Fuzzy approximate continuity

M. Burgin* and O. Dumanb

aDepartment of Mathematics, University of California, Los Angeles, California, USA
bFaculty of Arts and Sciences, Department of Mathematics, TOBB Economics and Technology University, Söyüközü, Ankara, Turkey

Abstract. Approximately continuous functions were first introduced in connection with problems of differentiation and integration. The main idea of the approximate continuity of a function f is that the continuity conditions are true not necessarily everywhere but only almost everywhere with respect to some measure, e.g. Borel measure or Lebesgue measure. At the same time, it is known that functions that come from real life sources, such as measurement and computation, do not allow, in a general case, to test whether they are continuous or even approximately continuous in the strict mathematical sense. To overcome these limitations, fuzzy approximate continuity of functions is introduced and studied in this paper.

Keywords: Approximate continuity, fuzzy convergence, density, fuzzy approximate continuity, continuity defect

1. Introduction

One of the standard ways to define continuity for real functions is based on convergence of real number sequences. At the same time, there are different generalizations of the concept of convergence and it is an interesting mathematical problem to define a more general structure of continuity based on a more general concept of convergence. In some cases, it is done. For instance, fuzzy convergence in the sense of neoclassical analysis gives birth to fuzzy continuity (see [8]).

The concept of statistical convergence, introduced by Steinhaus [21] and Fast [15] and later reintroduced by Schoenberg [20], extends the concept of conventional continuity, we come to the following definition. A function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is statistically continuous at a point \( a \in \mathbb{R} \) if \( f(x) \) is defined at \( a \) and for any sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \) that converges to \( a \), the point \( f(a) \) is the statistical limit of the sequence \( \{f(a_i)\} \) or formally, for any sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \), \( \lim_{n \to \infty} a_i = a \) implies \( \lim_{n \to \infty} f(a_i) = f(a) \). Taking statistical continuity instead of conventional continuity, we come to the following definition. A function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is statistically continuous at a point \( a \in \mathbb{R} \) if for every sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \) or formally, for any sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \) that converges to \( a \), the point \( f(a) \) is the statistical limit of the sequence \( \{f(a_i)\} \) or formally, for any sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \) that converges to \( a \), the point \( f(a) \) is the statistical limit of the sequence \( \{f(a_i)\} \) for any sequence \( l = \{a_i \in \mathbb{R} : i = 1, 2, 3, \ldots\} \).
Lemma 2.1. It has to be $d(i \in N; |j_i - \alpha| \geq \varepsilon) = 0$. This contradiction shows that any sequence $\{f(a_i) \in \mathbb{R}; i = 1, 2, 3, \ldots\}$ converges to $f(a)$ when the sequence $\{\alpha_i \in \mathbb{R}; i = 1, 2, 3, \ldots\}$ converges to $\alpha$. Consequently, $f$ is a continuous at the point $a$ function.

Thus, we need a different definition of continuity that is true up to some extent in a statistical or probabilistic way. Such a concept exists and is called approximate continuity.

Approximate limits of functions and approximately continuous functions were first introduced by Denjoy [13] in his work on derivatives. Later they were utilized by Khinchin [17] who introduced concept of an approximate derivative and Denjoy [14] in the study of the Lebesgue and Denjoy–Khinchin integrals. A function $f(x)$ is approximately continuous if and only if it is continuous in the density topology. Later different authors studied and utilized approximately continuous functions (cf., for example, [19]). These functions have many good properties. For example, they have the Darboux property and belong to the first Baire class. Moreover, any bounded approximately continuous function is a derivative of some function.

The goal of this paper is to extend the concept of approximately continuous functions to a more general situation where imprecision, vagueness and fuzziness of real processes and systems is taken into account. To achieve this goal, here we define fuzzy approximate continuity and prove all results not only for the Lebesgue measure but also for any continuous, or equivalently, $\alpha$-additive measure $\mu$. The new development of approximate continuity is based on neoclassical analysis [3–12].

2. Fuzzy approximate continuity at a point

Let us consider the Lebesgue measure $\mu$ on the real line $\mathbb{R}$.

Definition 2.1. The density of a measurable subset $A$ of $\mathbb{R}$ at a point $a \in \mathbb{R}$ is the number

$$b_\mu(A) = \lim_{\epsilon \to 0} \mu([a-\epsilon, a+\epsilon] \cap A)/(2\epsilon)$$

provided the limit exists.

It follows from the definition that if $b_\mu(A)$ exists, then $b_\mu(A) = 1 - b_\mu(A')$ where $A'$ is the complement of $A$, i.e., $A' = \mathbb{R} \setminus A$.

Lemma 2.1. For any sets $A$ and $B$, if $b_\mu(A)$ and $b_\mu(B)$ exist, then we have:

a) if $A \subseteq B$, then $b_\mu(A) \leq b_\mu(B)$,

b) $b_\mu(A \cup B) \leq b_\mu(A) + b_\mu(B)$,

c) if $B \subseteq A$, then $b_\mu(A \setminus B) = b_\mu(A) - b_\mu(B)$,

d) $b_\mu(A \cap B) \leq \min \{b_\mu(A), b_\mu(B)\}$.

Using this density, Denjoy [13] introduced the notion of approximate continuity in the following way.

