A note on perturbed fixed slope iterations

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Abstract

An approximation to the exact derivative leads to perturbed fixed slope iterations in the context of Inexact Newton methods. We prove an a posteriori convergence theorem for such an algorithm, and show an application to nonlinear differential boundary value problems. The abstract setting is a complex Banach space.

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1. The fixed slope iteration

Let us consider the following abstract formulation of a nonlinear equation: given a nonempty open set $\Omega$ in a Banach space $X$ whose norm is denoted by $\| \cdot \|$, find $x_\infty \in \Omega$ such that $F(x_\infty) = 0$, (1)

where $F : \Omega \rightarrow X$ is a Fréchet differentiable (most probably nonlinear) operator.

We recall that the standard Fixed slope iteration for solving (1) is defined by

$\xi_0 \in \Omega,$

$\xi_{k+1} := \xi_k - F'(\xi_0)^{-1}F(\xi_k)$ for all $k \geq 0,$ (2)

where $F$ is supposed to have an invertible Fréchet derivative at $\xi_0$. 

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Let \( \mathcal{L}(X) \) denote the Banach algebra of bounded linear operators from \( X \) into itself. Its norm is also denoted by \( \| \cdot \| \).

The following a posteriori result gives sufficient conditions for the existence and uniqueness of a solution of (1) at a vicinity of \( \xi_0 \), and for the convergence of the sequence defined by (2) towards it (cf. [1,2]).

**Theorem 1.** Suppose that \( \Omega, F, \xi_0 \in \Omega, \ell > 0, \mu > 0 \) and \( c > 0 \) satisfy

1. \( 4\mu\ell c \leq 1 \),
2. \( \Omega \) includes the closed neighborhood \( N_0 := \{ \xi \in X : \|\xi - \xi_0\| \leq r \} \), where
   \[
   r := \frac{1 - \sqrt{1 - 4\mu\ell c}}{2\mu\ell},
   \]
3. \( F'(\xi) \) exists for all \( \xi \in N_0 \),
4. \( F'(\xi_0) \) is invertible, and \( \|F'(\xi_0)^{-1}\| \leq \mu \),
5. \( \|\xi_1 - \xi_0\| \leq c \),
6. \( \ell \) is a Lipschitz constant for \( F' : N_0 \to \mathcal{L}(X) \).

Then (1) has a unique solution \( x_\infty \) in \( N_0 \) and the sequence defined by (2) converges to \( x_\infty \) at the following rate: for all \( k \geq 0 \),

\[
\|\xi_k - x_\infty\| \leq \frac{c\beta^k}{1 - \beta},
\]

where

\[
\beta := \frac{1}{2}(1 - \sqrt{1 - 4\mu\ell c}) \in \left[ 0, \frac{1}{2} \right].
\]

**2. A perturbed fixed slope iteration**

A perturbed Fixed slope iteration for solving (1) may be defined as

\[
x_0 \in \Omega, \quad x_{k+1} := x_k - B^{-1}F(x_k) \quad \text{for all } k \geq 0,
\]

where \( B \in \mathcal{L}(X) \) is invertible and sufficiently close to \( F'(x_0) \). The following a posteriori result gives sufficient conditions for the existence and uniqueness of a solution of (1) at a vicinity of \( x_0 \), and for the convergence of the sequence defined by (3) to it.

**Theorem 2.** Suppose that \( \Omega, F, B, x_0 \in \Omega, \delta \geq 0, \ell > 0, m > 0 \) and \( c > 0 \) satisfy

1. \( m\delta < 1 \) and \( 4m\ell c \leq (1 - m\delta)^2 \),
2. \( \Omega \) includes the closed ball \( B_0 := \{ x \in X : \|x - x_0\| \leq \rho \} \), where
   \[
   \rho := \frac{1 - m\delta - \sqrt{(1 - m\delta)^2 - 4m\ell c}}{2m\ell},
   \]
3. \( F'(x) \) exists for all \( x \in B_0 \),
4. \( \|F'(x_0) - B\| \leq \delta \), \( B \in \mathcal{L}(X) \) is invertible and \( \|B^{-1}\| \leq m \),
5. \( \|x_1 - x_0\| \leq c \),
6. \( \ell \) is a Lipschitz constant for \( F' : B_0 \to \mathcal{L}(X) \).
Then (1) has a unique solution $x_\infty$ in $B_0$ and the sequence defined by (3) converges to $x_\infty$ at the following rate: for all $k \geq 0$,
\[ \|x_k - x_\infty\| \leq \frac{c\gamma^k}{1 - \gamma}, \]
where
\[ \gamma := \frac{1}{2}(1 + m\delta - \sqrt{(1 - m\delta)^2 - 4m\ell c}) \in [0, 1[. \]

