a stabilizer group $S' = S_S \cup S_I$ and a logical group $L$, is the $[[n, c, d'; k]]$ EAQEC code with $L \cup S_I$ being the stabilizer group and $S_S$ being the logical group for some minimum distance $d'$.

We discuss how to determine the minimum distance $d'$ of the dual code in the next section.

IV. THE MACWILLIAMS IDENTITY AND THE LINEAR PROGRAMMING BOUND

We now derive the MacWilliams identity for general EAQEC codes. Let

$$S_w = \{ M \in S : w t(M) = w \},$$

where $w t(M)$ is the weight of the $n$-fold Pauli operator $M$. Suppose that the weight enumerators of $S' = S_S \cup S_I$ and $L' = L \cup S_I$ are respectively $A(z) = \sum_{w=0}^{n} A_w z^w$ and $B(z) = \sum_{w=0}^{n} B_w z^w$, where

$$A_w = \sum_{E \in S'_w} 1 = |S'_w|, \quad B_w = \sum_{E \in L'_w} 1 = |L'_w|.$$

The following lemma is similar to (7.17) and (7.18) in Gottesman’s thesis [9].

**Lemma 3.** For an operator $E \in G^n$ (where $G^n$ is the Pauli group for $n$ qubits),

$$\sum_{M \in S'} (-1)^{E \otimes M} = \begin{cases} |S'|, & \text{if } E \in L'; \\ 0, & \text{if } E \notin L'. \end{cases} \quad (6)$$

where

$$E \otimes M \triangleq \begin{cases} 0, & \text{if } E \text{ and } M \text{ commute;} \\ 1, & \text{if } E \text{ and } M \text{ anti-commute.} \end{cases}$$

**Proof:** We only give the concept of the proof and a more rigorous proof can be provided. If $E \in L'$, the result is obvious. If $E \notin L'$, $E$ anti-commutes with half of the elements in $S'$. Indeed, suppose $C(z)$ is the weight enumerator of $S_I$. The minimum distance $d$ of an EAQEC code is the minimum weight of an element in $N(S_S \cup S_I)/S_I$ [13]. Given the weight enumerator $A(z)$ of $L \cup S_I$ and the minimum distance $d$ of the EAQEC code, we have $C_w = A_w$ for $w \in \{0, \ldots, d - 1\}$. Equivalently, $d$ is the smallest positive integer $w$ such that $A_{w} - C_w > 0$. However, the MacWilliams identity only gives a lower bound on the minimum distance $d'$ of the dual code—if $B(z)$ is the weight enumerator of $S_S \cup S_I$ and $w'$ is the smallest positive integer such that $B_{w'} > 0$, then $d' \geq w'$. More characteristics of the weight enumerator of $S_I$ are required to determine the minimum distance of the dual code. If the dual code is nondegenerate, then $d' = w'$.

In the case of maximal-entanglement EAQEC codes, $S_I = \emptyset$.

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**Fig. 1.** (a) The encoder for a $[[9, 1, 9; 8]]$ entanglement-assisted repetition code consists of a periodic cascade of CNOT gates. The encoder for arbitrary $[[n, 1, n - 1]]$ entanglement-assisted repetition codes extends naturally from this design. We can implement these encoders with a simple quantum shift-register circuit that uses only one memory qubit [27]. (b) Considering the circuit in (a) but changing the information qubit to an ebit and all of the ebits to information qubits gives the encoder for the dual of the entanglement-assisted repetition code, namely, the $[[9, 8, 2; 1]]$ entanglement-assisted accumulator code. This circuit naturally extends to encode the $[[n, n - 1, 2; 1]]$ entanglement-assisted accumulator codes. Simple variations of the above circuit encode the even $n$ repetition and accumulator EAQEC codes, and we discuss them in Section V-B.
and there is no degeneracy. If we exchange the roles of $S_S$ and $L$, we obtain an $[[n, n - k, d'; k]]$ EAQEC code, which is the dual of an $[[n, k, d; n - k]]$ EAQEC code. The minimum distance $d'$ of this $[[n, n - k, d'; k]]$ EAQEC code is the minimum weight of a nontrivial element in $N(L) = S_S$. Thus $d'$ can be determined from the MacWilliams identity and the weight enumerator of the $[[n, k, d; n - k]]$ EAQEC code, as in the following example.

**Example 1.** The dual of the $[[n, 1, n; n - 1]]$ repetition code is the $[[n, n - 1, 2; 1]]$ accumulator code whenever $n$ is odd. The coefficients of $B(z)$ for the odd-$n$ $[[n, 1, n; n - 1]]$ repetition code are $B_B(n) = (1, 0, 0, \cdots, 0, 3)$. Using the MacWilliams identity, we obtain the weight enumerators of these dual EAQEC codes:

$$A(3) = (1, 0, 9, 6),$$
$$B(3) = (1, 0, 0, 3),$$
$$A(5) = (1, 0, 30, 60, 105, 60),$$
$$B(5) = (1, 0, 0, 0, 0, 3),$$
$$A(7) = (1, 0, 63, 210, 735, 1260, 1281, 546),$$
$$B(7) = (1, 0, 0, 0, 0, 0, 0, 3),$$
$$A(9) = (1, 0, 108, 504, 2646, 7560, 15372, 19656, 14769, 4920),$$
$$B(9) = (1, 0, 0, 0, 0, 0, 0, 0, 3),$$
$$A(11) = (1, 0, 165, 990, 6930, 27720, 84546, 180180, 270765, 270600, 162393, 44286),$$
$$B(11) = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$$

We have also verified the correctness of the above weight enumerators by counting the logical operators with a computer program.