Let $f$ be a mapping from $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$.

Definition 2.2. A function $f(x)$ is called approximately continuous at a point $a \in X$ if for every $\varepsilon > 0$, the following equality holds:

$$b_\mu(x; |f(x) - f(a)| \geq \varepsilon) = 0.$$ 

This concept extends the concept of a continuous at a point function.

Example 2.1. Let us consider the following function

$$f(x) = \begin{cases} 2x & \text{when } x \neq 1/n \\ 2 & \text{when } x = 1/n. \end{cases}$$

This function is discontinuous and even not fuzzy continuous at $x = 0$, but it is approximately continuous at $x = 0$.

This example reflects a more general situation.

Proposition 2.1. If some interval $I$ that contains a point $a$ has not more than a countable set of removable discontinuities of a function $f(x)$ and $a$ is not one of them, then the function $f(x)$ is approximately continuous at a point $a$.

Proof. Consider the function $g(x)$ obtained by removing all discontinuities of a function $f(x)$ in the interval $I$. Then $g(x)$ is continuous at the point $a$. Consequently, $b_\mu(x; |g(x) - g(a)| \geq \varepsilon) = 0$. At the same time, if $\Phi_{\mu}(\alpha, a, f) = [m[a-\alpha < x < a + \alpha]; |f(x) - f(a)| \geq \varepsilon]/(2\alpha)$ and $\Phi_{\mu}(\alpha, g, a) = [m[a-\alpha < x < a + \alpha]; |g(x) - g(a)| \geq \varepsilon]/(2\alpha)$, then $\Phi_{\mu}(\alpha, f, a) = \Phi_{\mu}(\alpha, g, a)$ when $[a-\alpha, a + \alpha] \subseteq I$ because the Lebesgue measure of any countable set is equal to zero. As a result, we have

$$b_\mu(x; |f(x) - f(a)| \geq \varepsilon) = b_\mu(x; |g(x) - g(a)| \geq \varepsilon) = 0.$$ 

Thus, the function $f(x)$ is approximately continuous at the point $a$. Proposition is proved. □

To define fuzzy approximately continuous functions, we use the following constructions:

$$\Phi_{\mu}(\alpha, A; a) = [m[a-\alpha < x < a + \alpha]; a \in A]/(2\alpha),$$

$$E_\mu(f; a) = |x; |f(x) - f(a)| \geq \varepsilon |/\mu(I).$$

$$E_\mu(f; a) = [a-\alpha < x < a + \alpha; |f(x) - f(a)| \geq \varepsilon |/\mu(I).$$
Then, the next results are immediately obtained from the above definitions.

**Lemma 2.2.** For any real number \(k \neq 0\) and any real functions \(f\) and \(g\), we have:

a) \(E_k(f; a) = E_k(-f, a)\),

b) \(E_k(f \pm g, a) \leq E_k(f, a) \cup E_k(g, a)\),

c) \(E_k(f, a) \cup E_k(g, a) \geq E_k(f, a) \cap E_k(g, a)\),

d) \(E_k((\lambda_k f), a) = E_k(f, a)\) for any \(\lambda \neq 0\) and any real number \(k\).

Let \(\alpha\) be an arbitrary positive real number.

**Corollary 2.1.** For any real number \(k \neq 0\) and any real functions \(f\) and \(g\), we have:

a) \(\Phi_\alpha(f; a) = \Phi_\alpha(-f, a)\),

b) \(\Phi_\alpha(f \pm g, a) \leq \Phi_\alpha(f, a) + \Phi_\alpha(g, a)\),

c) \(\Phi_\alpha(f, a) + \Phi_\alpha(g, a) \geq \Phi_\alpha(f, a) + \Phi_\alpha(g; a)\),

d) \(\Phi_\alpha(f, (\lambda_k g); a) = \Phi_\alpha(f, a)\) for any \(\lambda \neq 0\) and any real number \(k\).

It follows from Definition 2.2 that if \(f\) is approximately continuous at the point \(a\) if and only if for every \(\varepsilon > 0\), we have

\[
0 = \lim_{n \to \infty} \Phi_\alpha(f; a).
\]

Let \(\Upsilon\) be a non-negative real number.

**Definition 2.3.** ([4]). A number \(\Upsilon\) is called an \(r\)-limit of a sequence \(l\) (it is denoted by \(\Upsilon = \lim_{i \to \infty} f (x)\)) if for any sequence \(l = (a_0, a_1, a_2, \ldots)\), the inequality \(|a_i - a_j| < \Upsilon r + \varepsilon\) is valid for all \(i, j\), i.e., there is such \(\Upsilon\) that for any \(i > n\), we have \(|a_i - a_j| < \Upsilon r + \varepsilon\).

**Definition 2.4.** ([7]).

a) A number \(b\) is called an \(r\)-limit of a function \(f(x)\) at a point \(a\) in the set \(\mathbb{R}\) (it is denoted by \(b = \lim_{x \to a} f (x)\)) if for any sequence \(l = (a_0, a_1, a_2, \ldots)\), the condition \(a_i \leq b\) implies \(b = \lim_{x \to a} f (x)\).

b) A function \(f(x)\) \(r\)-converges at a point \(a\) if it has an \(r\)-limit at this point.

