On some third parts of nearly complete digraphs

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Abstract

For the complete digraph $D_{Kn}$ with $n \geq 3$, its half as well as its third (or near-third) part, both non-self-converse, are exhibited. A backtracking method for generating a $t$th part of a digraph is sketched. It is proved that some self-converse digraphs are not among the near-third parts of the complete digraph $D_{K5}$ in four of the six possible cases. For $n = 9 + 6k$, $k = 0, 1, \ldots$, a third part $D$ of $D_{Kn}$ is found such that $D$ is a self-converse oriented graph and all $D$-decompositions of $D_{Kn}$ have trivial automorphism group. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $t$ be an integer, $t \geq 2$. One of the most significant results in the graph decom-
position theory is that a $t$th part of a complete graph exists whenever the size of the graph is divisible by $t$. In case $t = 2$ this is the well-known result on the existence (and characterization) of self-complementary graphs due to Sachs [11] and Ringel [9]. The result in its generality is proved in Harary et al. [4] and also Schönheim and Bialostocki [12]. However, the characterization problem (solved if $t = 2$) remains open for $t > 2$ (and the complete graph’s order large enough).

Given a complete digraph $D_{Kn}$ with $n$ vertices (and $n(n-1)$ arcs), the numerical divisibility condition $t \mid n(n-1)$ is also known [5] to ensure that the class $D_{Kn}/t$ of $t$th parts of $D_{Kn}$ is nonempty. Furthermore, then a self-converse $t$th part exists. Recall that a digraph is called self-converse if it is isomorphic to its converse (obtainable by reversing the orientation of each arc). If $t = 2$, all these parts, halves, are known as self-complementary digraphs, see Read [8] for their characterization. Hence, in contrast to the undirected case, there exists a self-complementary digraph of any order $n$. One of them is the transitive tournament, $T_n$, which is a self-converse oriented graph (without

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Note that conversing and complementing can be considered as identical operations on a tournament. Consequently, all self-complementary oriented graphs are self-converse tournaments.

**Theorem 1.** For $n \geq 3$ there is a self-complementary digraph $D$ of order $n$ which is not self-converse (and necessarily includes a 2-dicycle).

**Proof.** Let $D$ include (1) a transitive tournament $T_{n-2}$ on $n-2$ vertices, (2) two more vertices, say $u$ and $v$, (3) either one of the two $u-v$ arcs, and (4) all $u-T_{n-2}$ arcs. Then $D$ is clearly a self-complementary digraph. The maximum degree $\Delta(D) = 2n-3$, which is odd, and $u$ is the only vertex of that odd degree. Hence $D$ is not self-converse.

We are going to deal with the next case, namely $t = 3$. Then, if $n \equiv 2 \pmod{3}$, the divisibility condition is not satisfied. In this case we consider the third parts of the corresponding nearly complete digraph obtained from $D_{Kn}$ either by adding a surplus $S$ comprising a copy of an arc $a$ or by deleting a remainder $R$, where $R$ is a set of two arcs. Then, following Skupień [13], the classes of such third parts are denoted $[D_{Kn}/3]_S$ and $[D_{Kn}/3]_R$, and are called the ceiling third class and the floor third class, respectively. Call elements of these classes (also if $S = \emptyset = R$) to be near-third parts of $D_{Kn}$. More precisely, these are ceiling third parts and floor-R third parts, respectively. Thus, for $n = 3k + 2 \geq 5$ and $k \in \mathbb{N}$, we have six such classes: one ceiling class and five floor classes, namely with the remainder $R$ inducing $2P_2$ (a 2-matching), 2-dicycle $\tilde{C}_2$, or (one of the three oriented paths $P_3$, namely) dipath $\tilde{P}_3$, gutter $P^\vee$ or roof $P^\wedge$, see Fig. 1 for definitions.

The main purpose of this note is to sketch a backtracking method for generating $t$th parts of a digraph, parts being self-converse (or not) and/or oriented graphs (or not). We specify this method to get information on near-$t$th parts of $D_{Kn}$ for $t = 3$ and either $n = 5$ or $n = 9$.

We are able to prove [7] the following general result.