The significance of the MacWilliams identities is that linear programming techniques can be applied to find upper bounds on the minimum distance of EAQEC codes. For a general non-degenerate $[[n, k, d; c]]$ code, we obtain the linear programming bound on its minimum distance from the weight enumerators of $S_S \cup S_I$ and $L \cup S_I$; however, there is no degeneracy for a maximal-entanglement code, and hence the linear programming bound is the upper bound on the minimum distance of such codes. For an $[[n, k, d; n - k]]$ EAQEC code, it must have $B_w = 0$ for $w = 1, \cdots, d - 1$. If we cannot find any solutions to the integer programming problem with the following constraints:

$$A_0 = B_0 = 1;$$
$$A_w \geq 0, B_w \geq 0, \text{ for } w = 1, \cdots, n;$$
$$A_w \leq |S'|, B_w \leq |L|, \text{ for } w = 1, \cdots, n;$$
$$\sum_{w=0}^{n} A_w = |S'|, \sum_{w=0}^{n} B_w = |L|$$
$$B_w = \frac{1}{|S'|} \sum_{w'=0}^{n} \sum_{u=0}^{w} (-1)^{w-u} \binom{w'}{u} \binom{n-w'}{w-u} A_{w'},$$

for $w = 0, \cdots, n$;

$$B_w = 0, \text{ for } w = 1, \cdots, d - 1;$$

for a certain $d'$, this result implies that there is no $[[n, k, d; n - k]]$ EAQEC code. If $d'$ is the smallest of such $d'$s, then $d' - 1$ is an upper bound on the minimum distance of an $[[n, k, d; n - k]]$ EAQEC code. This bound is called the linear programming bound for EAQEC codes with maximal entanglement. If we change the constraint $B_w = 0$, for $w = 1, \cdots, d - 1$ with

$$A_w = 0, \text{ for } w = 1, \cdots, d - 1,$$

this gives the linear programming bound on the minimum distance of the dual code $[[n, n - k, d; k]]$.

**Example 2.** Consider the $[[8, 3, 5; 5]]$ EAQEC codes from the random optimization algorithm in [13]. The linear programming bound shows that there is no $[[8, 3, 5; 5]]$ EAQEC code with $d > 5$, and thus the $[[8, 3, 5; 5]]$ code is optimal.

**Example 3.** Consider the $[[15, 7; 6; 8]]$ EAQEC codes from the random optimization algorithm in [13]. The linear programming bound shows that no $[[15, 7, d; 8]]$ EAQEC code with $d > 7$ exists; however, it does not rule out the existence of the $[[15, 7, 7; 8]]$ code.

**V. Bounds on EAQECs**

This section establishes a table of upper and lower bounds on the minimum distance of maximal-entanglement EAQEC codes for $n \leq 15$. We begin by discussing the existence of arbitrary EAQEC codes, followed by some specific EAQEC code constructions.

The existence of an $[[n, k, d]]$ stabilizer code implies the existence of an $[[n, k, d' \geq d; c]]$ EAQEC code, since we can permute ancilla qubits with ebits and then optimize the encoding operator [13]. Therefore, the lower bound on the minimum distance of standard codes [13] can be applied here. Similarly, the existence of an $[[n, k, d; c]]$ EAQEC code where $c < n - k$ implies the existence of an $[[n, k, d' \geq d; c' > c]]$ EAQEC code. This basically establishes the existence of many $[[n, k; c]]$ EAQEC codes.

\[\text{A. Gilbert-Varshamov Bound for EAQEC Codes}\]

Consider the stabilizer group $S$ of an $[[n, k, d; c]]$ EAQEC code, which is a subgroup of the Pauli group $G^{n+c}$. We consider only the error operators in the group $G^n$ because the entanglement-assisted paradigm assumes that the ebits on Bob’s side of the channel are not subject to errors. Now the EAQEC code is defined in an $(n+c)$-qubit space, but only the first $n$ qubits suffer from errors. Following the argument of the quantum Gilbert-Varshamov bound [4], we can get the Gilbert-Varshamov bound for EAQECs. However, we will show that there are maximal-entanglement EAQEC codes with minimum distance higher than the value predicted by this bound for $n \leq 15$.

\[\text{We used the optimization software LINGO from LINDO Systems to solve the integer programming problem for these examples.}\]
Theorem 5. Given $n, d, c$, let

$$k = \left\lfloor \log_2 \left( \frac{2^{n+c}}{\sum_{j=0}^{d-1} 3^j \binom{n}{j} 2^k} \right) \right\rfloor.$$ 

The Gilbert-Varshamov bound for EAQECs states that if $0 \leq k \leq n - c$, then there exists an $[[n, k, d; c]]$ EAQEC code. Equivalently,

$$\sum_{j=0}^{d-1} 3^j \binom{n}{j} 2^k \geq 2^{n+c}.$$ 

B. Maximal-Entanglement EAQEC Repetition and Accumulator Codes for Even $n$

Lai and Brun proposed a construction of $[[n, 1, n; n-1]]$ entanglement-assisted repetition codes for $n$ odd in [18]. In the case of even $n$, that construction gives a series of $[[n, 0, n; n-2]]$ EA repetition codes with no information qubits.

In this section, we construct $[[n, 1, n-1; n-1]]$ EA repetition codes for $n$ even. We obtained these codes both with the techniques from Ref. [18] and by realizing that the duals of these codes are the $[[n, n-1, 1; 1]]$ EA accumulator codes for even $n$. Theorems 7 and 8 show that both of these code constructions are optimal, in the sense that $[[n, 1, n; n-1]]$ and $[[n, n-1, 2; 1]]$ EAQEC codes do not exist for even $n$. Thus, the results here complete our understanding of the dual classes of entanglement-assisted repetition and accumulator codes for arbitrary $n$.

Theorem 6. There are $[[n, 1, n-1; n-1]]$ entanglement-assisted repetition codes for $n$ even. The duals of these codes are the $[[n, n-1, 1; 1]]$ entanglement-assisted accumulator codes.