Definitions 2.3 and 2.4 extend the classical concept of the limit and allow one to extend the concept of a continuous function.

**Definition 2.5.** ([7]). A number \(b\) is called a left (right) \(r\)-limit or \(r\)-limit from the left (right) of a function \(f(x)\) at a point \(a\) if for any \(\varepsilon > 0\) there is \(\delta > 0\) such that for all \(x < a\) \((x > a)\), the inequality \(|x - a| \leq \delta\) implies the inequality \(|f(x) - b| \leq \varepsilon\).

A right \(r\)-limit \(b\) of \(f(x)\) at a point \(a\) is denoted by \(b = \lim_{x \to a, x < a} f(x)\) and a left \(r\)-limit \(c\) of \(f(x)\) at a point \(a\) is denoted by \(c = \lim_{x \to a, x > a} f(x)\).

This concept allows us to extend the concept of density to the concept of \(r\)-approximate density.

**Definition 2.6.** a) An \(r\)-approximate density of a measurable subset \(A\) of \(\mathbb{R}\) at a point \(a \in \mathbb{R}\) is defined as a number

\[
c = \lim_{r \to 0} \frac{\Phi_\alpha(A; a)}{r},
\]

where \(\Phi_\alpha(A; a)\) provided the limit exists.

b) The complete \(r\)-approximate density \(\Phi_\alpha(A; a)\) is defined as a set

\[
\Phi_\alpha(A; a) = \{c : c = \lim_{r \to 0} \frac{\Phi_\alpha(A; a)}{r}\).
\]

Properties of fuzzy limits obtained in ([8]) imply the following result.

**Proposition 2.2.** For any set \(A\), \(\Phi_\alpha(A; a)\) is a closed interval or is empty.

Relations and operations with real numbers induce similar relations and operations with sets of real numbers.

Let \(A, B \subseteq \mathbb{R}\).

**Definition 2.7.** \(A \leq B\) if \(\forall a \in A \forall b \in B (a \leq b)\).

**Definition 2.8.** a) The Minkowski sum \(A + B\) of the sets \(A\) and \(B\) is the set that consists of all sums \(a + b\) where \(a \in A\) and \(b \in B\), i.e., \(A + B = \{a + b : a \in A, b \in B\}\).

b) The Minkowski difference \(A \ominus B\) of the sets \(A\) and \(B\) is the set that consists of all sums \(a - b\) where \(a \in A\) and \(b \in B\), i.e., \(A \ominus B = \{a - b : a \in A, b \in B\}\).
Lemma 2.3. The following equalities hold:

a) \([a, b] \oplus [c, d] = [a + c, b + d]\),

b) \([a, b] \ominus [c, d] = [a - d, b - c]\).

Proposition 2.3. For any sets \(A\) and \(B\), if \(\delta_{a,\epsilon}(A)\) and \(\delta_{a,\epsilon}(B)\) are not empty, then we have:

a) If \(A \subseteq B\), then \(\delta_{a,\epsilon}(A) \subseteq \delta_{a,\epsilon}(B) \oplus \{r\}\).

b) \(\delta_{a,\epsilon}(A \cup B) \leq \delta_{a,\epsilon}(A) \oplus \delta_{a,\epsilon}(B) \oplus \{r\}\).

c) If \(A \subseteq B\), then \(\delta_{a,\epsilon}(A \setminus B) \oplus \{r\} \geq \delta_{a,\epsilon}(A) \ominus \delta_{a,\epsilon}(B)\).

d) \(\delta_{a,\epsilon}(A \cap B) \leq \min \{\delta_{a,\epsilon}(A), \delta_{a,\epsilon}(B)\} \oplus \{r\}\).

Proof. a) Since \(A \subseteq B\), we have

\[
[a - \alpha < x < a + \alpha; \; a \in A] \subseteq [a - \alpha < x < a + \alpha; \; a \in A]
\]

This yields

\(\Phi(\alpha; A; a) \subseteq \Phi(\alpha; B; a)\).

Now taking \(r\)-limit as \(\alpha \to 0^+\) on the both sides of the above inequality, we may write

\(r\)-\(\text{lim}_{\alpha \to 0^+} \Phi(\alpha; A; a) \leq r\)-\(\text{lim}_{\alpha \to 0^+} \Phi(\alpha; B; a)\).

This completes the proof of a).

b) Since

\(\Phi(\alpha; A \cup B; a) \leq \Phi(\alpha; A; a) \oplus \Phi(\alpha; B; a)\),

taking \(r\)-limit as \(\alpha \to 0^+\), we easily get b).

The proofs of c) and d) are similar. \(\square\)

Remark 2.1. The following example demonstrates that in contrast to approximate density (cf. Lemma 2.1), \(A \subseteq B\) does not imply \(\delta_{a,\epsilon}(A) \subseteq \delta_{a,\epsilon}(B)\) in a general case.

Example 2.3. Let us consider sets \(A = [0, 1]\) and \(B = \{1/2, 3/4, 5/6\}\). Then \(\delta_{1/2,\epsilon}(A) = [1/6, 5/6]\) and \(\delta_{1/2,\epsilon}(B) = [1/12, 7/12]\).