**Result 1.** With some (not too many) exceptions (to be specified below), for any order $n$ and any admissible remainder $R$ and surplus $S$, there exist both a ceiling third part and floor-R third part, say $D$ in either case, of the complete digraph $D_{Kn}$ such that $D$ has any fixed property of the following ones. (i) $D$ is an oriented graph, (ii) $D$ is self-converse, (iii) $D$ is a self-converse oriented graph, (iv) $D$ is a self-converse digraph with 2-dicycle. There is no exception if $D$ is to have property (i).
Theorem 2. For \( n = 5 \) and \( R \) being a non-self-converse remainder (a gutter or roof), the floor class \( \mathcal{D}K_5/3 \) contains no self-converse digraph.

Remark 1. A computer was used [6] to confirm that Theorem 2 is also true for \( n = 8 \). Theorem 2 remains unsettled for \( n = 3k + 2 \geq 11, \ k \in \mathbb{N} \).

However, we have resisted temptation of transforming Theorem 2 into a corresponding conjecture on \( n = 5 + 3k \geq 8 \). The reason is that sporadic exclusions are likely, cf. Remark 2 below.

Theorem 3. For \( n = 5 \) and \( R = 2a \), the floor class \( \mathcal{D}K_5/3 \) does not contain a self-converse digraph with dicycle \( \tilde{C}_2 \).

Theorem 4. For \( n = 5 \), there is no ceiling third part of the complete digraph \( \mathcal{D}K_5 \) which could be a self-converse oriented graph.

Remark 2. If \( n = 8 \) and \( \langle R \rangle = \tilde{P}_3 \), a computer was used [6] in order to see that the floor class \( \mathcal{D}K_8/3 \) does not contain a self-converse oriented graph. Clearly, \( \mathcal{D}K_3/3 \) comprises three digraphs, none of which is a self-converse oriented graph.

\[
\mathcal{D}K_3/3 = \{ \tilde{C}_2, P^\vee, P^\wedge \}.
\]

In all remaining cases, one can prove that the near-third parts which are self-converse oriented graphs or self-converse digraphs with 2-dicycle exist, see Result 1 and [7].

Two decompositions of a fixed structure (multigraph or digraph) are said to be essentially distinct if there is no relabelling of vertices followed by reordering of parts which could make one decomposition coincide with another one.

As is claimed by Robinson [10] Schönheim (unpublished) discovered that the (undirected) path with five edges, \( P_6 \), is the smallest example of a \( t \)th part \( F \) of a complete graph \( K_n \) such that there is an \( F \)-decomposition of \( K_n \) whose automorphism group is trivial. In this example \( F = P_6, \ n = 6 \) and \( t = 3 \). However, \( P_6 \) admits a \( P_6 \)-decomposition of \( K_6 \) whose automorphism group is the cyclic group \( \mathbb{Z}_3 \). So is the case of Robinson’s extension of \( P_6 \) to a third part of \( K_n \) for \( n = 3k + 6 \) or \( 3k + 7, k = 0, 1, \ldots \). In case of digraphs we find a stronger example.

Theorem 5. For \( n = 9 + 6k, k = 0, 1, \ldots \), among third parts of the complete digraph \( \mathcal{D}K_n \) there is one, \( D_n \), such that \( D_n \) is a self-converse oriented graph which admits two essentially distinct \( D_n \)-decompositions of \( \mathcal{D}K_n \) and both \( D_n \) and all \( D_n \)-decompositions of \( \mathcal{D}K_n \) have trivial automorphism groups.

Theorem 6. For each \( n \geq 3 \) and for any admissible remainder \( R \) and any admissible surplus \( S \), there exists a non-self-converse oriented graph which is a near-third part of the digraph \( \mathcal{D}K_n \).
2. Notation and terminology

We use standard notation and terminology of graph theory [1–3] unless otherwise stated.

The word *line* stands for an edge or an arc. Graph means a simple graph (loopless and without multiple edges). Multigraph and digraph may have multiple lines, loops being forbidden.