Proof: Suppose $H_{(n-1)}$ is an $(n-2) \times (n-1)$ parity-check matrix of a classical $[n-1, 1, n-1]$ binary repetition code:

$$H_{(n-1)} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$ 

We define two $(n-1) \times n$ matrices

$$H_1 = \begin{bmatrix} \vdots & H_{(n-1)} & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{bmatrix},$$

and

$$H_2 = \begin{bmatrix} \vdots & H_{(n-1)} & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$ 

Consider a simplified check matrix of the form

$$H' = \begin{bmatrix} O & H_2 \\ H_1 & O \end{bmatrix}.$$ 

Consider the matrix $H_1H_2^T$. We have that

$$[H_1H_2^T]_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ for } j = 1, \ldots, n-1, \text{ or } i = j-2 \text{ for } j = 3, \ldots, n-2 \\ 0, & \text{else} \end{cases}.$$ 

For example, when $n = 6$,

$$H_1H_2^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

Thus the number of symplectic pairs in $H'$ is as follows [17]:

$$\frac{1}{2} \text{rank}(H' \Lambda H'^T) = \text{rank}(H_1H_2^T) = n-1.$$ 

The simplified logical matrix is

$$L' = \begin{bmatrix} 00 \cdots 00 \\ 01 \cdots 1 \\ 11 \cdots 11 \\ 00 \cdots 0 \end{bmatrix},$$

which implies the minimum distance is $n-1$. Therefore, $H'$ and $L'$ define an $[[n, 1, n-1; n-1]]$ EAQEC code. One obtains the dual $[[n, n-1, 1; 1]]$ codes simply by swapping the roles of the logical matrix and the simplified check matrix.

This completes the family of entanglement-assisted repetition and accumulator codes for any $n$. The encoding circuit of Figure 1 encodes these even-$n$ repetition codes with the exception that the last qubit is removed, the last CNOT in the first string does not act, and the last CNOT in the second string does not act.

The coefficients of $B(z)$ of the even-$n$ $[[n, 1, n-1; n-1]]$ repetition code are $B_{(n)} = (1, 0, \cdots, 0, 1, 2)$. Using the MacWilliams identity, we obtain the weight enumerators of these dual even-$n$ EAQEC codes:

$$A_{(4)} = (1, 1, 15, 27, 20),$$

$$B_{(4)} = (1, 0, 0, 1, 2),$$

$$A_{(6)} = (1, 1, 40, 130, 305, 365, 182),$$

$$B_{(6)} = (1, 0, 0, 0, 0, 1, 2),$$

$$A_{(8)} = (1, 1, 77, 357, 1435, 3395, 5103, 4375, 1640),$$

$$B_{(8)} = (1, 0, 0, 0, 0, 0, 0, 1, 2),$$

$$A_{(10)} = (1, 1, 126, 756, 4326, 15246, 38304, 65604, 73809, 49209, 14762),$$

$$B_{(10)} = (1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2),$$

$$A_{(12)} = (1, 1, 187, 1375, 10230, 47850, 168630, 432894, 811965, 1082565, 974303, 531443, 132860),$$

$$B_{(12)} = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2).$$

The even-$n$ $[[n, 1, n-1; n-1]]$ repetition codes do not saturate the quantum singleton bound or the linear programming bounds. Were an even-$n$ $[[n, 1, n; n-1]]$ code to exist, it would have a weight enumerator $B(z)$ with $B_0 = 1, B_n = 3$, and $C_w = 0$ for $w \neq 0, n$. The weight enumerator of its dual
would also have the coefficients

\[ A_w = \frac{1}{4} \left( 3^w + 3(-1)^w \right) \binom{n}{w}, \]

which are positive integers for \( w = 0, \ldots, n \). It would only be able to correct up to \( \left\lfloor \frac{n-1}{2} \right\rfloor \) channel qubit errors, which is the same number of errors that our even-\( n \) repetition codes can correct. Though, we prove below that even-\( n \) \([n, 1, n; n-1]\) codes do not exist, and thus our even-\( n \) repetition codes from Theorem 6 are optimal.

**Theorem 7.** There is no \([n, 1, n; n-1]\) EAQEC code for \( n \) even.

**Proof:** We prove it by contradiction. Suppose there is an \([n, 1, n; n-1]\) EAQEC code for \( n \) even with a \( 2 \times 2n \) logical matrix

\[
\begin{bmatrix}
u^1 & v^1 \\
u^2 & v^2 \\
\end{bmatrix},
\]

where \( u^1, u^2, v^1, \) and \( v^2 \) are binary vectors of length \( n \). These vectors should satisfy the following condition in order for the above matrix to be a valid logical matrix:

\[ u^1 \cdot v^2 + u^2 \cdot v^1 = 1 \mod 2. \]

Let \( \text{gw}(\cdot) \) be the “general weight” function defined by

\[ \text{gw}(u|v) \equiv \sum_{i: u_i = v_i = 1} 1, \]

where \( u_i \) denotes the \( i \)th bit of the binary \( n \)-tuple \( u \). The above binary vectors should satisfy the further constraints

\[ \text{gw}(u^1|v^1) = \text{gw}(u^2|v^2) = \text{gw}(u^1 + u^2|v^1 + v^2) = n, \]

in order for the code to have distance \( n \) as claimed.

We now exploit the above constraints in order to obtain a contradiction. We first partition the first row of the matrix into subsets \( A, B, \) and \( C \) of \( X, Y, \) and \( Z \) operators, respectively. There should not be any identity operators in the first row in order for the code to have distance \( n \). Up to permutations on the qubits (under which the distance is invariant), the logical matrix has the following form:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
u_A^1 & u_B^1 & u_C^1 & v_A^2 & v_B^2 & v_C^2
\end{bmatrix},
\]

where \( 1 \) is a vector of all ones, \( 0 \) is a vector of all zeros, and we have split up the vector \( (u^2|v^2) \) into different components corresponding to the subsets \( A, B, \) and \( C \). Consider the vector \( u_A^1 \). Suppose that a component \( (u_A^2) \) should equal 1 so that the code’s distance is not less than \( n \). Now suppose that \( (u_A^2) = 0 \). Then \( (v_A^2) \) should also equal 1 so that the code’s distance is not less than \( n \). Otherwise, we could add \((1|0)\) to \((u_A^2|v_A^2)\) and obtain \((0|0)\) as a component of another logical operator, and such a result would imply that the code’s distance is less than \( n \). Now suppose that \( (u_B^2) = 0 \). Then \( (v_B^2) \) should equal 1, by reasoning similar to the above. Thus, the logical matrix should have the following form in order for the code’s distance to be equal to \( n \):