The following proposition directly follows from Definitions 2.1 and 2.4.

Proposition 2.4. The 0-approximate density of any set \(A\) at a point \(a\) is unique and coincides with the approximate density of \(A\) at the same point.

The concept of \(r\)-approximate density allows us to give the following definition.

Definition 2.9. A function \(f(x)\) is called \(r\)-approximately continuous at a point \(a \in X\) if, for every \(\epsilon > 0\), we have \(0 \in \delta_{a,\epsilon}(E_f(a))\), or equivalently,

\(0 = r\)-\(\text{lim}_{x \to a} \Phi(\alpha; f; a)\)

holds.

This concept extends concepts of continuous at a point and approximately continuous at a point functions.

Example 2.4. Let us consider the following function

\[
f(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ 0 & \text{when } x < 0. \end{cases}
\]

This function is discontinuous and even not approximately continuous at \(x = 0\), but it is \((1/2)\)-approximately continuous and fuzzy continuous at \(x = 0\).

Definitions 2.2 and 2.9 directly imply the following results.

Lemma 2.4. A function \(f(x)\) is approximately continuous at the point \(a\) if and only if it is \(0\)-approximately continuous at the same point.

Remark 2.2. Concepts of \(r\)-continuity and \(r\)-approximate continuity are independent as there are functions that are \(r\)-continuous at a point \(a\) but not \(r\)-approximately continuous at this point and there are functions that are \(r\)-approximately continuous at a point \(a\) but not \(r\)-continuous at this point. It is demonstrated in the following examples.

Example 2.5. Let us consider the following function

\[
f(x) = \begin{cases} 1 & \text{when } x \geq 0 \text{ and } x \neq 1/2 \\ 0 & \text{when } x < 0 \\ 0 & \text{when } x = 1/2. \end{cases}
\]

This function is discontinuous, approximately discontinuous and fuzzy discontinuous at \(x = 0\), but it is \((1/2)\)-approximately continuous at \(x = 0\).

Example 2.6. Let us consider the following function

\[
f(x) = \begin{cases} 1 & \text{when } x \geq 0 \text{ and } x \neq 1/2 \\ 0 & \text{when } x < 0 \\ 10 - 1/2 & \text{when } x = 1/2. \end{cases}
\]

This function is discontinuous, approximately discontinuous and \((1/2)\)-approximately continuous at \(x = 0\), but it is not \((1/2)\)-continuous at \(x = 0\). Although, \(f(x)\) is \(5\)-continuous at \(x = 0\).
Example 2.7. Let us consider the following function

\[ f(x) = \begin{cases} \frac{1}{10} & \text{when } x \text{ is irrational} \\ 0 & \text{when } x \text{ is rational} \end{cases} \]

This function is discontinuous and even not approximately continuous at \( x = 0 \), but it is \((1,\alpha)\)-continuous and fuzzy continuous at \( x = 0 \).

Lemma 2.5. If a function \( f(x) \) is \( r \)-approximately continuous at \( a \), then for each \( q \geq r \), it is \( q \)-approximately continuous at the same point.

This result is a direct corollary from Lemma 3.2.4 from [8], which states that if \( a = \text{r-lim}_l \), then \( a = \text{q-lim}_l \) for any \( q \geq r \).

Theorem 2.1. If functions \( f \) and \( g \) are \( r \)-approximately and \( q \)-approximately continuous at \( a \), respectively, then:

(i) functions \( f + g \) and \( f - g \) are \((r + q)\)-approximately continuous at \( a \);
(ii) the function \( kf \) is \( r \)-approximately continuous at \( a \) for any real number \( k \);
(iii) if functions \( f \) and \( g \) are bounded in some neighborhood of \( a \), then the function \( f(x) - g(x) \) is \((r + q)\)-approximately continuous at \( a \).

Proof. (i) Let us consider functions \( f \) and \( g \) such that \( f \) is \( r \)-approximately continuous at a point \( a \) and \( g \) is \( q \)-approximately continuous at the point \( a \). Since \( f \) and \( g \) are \( r \)-approximately and \( q \)-approximately continuous at \( a \), respectively, for every \( \varepsilon > 0 \), we have

\[ 0 = r-lim_{x \to a} \Phi_{r/2}(a; f; a) = q-lim_{x \to a} \Phi_{q/2}(a; g; a). \]  \hspace{0.4cm} (1)

Besides, \[ |(f + g)(x) - (f + g)(a)| = |(f)(x) - f(a) + g(x) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| \]

Consequently, the inequality

\[ |f + g(x) - (f + g)(a)| \geq \varepsilon \]

implies that, at least,

\[ |f(x) - f(a)| \geq \varepsilon/2 \]

or

\[ |g(x) - g(a)| \geq \varepsilon/2 \]

Thus,

\[ \{a - \alpha < x < a + \alpha; |(f + g)(x) - (f + g)(a)| \geq \varepsilon\} \leq \{a - \alpha < x < a + \alpha; |f(x) - f(a)| \geq \varepsilon/2\} \]

\[ \cup \{a - \alpha < x < a + \alpha; |g(x) - g(a)| \geq \varepsilon/2\} \]