In order to fix our notation and terminology, let \( G \) stand for a digraph or a multigraph with \( n \) vertices and \( m \) lines, \( G = (V,E) \) where \( |V| = n \) and \( |E| = m \). In case \( G \) is a digraph, we use the notation \( E = A(G) \), the set of arcs, else \( E = E(G) \) is the set of edges. The *degree* \( \deg x \) of a vertex \( x \) of \( G \) is the number of lines incident with \( x \) in \( G \), \( \deg x = \deg_G x \). The *degree sequence* of \( G \) is a decreasing sequence whose terms make up the multiset of vertex degrees in \( G \). If \( G \) is a digraph then \( \deg x = \od x + \id x \) where \( \od x \) and \( \id x \) is the outdegree and indegree, respectively, of the vertex \( x \) in \( G \). Moreover, then the ordered pair \( (\od x, \id x) \) is called the *degree pair* of \( x \) in \( G \).

Given a multigraph \( G \), let \( \mathcal{D} G \) denote a digraph obtained from \( G \) by replacing each edge with two opposite arcs connecting endvertices of the edge. A digraph without 2-dicycle \( \mathcal{D} K_2 = \bar{C}_2 \) is called an oriented graph.

The converse of a digraph is obtained by the reversal of each arc. Notice that the gutter and roof (Fig. 1) are mutually converse. A digraph is called self-converse if it is isomorphic to its converse.

Let \( R = R_{G,t} \) and \( S = S_{G,t} \) be a set and possibly a multiset, respectively, of lines of \( G \) such that \( |R| = m \mod t \) and \( |S| = (t - |R|) \mod t \). Hence \( |S| = |R| = 0 \) if \( t \mid m \), else \( |R| + |S| = t \). The digraphs (multigraphs) induced by \( R \) and \( S \) are denoted by \( \langle R \rangle \) and \( \langle S \rangle \), respectively.

The symbol \( \cup \) when applied to digraphs or multigraphs stands for the vertex-disjoint union. The symbol \( + \) when applied to digraphs (multigraphs) stands for the line-disjoint union. Moreover, if \( E' \) is a set of (possibly copies of) lines, and \( E' \cap E = \emptyset \) then \( G + E' \) denotes the spanning superdigraph (spanning supermultigraph) of \( G \) with the line set \( E \cup E' \). Likewise, \( G - E' \) denotes the spanning digraph (spanning multigraph) obtained from \( G \) by removing a set \( E' \) of lines. We write \( G \pm E' = G \pm l \) if \( E' = \{ l \} \), \( l = e \) (edge) or \( l = a \) (arc).

Given a positive integer \( \lambda \), a \( \lambda \)-fold graph \( \lambda \)-fold digraph is one in which multiplicity of each line does not exceed \( \lambda \). The union of \( \lambda \) disjoint copies of \( G \) is denoted \( \lambda G \), \( \lambda l \) \((= \lambda e \text{ or } \lambda a)\) being the set of \( \lambda \) disjoint lines. On the other hand, \( \lambda K_n \) stands for the complete \( \lambda \)-fold graph on \( n \) vertices (obtained by replacing each edge of \( K_n \) by \( \lambda \) copies with the same endvertices).

By a decomposition of a (labelled) \( G \) we mean a family of line-disjoint substructures (subdigraphs or submultigraphs) of \( G \) which include all lines of \( G \). Those substructures are called parts of the decomposition or \( \partial \)-parts. By an \( H \)-decomposition of \( G \) we mean a decomposition of \( G \) into substructures all isomorphic to \( H \). We write \( H \models G \) and \( t \models G \) if there exists an \( H \)-decomposition of \( G \) into substructures all isomorphic to \( H \) and a decomposition of \( G \) into \( t \) mutually isomorphic substructures,
respectively. Moreover, the isomorphism class of those isomorphic \( \hat{\beta} \)-parts is called a \( \mathit{t} \)th part of \( G \).

\( G/\mathit{t} \) stands for the class of \( \mathit{t} \)th parts of \( G \) provided that \( \mathit{t}|m \). In general, given an \( R \) as described above, let \( [G/\mathit{t}]_R := (G - R)/\mathit{t} \). Then \( [G/\mathit{t}]_R \) is called the floor \( \mathit{t} \)th class (with remainder \( R \)). Similarly, \( [G/\mathit{t}]_S := (G + S)/\mathit{t} \) is called the ceiling \( \mathit{t} \)th class (with surplus \( S \)).