**Proof.** For $x \in B_0$, we set:
\[ G(x) := x - B^{-1}F(x), \]
\[ E(x) := F(x_0) + B(x - x_0) - F(x). \]
We remark that $x_\infty = G(x_\infty)$ if and only if $F(x_\infty) = 0$. We will apply the Fixed point Banach theorem to $G$. Let us prove that $B_0$ is invariant under $G$ and that $G : B_0 \to X$ is a contraction. Notice that
\[ E(x_0) = 0, \]
and that, for all $x \in B_0$,
\[ E'(x) = B - F'(x_0) + F'(x_0) - F'(x), \]
\[ \|E'(x)\| \leq \ell\|x - x_0\| + \delta \leq \ell\varrho + \delta, \]
\[ \|E(x)\| = \|E(x) - E(x_0)\| \leq (\ell\varrho + \delta)\|x - x_0\| \leq (\ell\varrho + \delta)\varrho, \]
\[ G(x) - x_0 = B^{-1}(E(x) - F(x_0)). \]
Hence
\[ \|G(x) - x_0\| \leq m(\ell\varrho + \delta)\varrho + c = \varrho. \]
Also, for all $x \in B_0$,
\[ G'(x) = I - B^{-1}F'(x) = B^{-1}(B - F'(x_0) + F'(x_0) - F'(x)) = B^{-1}E'(x), \]
so
\[ \|G'(x)\| \leq m(\ell\varrho + \delta) = \gamma < 1 \]
for all $x \in B_0$, which proves that $G : B_0 \to X$ is a contraction. The convergence rate is a well known result of successive approximation fixed point iterations. □

3. An application

Let $L$ be a linear closed unbounded operator whose domain $D$ is included in a complex Banach space $X$ normed by $\|\cdot\|$. Suppose that $L$ has a bounded inverse $T$. Let $N$ be a nonlinear Fréchet differentiable operator such that $N'$ is locally Lipschitz. For $y \in X$, we consider the abstract nonlinear boundary value problem

Find $x_\infty \in D$ such that $Lx_\infty + N(x_\infty) = y,$
or equivalently,
Find $x_\infty \in X$ such that $x_\infty + TN(x_\infty) = Ty.$
To enter the abstract setting of the previous sections, define the nonlinear Fréchet differentiable operator

\[ F : X \to X, \quad x \mapsto F(x) := x + T(N(x) - y). \]

Clearly, the first Fréchet derivative of \( F \) is defined by

\[ F'(x)h = h + TN'(x)h \quad \text{for all } x, h \in X. \]

Let \( x_0 \in X \) be given. Define \( B := I \) so that \( m = 1 \). Then

\[ \| F'(x_0) - B \| = \| TN'(x_0) \| \leq \delta := M \| N'(x_0) \|, \]

\[ \| x_1 - x_0 \| = \| B^{-1}F(x_0) \| \leq c := \| x_0 \| + M \| N(x_0) \| + M \| y \|. \]

Let \( v \) be a Lipschitz constant for \( N' \) on a closed ball centered at \( x_0 \) with radius \( r \). Then

\[ \ell := v \| T \| \]

is a Lipschitz constant for \( F' \) on the same ball. Suppose that

\[ r \geq q := \frac{1 - \delta - \sqrt{(1 - \delta)^2 - 4\ell c}}{2\ell}, \]

so \( \ell \) is a Lipschitz constant for \( F' \) on \( B_0 \). Now we are in position to check if the hypotheses of Theorem 2 are satisfied or not. Iterations (3) read

\[ x_{k+1} := T(y - N(x_k)), \quad k \geq 0, \]

or equivalently

\[ Lx_{k+1} := y - N(x_k), \quad x_{k+1} \in D, \quad k \geq 0. \]