This gives

\[ m[a - \alpha < x < a + \alpha; |(f + g)(x) - (f + g)(a)| \geq \varepsilon]/2a \leq m[a - \alpha < x < a + \alpha; |f(x) - f(a)| \geq \varepsilon/2]/2a + m[a - \alpha < x < a + \alpha; |g(x) - g(a)| \geq \varepsilon/2]/2a. \]

This means that

\[ \Phi_{r}(a; f + g; a) \leq \Phi_{r/2}(a; f; a) + \Phi_{r/2}(a; g; a) \]  \hspace{0.4cm} (2)

By (1), we conclude that, for a given \( \gamma > 0 \), there exist \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that for all \( a > 0 \), the inequalities \( a < \delta_1 \) and \( a < \delta_2 \) imply \( \Phi_{r/2}(a; f; a) < \gamma r \) and \( \Phi_{r/2}(a; g; a) < \gamma q \). Now let \( \delta = \min \{\delta_1, \delta_2\} \). Then, it follows from (2) that for all \( a > 0 \), the inequality \( a < \delta \) implies

\[ \Phi_{r}(a; f + g; a) < (r + q) + 2\gamma. \]

Thus, we obtain that

\[ 0 = (r + q)-lim_{x \to a} \Phi_{r}(a; f + g; a) \]

This means that \( f + g \) is \((r + q)\)-approximately continuous at \( a \).

(ii) For \( k \neq 0 \), the proof utilizes Corollary 2.1.1 and is similar to the proof of (i). For \( k = 0 \), the function \( kf \) is continuous at \( a \) and thus, \( r \)-approximately continuous at \( a \).

(iii) Let us consider functions \( f \) and \( g \) such that both are bounded in some neighborhood \( O \) of \( a \), \( f \) is \( r \)-approximately continuous at a point \( a \), and \( g \) is \( q \)-approximately continuous at the point \( a \). As \( f \) and \( g \) are bounded in the neighborhood \( O \), there is a number \( \varepsilon \) such that \( |f(x)| < \varepsilon \) and \( |g(x)| < \varepsilon \) for all \( x \) from \( O \). Since \( f \) and \( g \) are \( r \)-approximately and \( q \)-approximately continuous at \( a \), respectively, for every \( \varepsilon > 0 \), we have

\[ 0 = r-lim_{x \to a} \Phi_{r/2}(a; f; a) = q-lim_{x \to a} \Phi_{r/2}(a; g; a). \]  \hspace{0.4cm} (3)

Besides,

\[ |(f - g)(x) - (f - g)(a)| = |(f)(x) - f(a) + g(x) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| \]

Consequently,

\[ |f - g(x) - (f - g)(a)| \geq \varepsilon \]

implies that, at least,

\[ |f(x) - f(a)| \geq \varepsilon/2 \]

or

\[ |g(x) - g(a)| \geq \varepsilon/2 \]
When $x$ belongs to the neighborhood $O_a$, we have
\[ |(f \cdot g)(x) - (f \cdot g)(a)| \leq c \cdot |f(x) - f(a)| \]

Consequently, the inequality
\[ |f(x) - f(a)| \geq \epsilon \]
implies that, at least,
\[ |f(x) - f(a)| \geq \epsilon/(2c) \]

or
\[ |g(x) - g(a)| \geq \epsilon/(2c) \]

Thus,
\[ |a - \alpha < x < a + \alpha; |(f \cdot g)(x) - (f \cdot g)(a)| \geq \epsilon| \\
\leq |a - \alpha < x < a + \alpha; |f(x) - f(a)| \geq \epsilon/(2c)|/2a \\
\cup |a - \alpha < x < a + \alpha; |g(x) - g(a)| \geq \epsilon/(2c)|/2a \]

when $|a - \alpha, a + \alpha| \subseteq O_a$. For sufficiently small $\alpha$, this gives
\[ m[a - \alpha < x < a + \alpha; |(f \cdot g)(x) - (f \cdot g)(a)| \geq \epsilon]/2a \]

This means that
\[ \Phi_{q}(a; f \cdot g; a) \leq \Phi_{q/2}(a; f; a) + \Phi_{q/2}(a; g; a) \tag{4} \]

By (3), we conclude that, for a given $\gamma > 0$, there exist $b_1 > 0$ and $b_2 > 0$ such that for all $a > 0$, the inequalities $a < b_1$ and $a < b_2$ imply $\Phi_{q/2}(a; f; a) < \gamma$ and $\Phi_{q/2}(a; g; a) < \gamma$. Now let $\delta = \min\{b_1, b_2\}$. Then, it follows from (4) that for all $a > 0$, the inequality $a < \delta$ implies
\[ \Phi_{q}(a; f \cdot g; a) < (r + q) + 2\gamma. \]

Thus, we obtain that
\[ 0 = (r + q) - \lim_{a \to 0+} \Phi_{q}(a; f \cdot g; a). \]

This means that $f \cdot g$ is $(r \cdot q)$-approximately continuous at $a$.

Theorem is proved. \qed

**Corollary 2.2.** Let $f(x)$ and $g(x)$ be approximately continuous functions at a point $a$, then

(i) if $f \cdot g$ and $f - g$ are approximately continuous at $a$, then

(ii) the function $k$ is approximately continuous at $a$ for any real number $k$.