The ordered pair \((v_1, v_2)\) of vertices \( v_1 \) and \( v_2 \) (or the symbol \( v_1 \to v_2 \)) denotes the arc which goes from \( v_1 \) to \( v_2 \).

We use the symbols \( P_n \) and \( \overline{P}_n \) to denote an undirected \( n \)-path and \( n \)-dipath, respectively, both on \( n \) vertices. Similarly, a cycle \( C_n \) and a dicycle \( \overline{C}_n \) on \( n \) vertices are named \( n \)-cycle (\( n \geq 3 \)) and \( n \)-dicycle (\( n \geq 2 \)), respectively.

If \( \phi \) is a permutation of the vertex set \( V(G) \), let \( \phi' \) be the induced permutation which acts on lines, e.g., \( \phi'(i, j) = (\phi i, \phi j) \). Then \( \phi G \) stands for the image of \( G \) under \( \phi \), \( \phi G := (V, \phi'[E]) \) where \( \phi'[E] \) is the image of the line set \( E \) under \( \phi' \). An automorphism of \( G \) is a permutation \( \phi \) of \( V(G) \) such that \( \phi G \cong G \). Given a self-converse digraph \( G \) on \( n \) vertices, we use the symbol \( \varphi \) (= \( \varphi_G = \varphi_n \)) to denote a \textit{conversing permutation}, that is, a permutation of \( V(G) \) such that \( \varphi G \) is isomorphic to the converse digraph of \( G \).

3. Outline of algorithm

Let \( H \) be a given digraph on \( n \) vertices whose size is divisible by a given integer \( \mathit{t} \). We are going to describe a backtracking algorithm for generating a \( \mathit{t} \)th part \( D \) of \( H \). We admit that the digraph \( D \) may be an oriented graph (or not) and/or self-converse (or not).

If \( G \) is a digraph then \( \check{G} \) stands for the \textit{underlying multigraph} such that both the vertex sets and the multiplicities of adjacency of any two vertices in \( G \) and \( \check{G} \) coincide. On the other hand, \textit{underlying graph} of \( G \) is the spanning (simple) graph of \( G \) in which any two vertices are adjacent whenever they are so in \( G \).

A decomposition of a multigraph \( \check{H} \) on \( n \) vertices into \( t \) \( \check{\beta} \)-parts, all isomorphic to a multigraph \( G \) on \( n \) vertices, can be represented by a \( \check{t} \times n \) matrix, called a \textit{decomposition matrix}, each row of which is a permutation of the degree sequence of \( G \) and each column is a partition of the corresponding vertex degree in \( \check{H} \). Assume that the interchanging of the two columns and/or the interchanging of the two rows repeatedly produces an \textit{equivalent matrix}. A decomposition matrix, \( M \), is a \textit{standard decomposition matrix} if the concatenation of the consecutive columns of the matrix \( M \) is a lexicographical maximum sequence among all matrices equivalent to \( M \). Hence, the standard \( M \) is characterized by the following properties: (i) the first row and the first column are both decreasing, (ii) the first column is the lexicographical maximum among the orderings of the columns each of which includes \( A(G) \) as an entry, and (iii) columns (with the same first entry) appear one after another in anti-lexicographical order.

Given digraphs \( D \) and \( H \), a \( D \)-decomposition of \( H \) can be represented by a \textit{D-decomposition matrix} whose entries are degree pairs of vertices of \( D \), rows represent
parts of the decomposition and each of the rows is a permutation of the multiset of degree pairs in $D$, each column of the matrix represents a vertex of $H$ and is a vector partition of the degree pair of the vertex of $H$. Thus a $D$-decomposition matrix is a complex matrix and is called a directed decomposition matrix.