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
u_A^1 & u_B^1 & 1 & 1 & u_B^2 & v_C^2
\end{bmatrix},
\]

where \( \bar{u}_B^2 \) is the binary complement of \( u_B^2 \). Now, the symplectic product of the above two vectors is

\[ (|A| + |B| + |C|) \mod 2 = n \mod 2 = 0, \]

which contradicts the assumption that the original matrix is a valid logical matrix. \( \blacksquare \)

**Theorem 8.** There is no \([n, n-1, 2; 1]\) EAQEC code for \( n \) even.

**Proof:** We prove the theorem by contradiction, in a fashion similar to the proof of the previous theorem. Suppose there is an \([n, n-1, 2; 1]\) EAQEC code for \( n \) even, and suppose its \( 2 \times 2n \) simplified check matrix \([H_X|H_Z]\) has the form

\[
\begin{bmatrix}
u^1 & v^1 \\
u^2 & v^2
\end{bmatrix},
\]

where \( u^1, u^2, v^1, \) and \( v^2 \) are binary vectors of length \( n \). These vectors should satisfy the following condition in order for the above matrix to be a simplified check matrix of a maximal-entanglement EAQEC code with one ebit:

\[ u^1 \cdot v^2 + u^2 \cdot v^1 = 1 \mod 2. \]

We now partition the first row of the simplified check matrix into subsets \( A, B, \) and \( C \) of \( X, Y, \) and \( Z \) operators, respectively. Up to permutations on the qubits (under which the distance is invariant), the simplified check matrix has the following form:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
u_A^1 & u_B^1 & v_C^1 & v_A^2 & v_B^2 & v_C^2
\end{bmatrix},
\]

where we have split up the vector \( (u^2|v^2) \) into different components corresponding to the subsets \( A, B, \) and \( C \). The code has a minimum distance two by assumption and is non-degenerate because it is a maximal-entanglement EAQEC code. Therefore, no column of the above matrix should be equal to the all-zeros vector. Were it not so, then the code would not be able to detect every single-qubit \( X \) or \( Z \) error and would not have distance two as claimed. These constraints
restrict the simplified check matrix to have the following form:

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

Also, no column of the entrywise addition of the matrix to the left of the vertical bar and the matrix to the right should be equal to the all-zeros vector. Were it not so, then the code would not be able to detect every single-qubit error and would not have distance two as claimed. These constraints further restrict the simplified check matrix to be as follows:

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

Now, the symplectic product of the above two vectors is

\[(|A| + |B| + |C|) \text{ mod } 2 = n \text{ mod } 2 = 0,
\]

which contradicts the assumption that the original matrix is a simplified check matrix for a maximal-entanglement EAQEC code with one ebit.

Interestingly, observe that the non-existent logical matrix in \[8\] has the same form as the non-existent simplified check matrix in \[9\]. Were either type of code to exist, we would expect them to be duals of each other, but they both fail to exist because they cannot satisfy the dual constraints imposed on them!

**C. Existence of Other EAQEC Codes**

The following theorem is similar to Theorem 6 in Ref. \[8\]. It shows how to obtain new EAQEC codes from existing ones. These results are helpful in our search for lower bounds on the minimum distance of maximal-entanglement EAQEC codes.

**Theorem 9.** Suppose an \([n, k, d; c]\) code exists. Then

1) An \([n + 1, k, d; c + 1]\) code exists.
2) An \([n, k - 1, d' \geq d; c + 1]\) code exists.

**Proof:**

1) Suppose \(H = [H_X \mid H_Z]\) is a simplified check matrix of an \([n, k, d; c]\) code. Then the simplified check matrix

\[
H' = \begin{bmatrix}
00\cdots0 & 00\cdots0 & 1 \\
00\cdots0 & 100\cdots0 & 0 \\
H_X & H_Z & \vdots \\
0 & 0 & 0
\end{bmatrix}
\]

defines an \([n+1, k, d; c+1]\) code. We have the stabilizer group \((S \otimes I) \cup \{X_{n+1}, Z_{n+1}\}\), where \((S \otimes I) = \{E \otimes I : E \in S\}\).

2) It is obtained by moving a symplectic pair from the logical group to the stabilizer group.

**D. The Plotkin bound for EAQEC Codes**

The Plotkin bound for EAQEC codes is similar to the Plotkin bound for classical codes \[23\]. It is again helpful in our efforts to bound the minimum distance of maximal-entanglement EAQEC codes.

**Theorem 10.** The Plotkin bound for any \([n, k, d; c]\) code is

\[d \leq \frac{3nM}{4(M - 1)},\]

where \(M = 2^{2k}\).

**Proof:** The proof is based on the proof of the classical Plotkin bound in \[23\]. Let \(M = 2^{2k}\) be the number of logical operators in the code. We bound the quantity \(\sum_{u,v \in L} \text{wt}(u \cdot v)\) in two different ways. First, we lower bound it. There are \(M\) choices for \(u\), and for each choice of \(u\), there are \(M - 1\) choices for \(v\). Furthermore, for a code of minimum distance \(d\), \(\text{wt}(u \cdot v) \geq d\) for any \(u \neq v\). So the following lower bound holds

\[M(M - 1)d \leq \sum_{u,v \in L} \text{wt}(u \cdot v) = \sum_{u,v \in L} \text{wt}(u \cdot v).
\]

The equality holds because \(\text{wt}(u \cdot v) = 0\) if \(u = v\). Now we obtain an upper bound on the quantity. We form an \(M \times n\) matrix whose rows are the elements in the logical group \(L\). Let \(m_1, m_2, m_3, m_4\) be the number of \(I, X, Y,\) and \(Z\) operators in column \(j\) of this matrix, respectively. So the equality \(\sum_{j=1}^{4} m_j = M\) holds for all \(j \in \{1, \cdots, n\}\). Each choice of a particular Pauli operator and some other Pauli operator in the same column contributes exactly 2 to the sum \(\sum_{u,v \in L} \text{wt}(u \cdot v)\). Thus, the first equality below holds for this reason, and the second holds by applying \(\sum_{l=1}^{4} m_l = M\):

\[
\sum_{u,v \in L} \text{wt}(u \cdot v) = \sum_{j=1}^{n} m_j (M - m_j) = \sum_{j=1}^{n} \left( M^2 - 4 \sum_{l=1}^{4} (m_l)^2 \right) \leq \sum_{j=1}^{n} \left( M^2 - M^2 / 4 \right) = \frac{3nM^2}{4}.
\]

The first inequality follows by applying \(\sum_{l=1}^{4} m_l / 4 = M/4\) and convexity of the squaring function:

\[
(M/4)^2 = \left( \frac{\sum_{l=1}^{4} m_l / 4}{4} \right)^2 \leq \frac{\sum_{l=1}^{4} (m_l)^2 / 4}{4}.
\]

Combining the lower and upper bounds gives us the Plotkin bound.