These results immediately follows from Theorem 2.1

(i) when we take $r = q = 0$.

Taking $k = 1$ in Theorem 2.1 (ii), we obtain the following result.

**Corollary 2.3.** If $f$ is an $r$-approximately continuous at $a$ point, then $f \cdot g$ is $r$-approximately continuous at $a$.

**Corollary 2.4.** If $f$ is approximately continuous at $a$, then for each $r > 0$, it is $r$-approximately continuous at the same point.

**Remark 2.3.** Examples 2.3 and 2.4 show that the converse statements to the statements of Lemma 2.5 and Corollary 2.4 are not always true.

**Definition 2.9.** Let $f(x)$ be $(r \cdot q)$-approximately continuous at any point from its domain.

**Lemma 2.6.** A function $f(x)$ is $r$-approximately continuous at a point $a$ if and only if for every $r > 0$, we have
\[ 1 = r \cdot \lim_{a \to 0+} [m[a - \alpha < x < a + \alpha; |f(x)| - f(a)| < \epsilon]]/(2a). \]

**Proposition 2.6.** Any function $f(x)$ is $1$-approximately continuous at any point from its domain.

**Proof.** Let us consider an arbitrary function $f(x)$ and a point $a$. As $0 \leq m[a - \alpha < x < a + \alpha; |f(x)| - f(a)| < \epsilon]] \leq 2a$, we have $0 \leq m[a - \alpha < x < a + \alpha; |f(x) - f(a)| < \epsilon]/(2a) \leq 1$. It means that the distance of all values to the point $1$ is not larger than $1$. By Definition 2.3, it means that $1 = 1 \cdot \lim_{a \to 0+} [m[a - \alpha < x < a + \alpha; |f(x) - f(a)| < \epsilon]]/(2a)$. By Lemma 2.6, the function $f(x)$ is $1$-approximately continuous at the point $a$. \qed

**Definition 2.10.** Let the left approximate continuity defect of a function $f(x)$ at a point $a$ is defined as
\[ \delta_{pq}(f; a) := \inf_{r, q \geq 0} r \cdot \lim_{a \to 0+} \Phi_{q}(a; f; a). \]

**Lemma 2.7.** If $q = \delta_{pq}(f, a)$, then the function $f(x)$ is $q$-approximately continuous at $a$ and if $p < q$, then it is not $p$-approximately continuous at $a$.

**Proof.** As $q = \inf_{r, q \geq 0} [r \cdot \lim_{a \to 0+} \Phi_{q}(a; f; a)]$, by Definition 2.3, for every $k > 0$, there exists a positive number $r$ such that $r \cdot q < 2k$ and that $0 = r \cdot \lim_{a \to 0+} \Phi_{q}(a; f; a)$ for every $k > 0$. By Definition 2.2, for any sequence $(\{a_{i}; i = 1, 2, 3, \ldots\})$ satisfying the condition $0 = \lim_{i} a_{i}$, we obtain the equality
\[ 0 = r \cdot \lim_{a \to 0+} \Phi_{q}(a; f; a) \]

\[ 0 = \lim_{a \to 0+} \Phi_{q}(a; f; a) \]

\[ 0 = \lim_{a \to 0+} \Phi_{q}(a; f; a) \]

\[ 0 = \lim_{a \to 0+} \Phi_{q}(a; f; a) \]
which holds for every $\varepsilon > 0$. This implies that
\[ |\Phi_r(f, a)| < r + k/2 < q + k/2 = q + k. \]
Since $k$ was arbitrary, we get $0 = q - \lim_{x \to a} \Phi_r(f, a)$, which means that $f(x)$ is $q$-approximately continuous at $a$. If $p < q$, then $f(x)$ cannot be $p$-approximately continuous at $a$ because $q = \inf \{ r : 0 = r - \lim_{x \to a} \Phi_r(f, a) \}$.

Lemma is proved. □

Let us consider functions $f(x)$ and $g(x)$.

**Theorem 2.2.** We have:

(i) $\delta_q(f + g, a) \leq \delta_q(f, a) + \delta_q(g, a)$,

(ii) $\delta_q(f - g, a) \leq \delta_q(f, a) + \delta_q(g, a)$,

(iii) $\delta_q(kf, a) = \delta_q(f, a)$ for any point $a$ and any real number $k \neq 0$.

**Proof.** (i) As it is demonstrated in the proof of Theorem 2.1, for every $e \in \mathbb{R}^n$, we have

\[ \Phi_e(f + g, a) \leq \Phi_e(f, a) + \Phi_e(g, a). \]

This yields

\[ \inf \{ r : 0 = r - \lim_{x \to a} \Phi_e(f + g, a) \} \]

\[ \leq \inf \{ r : 0 = r - \lim_{x \to a} \Phi_e(f, a) \}

+ \inf \{ r : 0 = r - \lim_{x \to a} \Phi_e(g, a) \}.