**Algorithm**

**Input:** An integer $t \geq 2$ and a digraph $H$ with $n$ vertices and with size $|A(H)|$ divisible by $t$.

Assume that $V(H) = \{1, 2, \ldots, n\}$ and we are given the corresponding $n$-sequence of the degree pairs $(\text{od}_i, \text{id}_i)$ and that of degrees $\text{deg}_i = \text{od}_i + \text{id}_i$, $i = 1, 2, \ldots, n$ so that degrees weakly decrease. Moreover, the vertex $i$, $i \in \mathbb{N}$, can be denoted by $v_i$. The vertex labels can be interchanged in Step 5 below.

**Output:** A $t$th part $D$ (or all $t$th parts) of $H$ (such that
(α) $D$ is an oriented graph (or not) or
(β) $D$ is self-converse).

1. Find the size $m := |A(D)| = (1/t) \sum_i \text{od}_i$ of $D$.
2. Find a decreasing composition of the integer $2m$ into $n$ summands of which some can be 0. Stop if none exists.

The composition is to be good as a degree sequence of the would-be corresponding multigraph $G$, $G = \tilde{D}$. Hence the composition is to be graphical or two-fold graphical. If $D$ is to be self-converse then the repetition number of any odd term in the composition is to be clearly even.

3. Find a standard decomposition matrix, $M$, of $\tilde{H}$ such that the first row of $M$ coincides with the given composition.

4. Find an admissible multigraph $G$ as a realization of the composition such that the matrix $M$ represents a $G$-decomposition of an isomorph of the multigraph $\tilde{H}$.

5. Make vertex labels of $\tilde{H}$ (and also of $H$) coincide with those of the isomorph of $\tilde{H}$ in question.

6. Transform the matrix $M$ into a directed decomposition matrix, $\mathcal{D}M$, of the digraph $H$.

7. Find an admissible digraph $D$ such that $\mathcal{D}M$ is a $D$-decomposition matrix of the digraph $H$. Output $D$ and stop.

**Remark 3.** Backtracking in the above algorithm consists of the following. If performing a given step $i$ is unsuccessful and $i > 2$ then another object is searched for at the preceding step $i - 1$.

4. Non-existence of some third parts for $n = 5$

Parts of the proofs of Theorems 2–4 are summarized in Table 1. Refer to the preceding section for the terminology. Entries in each column are either the number of indicated objects or objects themselves, which are compositions or $\tilde{c}$-parts, the latter
Table 1

<table>
<thead>
<tr>
<th>( n = 5 )</th>
<th>( t = 3 )</th>
<th>Compositions</th>
<th>M's</th>
<th>Multigraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Good</td>
<td>With M</td>
<td>All</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>12</td>
<td>6</td>
<td>4,3,3,2,0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4,3,3,1,1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3,3,2,2,2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>12</td>
<td>6</td>
<td>4,3,3,2,0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3,3,2,2,2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Theorem 4</td>
<td>14</td>
<td>2</td>
<td>4,3,3,2,2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3,3,3,3,2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 2. Possible \( \varepsilon \)-parts.

being depicted in Fig. 2. It follows from the table in case of Theorem 3 that five multigraphs found in Step 4 of Algorithm admit a self-converse orientation but none is in the floor-matching third class \( [^3K_3/3]_{2e} \). This ends the proof of Theorem 3.

Completing the proof of Theorem 2. There are the following two decomposition matrices \( M_1, G \) for \( \varepsilon \)-parts \( G_1 \) and \( G_2 \):

\[
[M_1]_{1 \times 2} = \begin{bmatrix}
3 & 3 & 2 & 2 & 2 \\
3 & 3 & 2 & 2 & 2 \\
2 & 2 & 3 & 3 & 2 \\
3 & 2 & 3 & 2 & 2 \\
3 & 3 & 3 & 3 & 2
\end{bmatrix}.
\]

Notice that \((4,4), (4,4), (4,3), (4,3), (2,4)\) is a corresponding sequence of the degree pairs of vertices in \( \mathcal{D}K_5 - R \), where \( R \) comprises the arcs of a roof, namely, \( 5 \rightarrow 3, 5 \rightarrow 4 \). The case \( R \) being a gutter can be obtained by reversing the orientation of each arc.