Since the proof is independent of the number of ebits \(c\), the Plotkin bound applies to arbitrary EAQEC codes. However, note that \(c\) does not appear in the bound, and consequently, this bound best describes the characteristics of maximal-entanglement EAQEC codes. However, when \(k\) increases, the bound is approximately \(\frac{3}{8}n\), Hence, this bound is useful only for small values of \(k\).
E. Table of Lower and Upper Bounds on the Minimum Distance of Maximal-Entanglement EAQEC Codes

Combining all the results in this and the previous section, we establish Table 1, which details lower and upper bounds on the minimum distance of maximal-entanglement EAQEC codes with length \( n \leq 15 \). The lower bounds in Table 1 are slightly higher than the Gilbert-Varshamov bounds for \( n \leq 15 \). The matching upper and lower bounds for \( k = 1 \) are from the family of entanglement-assisted repetition codes. The upper bounds for \( n \leq 15 \) and \( k \geq 2 \) are from the linear programming bound, which is generally tighter than the singleton bound:

\[
n - k + c \geq 2(d - 1),
\]

and the Hamming bound for non-degenerate EAQECs [14]:

\[
\sum_{j=0}^{t} 3^j \binom{n}{j} \leq 2^{n-k+c}.
\]

The Plotkin bound and the linear programming bound match for \( k \leq 2 \) and \( n \leq 15 \). For \( k = 3 \) and \( n = 4, 5, 6, 9, 10, 11, 13, 14, 15 \), they also match. For \( k > 3 \), the Plotkin bound is not as tight as the linear programming bound, the singleton bound, or the Hamming bound.

The following codes are from [13]: \([7, 2, 5; 5]\) in Example 4; \([8, 3, 5; 5]\), \([13, 3, 9; 10]\) in Example 8; \([15, 7, 6; 8]\), \([15, 9, 5; 6]\), \([15, 8, 6; 7]\) in Example 7; \([7, 3, 4; 4]\), \([8, 2, 6; 6]\), \([10, 3, 6; 7]\) in Table 10.

From part 1) of Theorem 9, we have a \([14, 3, 9; 11]\) code. From part 2) of Theorem 9 and odd-\(n\) EA accumulator codes, we have the codes of minimum distance 2. From Theorem 8 and the even-\(n\) EA accumulator codes, we have the codes of minimum distance 1.

We used MAGMA [29] to find the optimal \([n, k, d]\) codes, and then applied the optimization algorithm in Ref. [13] to obtain the other lower bounds.

VI. THE WEIGHT ENUMERATOR BOUND ON THE BLOCK ERROR PROBABILITY UNDER MAXIMUM LIKELIHOOD DECODING

Since maximal-entanglement codes bear many similarities to classical codes, the block error probability when transmitting coded quantum information through the depolarizing channel can be upper bounded with the weight enumerator of a particular maximal-entanglement EAQEC code (similarly to the case for classical codes [24, 25]). This “weight enumerator bound” gives an idea of the performance of maximum-likelihood decoding of an arbitrary maximal-entanglement EAQEC code. We can also determine the expected performance when decoding a random EAQEC code with a maximum likelihood decoding rule. Below, we determine these bounds and plot them for the maximal-entanglement repetition and accumulator EAQEC codes. The result is that these codes perform comparably to a random EA code with respect to this upper bound.

**Theorem 11.** Suppose that the sender transmits an \([n, k; n-k]\) maximal-entanglement EAQEC code over a depolarizing channel with parameter \( p \), and furthermore, that the receiver decodes this code according to a maximum likelihood decoding rule. Then we have the following upper bound on the block error probability \( P_B \):

\[
P_B \leq B(\gamma) - 1,
\]

where \( B(z) \) is the weight enumerator of the maximal-entanglement EAQEC code and \( \gamma \) is the “Bhattacharyya parameter” for the depolarizing channel:

\[
\gamma \equiv 2 \sqrt{\frac{p}{3} (1 - p) + \frac{2}{3} p}.
\]

**Proof:** Suppose that Alice wants to send a \( k\)-qubit state \( |\phi\rangle \) to Bob and they share \( (n-k) \) pairs of maximally-entangled states \( |\Phi_+\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \). Let \( U \) be a Clifford encoder for the \([n, k; n-k]\) maximal-entanglement EAQEC code. Then the encoded state \( |\bar{\psi}\rangle \) is

\[
|\bar{\psi}\rangle = (U^A \otimes I^B) \left( |\phi\rangle \otimes (|\Phi_+\rangle_{AB})^{\otimes (n-k)} \right),
\]

where the superscript \( A \) or \( B \) indicates that the operator acts on the qubits of Alice or Bob, respectively. Then Alice transmits her qubits (entangled with Bob’s qubits) through \( n \) independent uses of a depolarizing channel \( \mathcal{E} \) where

\[
\mathcal{E}(\rho) = (1 - p) \rho + \frac{p}{3} (X\rho X + Y\rho Y + Z\rho Z),
\]

and \( \rho \) is the density operator of a single qubit. We assume that \( p < 3/4 \) because the channel is completely depolarizing when \( p = 3/4 \). Suppose that an error operator \( \mathcal{E} \in G^n \) occurs after the depolarizing channel, and that \( s^x, s^y \) are the binary vector representations of the error syndrome, each of length