Thus, we have

\[ \delta_q(f + g, a) \leq \delta_q(f, a) + \delta_q(g, a). \]

(iii) By Corollary 2.1 (d), we have

\[ \Phi_e(kf, a) = a \Phi_e(f, a). \]

This yields

\[ \inf \{ r : 0 = r - \lim_{x \to a} \Phi_e(kf, a) \} \]

\[ \leq \inf \{ r : 0 = r - \lim_{x \to a} \Phi_e(f, a) \}, \]

and hence

\[ \delta_q(kf, a) = \delta_q(f, a). \]

(ii) By (i), we have

\[ \delta_q(f - g, a) \leq \delta_q(f, a) + \delta_q(-g, a). \]

By (iii), we have

\[ \delta_q(-g, a) = \delta_q(g, a). \]

Consequently,

\[ \delta_q(f - g, a) \leq \delta_q(f, a) + \delta_q(g, a). \]

Theorem is proved. □

**Definition 2.11.** The left measure of approximate continuity of a function $f(x)$ at a point $a$ is defined by the number

\[ \mu_{p}(f, a) = 1/[1 + \delta_p(f, a)]. \]

**Corollary 2.5.** We have:

(i) $\mu_{p}(f + g, a) < \mu_{p}(f, a) + \mu_{p}(g, a)$,

(ii) $\mu_{p}(f - g, a) < \mu_{p}(f, a) + \mu_{p}(g, a)$,

(iii) $\mu_{p}(kf, a) = \mu_{p}(f, a)$ for any point $a$ and any real number $k \neq 0$.

**Definition 2.12.** A function $f(x)$ is called fuzzy approximately continuous at the point $a$ if $\delta_p(f, a) < 1$, or equivalently, $\mu_{p}(f, a) > 0$.

**Remark 2.4.** A fuzzy continuous at a function $f(x)$ is not necessarily fuzzy approximately continuous at the same point.

**Theorem 2.3.** A linear combination $\Sigma_{i=1}^{n} k_i f_i(x)$ of fuzzy approximately continuous functions $f_i(x)$ at the point $a$ is fuzzy approximately continuous at the same point if each function $f_i(x)$ is $r_i$-approximately continuous at the point $a$ for all $i = 1, 2, 3, \ldots, n$ and $\Sigma_{i=1}^{n} r_i < 1$.

**Proof.** Since each $f_i(x)$ is $r_i$-approximately continuous at the point $a$ ($i = 1, 2, 3, \ldots, n$), for a given $\gamma > 0$, there exist $\delta_i > 0$ ($i = 1, 2, 3, \ldots, n$) such that for all $a > 0$, the inequalities $a < \delta_i$ imply that $\Phi_{r_i}(a) \leq a < \delta_i + \delta_i$. Now let $\delta = \min \{ \delta_1, \delta_2, \ldots, \delta_n \}$. We know from Corollary 2.1 a) and d) that

\[ \Phi_{r}(a; \Sigma_{i=1}^{n} k_i f_i(x); a) \leq \Sigma_{i=1}^{n} \Phi_{r_i}(a; k_i f_i(x); a). \]

Then, for all $a > 0$, the inequality $a < \delta$ implies

\[ \Phi_{r}(a; \Sigma_{i=1}^{n} k_i f_i(x); a) < (\Sigma_{i=1}^{n} r_i) + \gamma. \]

Thus, we obtain that

\[ 0 = (\Sigma_{i=1}^{n} r_i) - \lim_{x \to a} \Phi_{r}(a; \Sigma_{i=1}^{n} k_i f_i(x); a). \]

By hypothesis, since $\Sigma_{i=1}^{n} r_i < 1$, we conclude

\[ \delta_{p}(a; \Sigma_{i=1}^{n} k_i f_i(x); a) < 1, \]

which gives that $\Sigma_{i=1}^{n} k_i f_i(x)$ is fuzzy approximately continuous function at the point $a$.

Theorem is proved. □

**Remark 2.5.** It is known that the sum and difference of fuzzy continuous functions are fuzzy continuous (see [2]). However, this is not true for fuzzy approximately
continuous functions as the following example demonstrates.

Example 2.8. Let us consider the following functions

\[
f(x) = \begin{cases} 
1 & \text{when } x > 0 \text{ and is irrational} \\
0 & \text{when } x \leq 0 \text{ or when } x > 0 \text{ and is rational.}
\end{cases}
\]

\[
g(x) = \begin{cases} 
1/3 & \text{when } x < 0 \text{ and is irrational} \\
0 & \text{when } x \geq 0 \text{ or when } x < 0 \text{ and is rational.}
\end{cases}
\]

Both functions are discontinuous but (\%)-continuous, fuzzy continuous and (\%)-approximately continuous at \(x=0\). At the same time, the sum \((f+g)(x)\) of these functions is not \(r\)-approximately continuous at \(x=0\) for any \(r<1\) and thus, it is not fuzzy approximately continuous.

This example also shows that conditions in Theorem 2.3 are essential.

Nevertheless, we have a weaker result.

Proposition 2.5. If \(f\) is a fuzzy approximately continuous at a point \(a\) function, then \(kf\) where \(k\) is any real number is also fuzzy approximately continuous at \(a\).

3. Locally and globally fuzzy approximately continuous functions

Let \(X\) be a subset of the real line \(\mathbb{R}\).

Definition 3.1. A function \(f(x)\) is called \(r\)-approximately continuous in \(X\) if it is \(r\)-approximately continuous at every point \(a \in X\).