We partition all entries of \( M \) into their corresponding degree pairs, so that a directed decomposition matrix, \( \mathcal{D}M \), could arise. Notice that the pair \((1,1)\) must be the degree pair of the common neighbour of degree-3 vertices in all self-converse orientations, \( D_1 \), of both \( G \). It is so because due to the structure of both \( G_i \) that neighbour is clearly the only vertex fixed by each conversing permutation of \( D \). Hence the degree pairs of the remaining degree-2 vertices must be \((2,0)\) and \((0,2)\), so that the fifth column of \( \mathcal{D}M \) could sum up to \((2,4)\). This eliminates the multigraph \( G_2 \) because \( D \) has no multiple arcs. Thus \( G = G_1 \) in what follows.
Let $M = M_{11}$. Then the first two columns of $\mathcal{D}M$ sum up to $(4,4)$ each. Hence there are essentially the following three possibilities:

$$
\begin{array}{c|c|c}
3,0,0,3 & 2,1,1,2 & 2,1,1,2 \\
0,3,3,0 & 1,2,2,1 & 2,1,1,2 \\
1,1,1,1 & 1,1,2,1 & 0,2,2,0 \\
\end{array}
$$

In the first two cases the second column cannot be completed because the entry $(1,1)$ cannot be repeated in the same row. On the other hand, the first two parts of the decomposition in the third case include the same orientation of the edge $v_1v_2$, a contradiction.

Let $M = M_{12}$. Then there are essentially the following six possibilities of the initial part of the complex matrix $\mathcal{D}M$ where stars * represent the third column.

$$
\begin{array}{c|c|c|c|c|c|c}
3,0,0,3 & 2,1,1,2 & 2,1,1,2 \\
0,3,1,1 & 1,2,1,1 & 2,1,2,0 \\
1,1,3,0 & 1,1,2,1 & 0,2,1,2 \\
2,1,1,2 & 1,1 & 2,1,1,2 & 1,1 & 2,1,1,2 & 2,0 \\
1,2,2,0 & * & 1,1 & 2,1,1,1 & * & 2,0 & 1,2,2,0 & * & 0,2 \\
1,1,1,2 & 2,1 & 0,2,2,1 & 1,2 & 1,1,1,2 & 2,1 \\
\end{array}
$$

The decompositions represented by $\mathcal{D}M$ have too many orientations of either the edge $v_1v_2$ in the first two cases or $v_1v_4$ in the fourth case. On the other hand, there are either too many (three) or too few (one) orientations of $v_1v_2$ in the third case and $v_1v_4$ in the last two cases. □

Instead of completing the proof on ceiling third class (Theorem 4) we only mention that one of the five standard matrices $M$ for $\Gamma_1$ can be quickly eliminated. In order to eliminate the remaining matrices (four for $\Gamma_1$ and one for $\Gamma_2$, see the above table) oriented $\bar{c}$-parts $D$ have to be considered in some detail. Details can be obtained upon request from the authors.

5. Non-symmetric decomposition

Proof of the case $n = 9$ in Theorem 5. Assume that $V(D_9) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A(D_9) = \{1 \rightarrow 2, 3, 5, 6; 2 \rightarrow 5, 6, 8, 9; 3 \rightarrow 2, 5, 7; 4 \rightarrow 1, 2, 3; 5 \rightarrow 4, 7, 8; 6 \rightarrow 3, 4; 7 \rightarrow 1, 2; 8 \rightarrow 1, 4; 9 \rightarrow 1\}$ (see Fig. 3). Hence $D_9$ is a self-converse oriented graph which is a balanced digraph, i.e., $\text{od}x = \text{id}x$ for each vertex $x$. The unique conversing permutation of $D_9$ is $\phi:= (12)(45)(67)$, which fixes the three vertices 9, 3, and 8 (Fig. 3). The degree sequence of $D_9$ is $(8^2, 6^3, 4^3, 2)$. Let $\theta$ be an automorphism of $D_9$. Then $\theta(9) = 9$ because 9 is the only vertex of degree 2. Hence or because the arc $1 \rightarrow 2$ is in $D_9$, the vertices 1 and 2, both of degree 8, are not similar. No two of the remaining vertices (even if their degree pairs coincide) are similar because of arcs incident to degree-8 vertices 1 and 2. Thus $\theta$ is the trivial automorphism.
Therefore, fixed points are precluded among nontrivial automorphisms of any \( D_9 \)-decomposition of \( \mathcal{D}K_9 \). In particular, there is no automorphism which fixes one part of a \( D_9 \)-decomposition and interchanges the other two.