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**Table 1**

Upper and lower bounds on the minimum distance of any \([n, k; n-k]\) maximal-entanglement EAQEC.
(n−k), that Bob observes by first decoding the qubits with a
decoding unitary Un and then performing Bell measurements
on the ebits. This implies that

\[ U^{†} \tilde{E}^{\dagger} |\psi\rangle = \tilde{L}_{0} |\phi\rangle \otimes \left( X^{s^{x}} Z^{s^{z}} \right)^{A} \otimes I^{B} (|\Phi^{+}\rangle)^{A_{2}} \otimes I^{n−k}, \]

and \( \tilde{E} = U(\tilde{L}_{0} \otimes X^{s^{x}} Z^{s^{z}}) U^{†} \) for some logical error \( \tilde{L}_{0} \in \mathcal{L}_{0} \),
where \( \mathcal{L}_{0} \) is the set of unencoded logical operators. Poulin
et al. devised a maximum-likelihood decoder for standard
stabilizer codes [30], and we can modify their decoder to be
a maximal-entanglement EQEC code maximum-likelihood
decoder \( L_{ML}(s^{x}, s^{z}) \), where

\[ L_{ML}(s^{x}, s^{z}) = \text{arg max}_{L \in \mathcal{L}_{0}} \Pr \{ L | s^{x}, s^{z} \}. \]

This decoder selects the most likely recovery operation on the
decoded qubits, given the syndrome information \( s^{x} \) and \( s^{z} \). We
can calculate the above conditional distribution by applying the
Bayes rule to the joint distribution \( \Pr \{ L, s^{x}, s^{z} \} \):

\[ \Pr \{ L | s^{x}, s^{z} \} = \frac{\Pr \{ L, s^{x}, s^{z} \}}{\sum_{L'} \Pr \{ L', s^{x}, s^{z} \}}, \]

where

\[ \Pr \{ L, s^{x}, s^{z} \} = \Pr \{ E \} |_{E = U(L \otimes X^{s^{x}} Z^{s^{z}}) U^{†}} \]

\[ = (1−p)^{n−w_{t}(E)} \left( \frac{p}{3} \right)^{w_{t}(E)} |_{E = U(L \otimes X^{s^{x}} Z^{s^{z}}) U^{†}} \]

\[ = (1−p)^{n} \left( \frac{p}{3(1−p)} \right)^{w_{t}(E)} \left|_{E = U(L \otimes X^{s^{x}} Z^{s^{z}}) U^{†}} \right. \]

The distribution \( \sum_{L'} \Pr \{ L', s \} \) is fixed over all choices of \( L \)
satisfying the constraint \( P = U(L \otimes X^{s^{x}} Z^{s^{z}}) U^{†} \) where \( U \) is
the encoding unitary. Since \( p < 3/4 \iff p/(3(1−p)) < 1 \),
the best choice of \( L \) for the maximum likelihood decoder
\( L_{ML}(s^{x}, s^{z}) \) is the one that selects a recovery operator \( L^{-1} = L \) such that \( E = U(L \otimes X^{s^{x}} Z^{s^{z}}) U^{†} \) with the minimum weight.
This minimum distance decoder is similar to a classical minimum
distance decoder, which is perhaps unsurprising because maximal-entanglement EQEC codes bear many similarities
to classical codes.

Given \( E_{0} \in \mathcal{L}_{0} \), let

\[ Q(E_{0}, \tilde{L}_{0}) = \{ s^{x}, s^{z} : \Pr \{ \tilde{L}_{0} E_{0}, s^{x}, s^{z} \} \geq \Pr \{ \tilde{L}_{0}, s^{x}, s^{z} \} \}. \]

Let \( \mathcal{L} \) be the set of encoded logical operators and \( \mathcal{S}' \) be the
set of simplified stabilizer generators. We can now bound the probability \( P_{B}(\tilde{L}_{0}) \) of a block error given that the error
operator \( \tilde{L}_{0} \) occurs under this decoding scheme:

\[ P_{B}(\tilde{L}_{0}) = \Pr \{ \text{ML decoder fails} | \tilde{L}_{0} \text{ occurs} \} \]

\[ = \Pr \{ L_{ML}(s^{x}, s^{z}) \neq \tilde{L}_{0} \} \]

\[ = \Pr \{ \tilde{L}_{0} \cdot L_{ML}(s^{x}, s^{z}) \neq I \} \]

\[ = \Pr \{ \tilde{L}_{0} \cdot L_{ML}(s^{x}, s^{z}) \in \mathcal{L}_{0} \} \]

\[ \leq \sum_{E_{0} \in \mathcal{L}_{0}} \Pr \{ \tilde{L}_{0} \cdot L_{ML}(s^{x}, s^{z}) = E_{0} \} \]

Since \( \sqrt{\frac{\Pr \{ \tilde{L}_{0} E_{0}, s^{x}, s^{z} \}}{\Pr \{ L_{0}, s^{x}, s^{z} \}} \geq 1 \) for \( s^{x}, s^{z} \in Q(E_{0}, \tilde{L}_{0}) \), we can
multiply each term in sum by this factor and then

\[ P_{B}(\tilde{L}_{0}) \leq \sum_{E_{0} \in \mathcal{L}_{0}} \sum_{s^{x}, s^{z} \in Q(E_{0}, \tilde{L}_{0})} \sqrt{\Pr \{ \tilde{L}_{0}, s^{x}, s^{z} \} \Pr \{ \tilde{L}_{0} E_{0}, s^{x}, s^{z} \}} \]

\[ \leq \sum_{E_{0} \in \mathcal{L}_{0}} \sum_{s^{x}, s^{z} \in Z_{2}^{n−k}} \Pr \{ \tilde{L}_{0}, s^{x}, s^{z} \} \Pr \{ \tilde{L}_{0} E_{0}, s^{x}, s^{z} \} \]