Lemma 3.1. If a function \(f(x)\) is \(r\)-approximately continuous in \(X\), then for each \(q \geq r\), it is \(q\)-approximately continuous in \(X\).

This result is a direct corollary from Lemma 2.5.

Lemma 3.2. If a function \(f(x)\) is \(0\)-approximately continuous in \(X\) if and only if \(f(x)\) is approximately continuous in \(X\).

This result is a direct corollary from Lemma 2.4.

Theorem 3.1. If \(f\) and \(g\) are \(r\)-approximately and \(q\)-approximately continuous in \(X\) functions, respectively, then

(i) functions \(f+g\) and \(f-g\) are \((r+q)\)-approximately continuous in \(X\);

(ii) the function \(kf\) is \(r\)-approximately continuous in \(X\) for any real number \(k\).

(iii) if functions \(f\) and \(g\) are bounded in \(X\), then the function \(f(x)g(x)\) is \((r+q)\)-approximately continuous in \(X\).

Indeed, if \(f\) and \(g\) are \(r\)-approximately and \(q\)-approximately continuous at any point of \(X\) functions, then by Theorem 2.1, functions \(kf, f+g, f−g\) have the same properties of fuzzy approximate continuity. Thus, we have statements (i) and (ii). In a similar way, boundedness of functions \(f\) and \(g\) gives the statement (iii).

Definition 3.2. A function \(f(x)\) is called locally fuzzy approximately continuous in \(X\) if for each point \(a \in X\), there is a number \(\varepsilon\) such that \(0 \leq r < 1 \text{ and } f(x)\) is \(r\)-approximately continuous at \(a\).

Theorem 2.1 implies the following result.

Theorem 3.2. (a) If \(f\) and \(g\) are locally fuzzy approximately continuous in \(X\) functions, then functions \(f+g, f−g\) and \(kf\) are locally fuzzy approximately continuous in \(X\).

(b) If in addition, \(f\) and \(g\) are bounded in \(X\), then the function \(f(x)g(x)\) is locally fuzzy approximately continuous in \(X\).

Theorem 2.3 implies the following result.

Theorem 3.3. A linear combination \(\sum_{i=1}^{n} k_if(x)\) of locally fuzzy approximately continuous functions \(f_i(x)\) in \(X\) is \((r_1, 2, 3, \ldots, n)\) locally fuzzy approximately continuous in \(X\) if \(\sum_{i=1}^{n} k_i < 1\) for all points \(a\) from \(X\).

Example 2.8 shows that conditions in Theorem 3.3 are essential.

Definition 3.3. A function \(f(x)\) is called globally fuzzy approximately continuous in \(X\) if there is a number \(r\) such that \(0 \leq r < 1\) and \(f(x)\) is \(r\)-approximately continuous at every point \(a \in X\).

Proposition 3.1. In a general case, global and local fuzzy approximate continuities do not coincide.

Indeed, let us consider the following function

\[
f(x) = \begin{cases} 
1 & \text{when } x \text{ is irrational} \\
0 & \text{when } x \text{ is rational.}
\end{cases}
\]

The function \(f(x)\) is locally fuzzy approximately continuous in \(\mathbb{R}\) but is not globally fuzzy approximately continuous.
continuous in X because at each point \( n+1 \), \( f(x) \) is not \( n \)-approximately continuous for all \( n = 1, 2, 3, \ldots \).

Lemma 3.1 implies the following results.

**Corollary 3.1.** Any approximately continuous in X function \( f(x) \) is locally and globally fuzzy approximately continuous in X.

**Corollary 3.2.** Any continuous in X function \( f(x) \) is locally and globally fuzzy approximately continuous in X.

Theorem 3.1 implies the following result.

**Theorem 3.4.** a) If \( f \) and \( g \) are globally fuzzy approximately continuous in X functions, then functions \( f \circ g \), \( f + g \), and \( k \) are globally fuzzy approximately continuous in X.

b) If in addition, \( f \) and \( g \) are bounded in X, then the function \( f \circ g(x) \) is globally fuzzy approximately continuous in X.

Theorem 2.3 and Theorem 3.3 imply the following result.

**Theorem 3.5.** A linear combination \( \sum_{i=1}^{n} k_i f_i(x) \) of globally fuzzy approximately continuous functions \( f_i(x) \) in X (\( i = 1, 2, 3, \ldots, n \)) is globally fuzzy approximately continuous X if there is a number \( k < 1 \) such that \( \sum_{i=1}^{n} k_i f_i(x) \) is k-approximately continuous for all points \( a \) from X.

4. Conclusion

We have introduced concepts of a fuzzy approximately continuous at a point function, locally fuzzy approximately continuous function and globally fuzzy approximately continuous function, and studied their properties. It would be interesting to find what properties of continuous functions and fuzzy continuous functions remain true for fuzzy approximately continuous functions. Results obtained in this paper open an approach to the development of approximately scalable topology [5, 6].

Fuzzy approximate continuity is related to fuzzy statistical convergence studied in [9–11]. It would be interesting to make relations between these two concepts explicit.

One more interesting direction for further research is a study of relations between fuzzy approximate continuity and fuzzy integration.

**References**