Suppose that \( \gamma \) is a permutation of degree 9 which generates a cyclic \( D_9 \)-decomposition of \( \mathcal{D}K_9 \). Because vertex degrees of \( \mathcal{D}K_9 \) are all 16 and 3 \( |16 \), the permutation \( \gamma \) has no fixed point. Moreover, \( \gamma^3 \) is the identity permutation because \( D_9 \) is asymmetric and a third part. Hence all cycles of \( \gamma \) have length 3. The vertex set of each cycle of \( \gamma \) clearly induces a third part of \( \mathcal{D}K_3 \) in \( D_9 \) which, by (1), is a gutter or roof. One can see, however, that the degree-6 vertex 3 cannot appear in any such cycle of \( \gamma \). Therefore no \( D_9 \)-decomposition of \( \mathcal{D}K_9 \) is cyclic.

Note that, for permutations \( \psi_2 = (16295743) \) and \( \psi_3 = (18376259) \), the digraphs \( D_9, \psi_2D_9 \) and \( \psi_3D_9 \) make up a \( D_9 \)-decomposition of \( \mathcal{D}K_9 \).

Another \( D_9 \)-decomposition is given by permutations \( \hat{\psi}_2 = (19465327) \), \( \hat{\psi}_3 = (14928367) \) which are chosen so that \( \hat{\psi}_i = \phi \psi_i \phi \) for \( i = 1, 2 \), where \( \phi \) is the converse permutation of \( D_9 \).

To prove that the two decompositions are essentially distinct, assume that the vertex labels in \( D_9 \), Fig. 3, are those in \( \mathcal{D}K_9 \). Then the first \( D_9 \)-decomposition above splits the vertex 1 of \( \mathcal{D}K_9 \) into the set \( \{1, 3, 9\} \) which comprises vertices 1, \( \psi_2^{-1}(1) = 3 \), and \( \psi_3^{-1}(1) = 9 \) in the three \( \hat{\psi} \)-parts. In the second decomposition all sequences each of which includes the vertex 9 and represents splitting of a vertex of \( \mathcal{D}K_9 \) are clearly \( (9, 1, 4), (4, 9, 1) \) and \( (2, 3, 9) \). None of them is an ordering of the set \( \{1, 3, 9\} \). Hence and because \( D_9 \) has no similar vertices, the two decompositions are essentially distinct.

**Remark 4.** No other \( D_9 \)-decomposition of \( \mathcal{D}K_9 \) exists, which was checked with the help of a computer using the above Algorithm.

### 6. Recursive constructions

In the constructions which follow, given a near-third part \( D_n \) of \( \mathcal{D}K_n \), we use the symbol \( \psi_i (=\psi_{n,i}) \) to denote the \( i \)th placing function for \( D_n \), \( \psi_i \) being a permutation.
Completing the proof of Theorem 5. Consider the following digraph $D_6$ presented in Fig. 4. Note that if the vertex labels are as in Fig. 4 then $A(D_6) = \{1 \to 3, 4, 5; 2 \to 1; 3 \to 2, 5; 4 \to 2; 5 \to 6; 6 \to 2, 4\}$. Moreover, there is a $D_6$-decomposition of $\mathcal{D}K_6$ such that $\psi_{6,3} = (145)(236)$ and $\psi_{6,3} = (\psi_{6,2})^2$ are placing functions for $D_6$.