\[ = \sum_{E \in \mathcal{L} \setminus \mathcal{I} \setminus M \in \mathcal{S}} \sum_{s^{x}, s^{z} \in Z_{2}^{n−k}} \Pr \{ \tilde{L}_{0} \} \Pr \{ \tilde{E} \} \Pr \{ \tilde{E} \}, \]

where \( \tilde{L}_{0} = U \tilde{L}_{0} U^{†}, E = U E_{0} U^{†}, \) and \( M = U(X^{s^{x}} Z^{s^{z}}) U^{†} \in \mathcal{S}' \). Observe that

\[ \sum_{E \in \mathcal{L} \setminus \mathcal{I} \setminus M \in \mathcal{S}} \Pr \{ \tilde{L}_{0} \} \Pr \{ \tilde{E} \} \Pr \{ \tilde{E} \} \]

\[ = \sum_{M \in \mathcal{G}} \Pr \{ (M) \} \Pr \{ (L)_{i} \} \Pr \{ (L)_{i} \} \]

\[ = \prod_{i=1}^{n} \Pr \{ (M)_{i} \} \Pr \{ (L)_{i} \} \Pr \{ (L)_{i} \} \]

\[ = \prod_{i=1}^{n} \Pr \{ (M)_{i} \} \Pr \{ (L)_{i} \} \Pr \{ (L)_{i} \} \]

\[ \text{It holds that} \ (\tilde{L}_{0})_{i} \neq I \text{ if} \ (E)_{i} \neq I \text{ and so} \]

\[ \sum_{(M)_{i} \in \mathcal{G}} \sqrt{\Pr \{ (M)_{i} \} \Pr \{ (L)_{i} \} \Pr \{ (L)_{i} \}} \]

\[ = 2 \sqrt{\frac{p}{3(1−p)} + \frac{2}{3} p} = \gamma. \]

Otherwise,

\[ \sum_{(M)_{i} \in \mathcal{G}} \sqrt{\Pr \{ (M)_{i} \} \Pr \{ (L)_{i} \} \Pr \{ (L)_{i} \}} = 1. \]

Consequently,

\[ P_{B}(\tilde{L}_{0}) \leq \sum_{E \in \mathcal{L} \setminus \mathcal{I}} \gamma^{w_{t}(\tilde{E} \tilde{L})} \]

\[ = \sum_{E \in \mathcal{L} \setminus \mathcal{I}} \gamma^{w_{t}(E)} \]

\[ = B(\gamma) − 1. \]

Therefore, the probability \( P_{B} \) of a block error is bounded by
\( B(\gamma) − 1 \) when taking the expectation over all \( \tilde{L}_{0} \).

The above theorem is similar to Theorem 7.5 in Ref. [25],
which determines an upper bound on the block error probability
when transmitting a classical linear code over a binary
symmetric channel.

**Theorem 12.** Let $U$ be the Clifford encoder for a random $[[n, k; n - k]]$ maximal-entanglement EAQEC code. Suppose that the sender transmits this code over a depolarizing channel with parameter $p$ and furthermore that the receiver decodes this code according to a maximum likelihood decoding rule. Then we have the following upper bound on the expected block error probability $P_B$:

$$ P_B = \mathbb{E}_U \{ P_B \} \leq \frac{2^{2k} - 1}{2^{2n} - 1} \left( (1 + 3\gamma)^n - 1 \right), $$

where $\gamma$ is defined in the previous theorem and the expectation is with respect to the choice of random code. In particular, if the rate $k/n$ satisfies the following upper bound:

$$ \frac{k}{n} < 1 - \frac{1}{2} \log_2 (1 + 3\gamma), $$

then the error probability decreases exponentially to zero in the asymptotic limit.

**Proof:** We first establish a method for choosing a random maximal-entanglement EAQEC code. A natural method for doing so is first to fix a basis of Pauli operators $X_1, Z_1, X_2, Z_2, \ldots, X_n, Z_n$, where the first $n - k$ anticommuting pairs correspond to the stabilizer operators for the $n - k$ ebits and the next $k$ anticommuting pairs correspond to the logical operators for the $k$ information qubits. We then select a Clifford unitary uniformly at random from the Clifford group (see Section VI-A-2 of Ref. [32] for a relatively straightforward algorithm for doing so) and apply it to the above fixed basis. This procedure produces $2n$ encoded operators $X_1, Z_1, X_2, Z_2, \ldots, X_n, Z_n$ that specify a random maximal-entanglement EAQEC code.

We now need to determine the expected weight enumerator $\mathbb{E}_U \{ B(z) \} = \sum_{w=0}^n \mathbb{E}_U \{ B_w \} z^w$ for a such random maximal-entanglement code because such a result allows to apply Theorem 11 in order to get an upper bound on the expected block error probability. Each coefficient $\mathbb{E}_U \{ B_w \}$ corresponds to the expected number of Pauli operators of weight $w$ that belong to the logical operator group of a random EA code. Equivalently, it corresponds to the expected number of Pauli operators of weight $w$ that commute with the entanglement subgroup of a random code. First, let us consider $\mathbb{E}_U \{ B_0 \}$. The identity operator is only the Pauli operator with weight zero. It commutes with all operators with unit probability. Thus, $\mathbb{E}_U \{ B_0 \} = 1$. Now, let us consider $\mathbb{E}_U \{ B_w \}$ with $w \geq 1$. We first determine the probability that a Pauli operator $g$ with non-zero weight commutes with the $2(n - k)$ encoded operators $X_1, Z_1, X_2, Z_2, \ldots, X_{n-k}, Z_{n-k}$ for a random EA code. To simplify the calculation, observe that applying a uniformly random Clifford unitary to the operators $X_1, Z_1, X_2, Z_2, \ldots, X_{n-k}, Z_{n-k}$ and then determining the probability that a fixed Pauli operator $g$ commutes with all of them is actually the same as keeping the basis fixed and applying a random Clifford to the operator $g$ itself. This holds because

$$ C f C^\dagger g = g C f C^\dagger \iff f C^\dagger g C + C^\dagger g C f = 0. $$