As an example, let us consider a nonlinear differential problem (cf. [1,3]) in the space \( X := C^0([0, 1], \mathbb{C}) \) of all complex-valued continuous functions defined on \([0, 1]\), equipped with the uniform convergence norm. Given \( y \in X \), the nonlinear differential boundary value problem

\[ \text{Find } x \in C^2([0, 1], \mathbb{C}) \text{ such that } -x'' + x^2 = y, \quad x(0) = x(1) = 0, \tag{5} \]

is equivalent to the nonlinear integral equation

\[ x(s) + \int_0^1 \kappa(s, t) x(t)^2 \, dt = w(s) := \int_0^1 \kappa(s, t) y(t) \, dt \quad \text{for all } s \in [0, 1], \]

where \( \kappa \) is the Green kernel defined by

\[ \kappa(s, t) := \begin{cases} s(1 - t), & \text{if } 0 \leq s \leq t \leq 1, \\ t(1 - s), & \text{otherwise}. \end{cases} \]

The nonlinear operator \( F : X \to X \) is then

\[ F(x)(s) := x(s) + \int_0^1 \kappa(s, t) x(t)^2 \, dt - w(s) \quad \text{for all } s \in [0, 1] \text{ and all } x \in X. \]

Choose the constant function

\[ x_0(s) := \frac{1}{2} \quad \text{for all } s \in [0, 1], \tag{6} \]
as the starting point of the sequence of iterations defined by (3), where \( B := I \) so \( m = 1 \). We remark that \( N'(x_0) = I \) too. Hence
\[
F'(x_0)h(s) = \int_0^1 \kappa(s, t) h(t) \, dt + h(s) \quad \text{for all } s \in [0, 1] \text{ and all } h \in X,
\]
and
\[
\| F'(x_0) - B \| \leq \delta := \max_{s \in [0,1]} \int_0^1 |\kappa(s, t)| \, dt = \frac{1}{8}.
\]
Also
\[
\| F'(u) - F'(v) \| \leq 2 \delta \| u - v \| = \frac{1}{4} \| u - v \| \quad \text{for all } u, v \in X,
\]
and
\[
\| x_1 - x_0 \| = \| B^{-1} F(x_0) \| \leq c := \frac{1}{2} + \frac{1}{4} \delta + \frac{1}{8} \| y \|.
\]
We conclude from Theorem 2 that if \( \| y \| \leq 1.875 \), then the sequence defined by (3) with starting point (6) converges to the unique solution of (5) in the closed ball with radius
\[
\varrho := 1.750 - \sqrt{\frac{1.875 - \| y \|}{2}}
\]
centered at the starting point. The iterations read as
\[
x_0(s) := \frac{1}{2} \quad \text{for all } s \in [0, 1],
\]
\[
x_{k+1}(s) := \int_0^1 \kappa(s, t) (y(t) - x_k(t)^2) \, dt \quad \text{for all } s \in [0, 1].
\]
Consider for example
\[
y(s) := 1 + \frac{1}{4} (1 - s)^2 s^2, \quad x_\infty(s) := \frac{1}{2} s(1 - s) \quad \text{for all } s \in [0, 1].
\]
Then, up to seven significant digits, \( \| y \| = 1.015625 \) and \( \varrho = 1.094495 \).

4. Remarks and bibliographical comments

We remark that the condition \( \| F'(x_0) - B \| \leq \delta < \frac{1}{m} \leq \frac{1}{\| B^{-1} \|} \) implies that \( F'(x_0) \) is invertible, although its inverse is neither needed in computations nor used in the proof of the previous theorem, and that each one of the six hypotheses of Theorem 2 reduces to the corresponding hypothesis of Theorem 1 when \( \delta = 0 \), and the same holds for its conclusion, since \( \delta = 0 \) implies \( \gamma = \beta \). Hence the proof of Theorem 2 carried out with \( \delta = 0 \) gives a proof of Theorem 1.

The following figure shows the Neperian logarithm of the relative error corresponding to the first ten iterates in the case of the application developed in Section 3. As predicted by theory, convergence is linear.
The most interesting aspects of perturbed fixed slope iterations are the following:

– Theoretically speaking, they handle in an abstract setting the error in computations associated to the exact fixed slope algorithm.

– Computationally speaking, they preserve the linear convergence predicted by the exact fixed slope algorithm, although the rate itself may be weaker.

– They reduce the complexity of the classical Newton–Kantorovich method in which the Fréchet derivative of the nonlinear operator \( F \) must be computed at each iterate, and with it, a linear problem must be solved for the next iterate. Certainly, this computational advantage has a price: the quadratic rate of convergence becomes linear.

A spectral application of Theorem 2 is developed in [4].

References