Applying the following recursion for $n \geq 15$. Let $v_1, v_2, \ldots, v_{n-6}$ be the vertices of a given $D_{n-6}$. Take a copy of $D_6$ (Fig. 4) with vertices $i = u_i; i = 1, \ldots, 6$, disjoint from $D_{n-6}$. Define

$$D_n = D_{n-6} \cup D_6 + \{(v_j, u_1), (v_j, u_3), (u_2, v_j), (u_4, v_j): j = 1, \ldots, n-6\}.$$ 

Let $\psi_2$ and $\psi_3$ be placing functions for $D_{n-6}$. Then one can see that $\psi_{n,i} = (\psi_i, \psi_{6,i}), i = 2, 3$ are required placing functions for $D_n$.

The proof that both $D_n$ and all $D_n$-decompositions of $\mathcal{D}K_n$ have trivial automorphism groups for any $n$ can imitate the lines of that for $n = 9$, see the preceding section.

Proof of Theorem 6. Let $D_n$ stand for a non-self-converse oriented graph which is a required near-third part of $\mathcal{D}K_n$. We can assume that $D_3 = P^\vee$, Fig. 1. We have a single digraph $D_4$ presented in Fig. 5. For $n = 5$, depending on $R$ and $S$, $D_5$ is one of digraphs $D_x^2$ with $x = 1, 2, 3, 4$ (Fig. 5) or $x = 5$ where $D_5$ is defined to be the converse of the digraph $D_3^2$. For $n \geq 4$ together with $D_n$ we present the placing functions $\psi_2$ and $\psi_3$ and the set $R$ or $S$ but only if the set is non-empty.

$$A(D_4) = \{1 \to 2; 2 \to 3; 3 \to 1, 4\}, \psi_2 = (134), \psi_3 = (\psi_2)^2.$$ 

$$A(D_x^1) = \{1 \to 4; 2 \to 1, 3; 3 \to 1; 4 \to 2; 5 \to 1\},$$ 

$$R = A(\tilde{C}_3) = \{3 \to 4; 4 \to 3\}, \psi_2 = (125), \psi_3 = (\psi_2)^2.$$
A(D_2^5) = \{1 \rightarrow 2, 4; 2 \rightarrow 3; 3 \rightarrow 1; 4 \rightarrow 2; 5 \rightarrow 1\},
R = A(D_3) = \{2 \rightarrow 5; 5 \rightarrow 4\}, \psi_2 = (13)(45), \psi_3 = (14)(25),
R = 2a = \{2 \rightarrow 5; 3 \rightarrow 4\}, \psi_2 = (1543), \psi_3 = (14)(235).
A(D_2) = \{2 \rightarrow 1; 3 \rightarrow 1, 2; 4 \rightarrow 1, 2; 5 \rightarrow 1\},
R = A(D_2^5) = \{2 \rightarrow 5; 4 \rightarrow 5\}, \psi_2 = (13)(45), \psi_3 = (14)(25).

Note that reversing all arcs in the above D_2^5-decomposition with R = A(D_2^5) gives the
D_5^5-decomposition of D_K5 with the remainder R = A(D_2^5).
A(D_2^5) = \{1 \rightarrow 2, 3, 4; 3 \rightarrow 2, 4; 4 \rightarrow 2; 5 \rightarrow 2\}, \psi_2 = (152)(34), \psi_3 = (\psi_2)^2,
S comprises a copy of the arc 3 \rightarrow 4.

Apply the following recursion for n \geq 6. Let v_1, v_2, \ldots, v_{n-3} be all the vertices of a
given D_{n-3}. Take a copy of the D_3 with vertices u_1, u_2, u_3 disjoint from D_{n-3}. Define
D_n = D_{n-3} \cup D_3 + \{v_j \rightarrow u_1, u_2 \rightarrow v_j, j = 1, \ldots, n - 3\}.

Let \psi_2 and \psi_3 be placing functions for D_{n-3}. Let \psi_{3,2} and \psi_{3,3} be placing function
for our copy of D_3 (\psi_{3,3} = (\psi_{3,2})^2). Then one can see that \psi_{n,i} := (\psi_i, \psi_{3,i}), i = 2, 3
are required placing functions for D_n, sets R and S being the same as those for D_{n-3}.

Remark 5. For any fixed n = 2 + 3k with k \in \mathbb{N}, there exist floor-R third parts of D_Kn
which are identical or isomorphic, or have isomorphic underlying multigraphs for any
two different remainders R.

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