Then a uniform distribution on the Clifford unitaries takes this operator $g$ to an arbitrary Pauli operator $g'$, and the distribution induced is just the uniform distribution on all of the $2^{2n - 1} - n$-qubit Pauli operators not equal to the identity (this reasoning is the same as that in Section VI-A-1 of Ref. [32]). At this point, the argument becomes purely combinatorial, and the only operators that commute with the above fixed basis are the ones with identity acting on the first $n - k$ qubits. Thus, there are $2^{2k}$ Pauli operators besides the identity that commute with the fixed basis, and we conclude that the probability that a fixed Pauli operator $g$ commutes with the random set $X_1, Z_1, X_2, Z_2, \ldots, X_{n-k}, Z_{n-k}$ is

$$ \frac{2^{2k} - 1}{2^{2n} - 1}. $$

Now we can calculate the expected number of operators that are in the logical subgroup. The number of Pauli operators with weight $w$ is $\binom{n}{w} 3^w$. Consequently, we have

$$ \mathbb{E}_U \{ B_w \} = \frac{2^{2k} - 1}{2^{2n} - 1} \binom{n}{w} 3^w, $$

which implies

$$ \mathbb{E}_U \{ B(z) \} = \mathbb{E}_U \{ B_0 \} = \sum_{w=1}^n \mathbb{E}_U \{ B_w \} z^w = \n \frac{2^{2k} - 1}{2^{2n} - 1} \sum_{w=1}^n \binom{n}{w} 3^w z^w = \n \frac{2^{2k} - 1}{2^{2n} - 1} ((1 + 3z)^n - 1).$$

Therefore, by exploiting the result in Theorem 11, an upper bound on the expected block error probability for general EAQEC codes with maximal entanglement is

$$ \mathbb{E}_U \{ P_B \} \leq \mathbb{B}(\gamma) - \mathbb{B}_0, $$

$$ = \n \frac{2^{2k} - 1}{2^{2n} - 1} ((1 + 3\gamma)^n - 1).$$

We can drive the expected error probability to be arbitrarily low in the large $n$ and $k$ limit by ensuring that

$$ \frac{k}{n} < 1 - \frac{1}{2} \log_2 (1 + 3\gamma). \tag{12}$$

This bound is not as tight as the entanglement-assisted hashing bound (the optimal limit), and Figure 2 displays how these two bounds differ.

We can plot the error probability bound in (10) as a function of $p$ for specific codes such as the repetition codes or the accumulator codes and then compare the results with the average error probability bound. Figure 3 provides such plots and compares their performance with a random EA code, with respect to these bounds.

**VII. DISCUSSION**

In this paper, we studied the properties of EAQEC codes, especially those with maximal entanglement. We defined the dual code of an EAQEC code and derived the MacWilliams identities for EAQEC codes, which are the same as for the classical case. Based on these identities, we found a linear programming bound on the minimum distance of a maximal-entanglement code. We also derived the Plotkin bound and
the Gilbert-Varshamov bound for general EAQEC codes, together with several theorems examining the existence of EAQEC codes. Combining these results, we provided a table of upper and lower bounds on the minimum distance of maximal-entanglement EAQEC codes for $n \leq 15$. Finally, we determined “weight enumerator bounds” on the block error probability when decoding according to a maximum-likelihood decoding rule, and we found that the performance of maximal-entanglement repetition and accumulator codes is comparable to the expected performance of random codes, with respect to this upper bound.

We proposed a construction of $[[n, 1, n-1; n-1]]$ entanglement-assisted repetition codes for $n$ even, which completes the family of entanglement-assisted repetition codes for any $n$. These entanglement-assisted repetition codes are the optimal codes that encode a single information qubit. We also constructed an explicit encoding circuit for these codes.

Most lower bounds in Table 1 are from the optimization algorithm [18]. However, these codes may not be close to optimal when $n + k$ becomes large, for the complexity of the encoding optimization algorithm increases exponentially with $n + k$, making full optimization impossible. To make the bounds in Table 1 tighter, we need to consider other code constructions to raise the lower bounds. One possibility is to extend the encoding optimization algorithm from the form in [18], by starting the optimization from a good EAQEC code and adding additional entanglement. From a given $[[n, k, d; e]]$ code, we can change any subsets of the $n - k - e$ ancilla qubits with ebits and then apply a similar encoding optimization algorithm. From a given “good” $[[n, k, d'; n - k]]$ code, we obtain an optimized $[[n, k, d'; n - k]]$ code much faster than using the original optimization over an $[[n, k, d]]$ code. Since

![Fig. 3](image-url)
we begin with a “good” $[[n, k, d; c]]$ code, the optimized
$[[n, k, d'; n-k]]$ code is expected to approach the upper bound.
Such a “good” code can be obtained from the construction in [13], using optimal classical quaternary codes.
On the other hand, we proved the non-existence of
$[[n, 1; n−1]]$ or $[[n, n−1; 2; 1]]$ codes for $n$ even, which
decreases the upper bound predicted by the linear program-
ming bound for $k = 1$ and $n$ even. We plan to explore the
existence of other $[[n, k, d; n-c]]$ codes to decrease the upper
bound.
Shadow enumerators, designed for self-dual codes, give
additional constraints on the weight enumerator of a stabilizer
code. However, there is no general form in the case of EAQEC
codes.
Finally, consider the possibility of a “self-dual” code
$[[n, n/2; d; n/2]]$ for $n$ even, such that the dual code is also an
$[[n, n/2; d; n/2]]$ code with the same weight enumerators.
That is, $B(z) = A(z)$. We conjecture that such self-dual codes
exist. If so, the two groups $S_2$ and $L$ may be equivalent up to
a permutation on the qubits.

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Fig. 2. The figure plots both the entanglement-assisted hashing bound
$1−1/2[H_2(p)+p\log_2 3]$ from Ref. [23] and the “asymptotic weight
enumerator bound” from [13] as a function of the depolarizing parameter.
The two bounds become close for high depolarizing noise. Interestingly, the
thresholds of the maximal-entanglement EA turbo codes from Ref. [22] are just shy of the asymptotic weight enumerator bound (see Figures 6(b) and 7(b) of that paper).