ESTIMATION OF CHAOTIC THRESHOLDS FOR THE RECENTLY PROPOSED ROTATING PENDULUM

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In this paper, we investigate the nonlinear behavior of the recently proposed rotating pendulum which is a cylindrically nonlinear system with irrational type having smooth and discontinuous characteristics depending on the value of a smoothness parameter. We introduce a cylindrical approximate system whose analytical solutions can be obtained successfully to reflect the nature of the original system without the barrier of irrationalities. Furthermore, Melnikov method is employed to detect the chaotic thresholds for the homoclinic orbits of the second-type, a pair of homoclinic orbits of the first and second-type and the double heteroclinic orbits under the perturbation of viscous damping and external harmonic forcing within the smooth regime. Numerical simulations show the efficiency of the proposed method and the results presented herein this paper demonstrate the predicated chaotic attractors of pendulum-type, SD-type and their mixture depending on the coupling of the nonlinearities.

Keywords: Rotating pendulum; SD oscillator; irrational nonlinearity; chaotic thresholds; singular closed orbits.

1. Introduction

Ever since Galileo found the law of isochronism through classical experiments [Galileo, 1592] and Huygens discovered the periodicity of pendulum [Huygens, 1658], more than four centuries have been focused on the research of pendulum. Much attention has been paid on the behavior of pendulum in physics [Formalskii, 2006; Mazaheri et al., 2012] and engineering [Massad et al., 2008; Hoedl et al., 2011]. Various kinds of pendulums have been presented and investigated in modern times, such as the mathematical pendulum [Liang & Feeny, 2006; Chernousko & Reshmin, 2007], double pendulum [Enolski et al., 2003; Dullin, 1994; Paul & Richer, 1994], spring pendulum [Zaki et al., 2002; Lee & Park, 1997; Narkis, 1997; Bayly & Virgin, 1993; Ioannis, 1999] and spherical pendulum [Miles, 1962, 1984; Kana, 1995; Zou, 1992; Bryant, 1993; Tritton, 1986; Georgiou & Schwartz, 2001] as well. Pendulum has been one of the earliest topics in science [Matthews, 2001] and its interest extends from science to philosophy [Matthews et al., 2005] and

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Nowadays, the conventional pendulum is widely used in engineering, such as energy harvesting [Wiercigroch, 2010] and robot design [Jeong & Takahashi, 2008; Kim et al., 2005] as well.

In recent years, the inverted pendulum subjected to a linear spring providing a follower force has drawn attention [Lobas et al., 2007] on nonlinear dynamics. Meanwhile, the approximation and numerical methods are developed to detect the bifurcations and the chaotic motions of the system, see [Yu & Zhu, 2005] for details. Recently, a novel model of rotating pendulum [Cao et al., 2011] coupled with smooth-and-discontinuous (SD) oscillator [Cao et al., 2006, 2008a; Tian et al., 2012; Han et al., 2012] exhibits both smooth and discontinuous dynamics depending on the value of a smoothness parameter. Numerical investigations show that the type of attractors are similar to those of pendulum or SD type or their mixture depending on the coupling of the nonlinearities.

The motivation of this paper is to detect the chaotic boundary of the recently proposed rotating pendulum by means of the approximate system which efficiently reflects the nonlinear dynamics of the original system. This approach enables us to investigate the nonlinear dynamics theoretically for both perturbed and unperturbed dynamics. The proposed cylindrical system bears significant similarities to the original system of double pitchfork bifurcation at $x = 0$ and $x = \pm \pi$, phase portraits and the bifurcation which successfully avoids the barrier of the associated irrational nonlinearity.

This paper is organized as follows. In Sec. 2, all the phase portraits of the unperturbed system are displayed with the homoclinic orbit of the second-type, a pair of homoclinic orbits of the first and second-type and double heteroclinic orbits for both smooth and discontinuous regimes. In Sec. 3, we introduce a cylindrical approximate system that is topologically equivalent to the original system within the smooth regime to investigate analytically the chaos of the original system. In Sec. 4, Melnikov method is used to obtain the analytical chaotic thresholds for the perturbed homoclinic orbit of the second-type, a pair of homoclinic orbits of the first and second-type and double heteroclinic orbits within the smooth regime. Numerical results show the efficiency of the obtained chaotic criteria in every region. Finally in Sec. 5 the paper is concluded with the summary and future interests for this cylindrical pendulum system.

2. The Rotating Pendulum Coupled with SD Oscillator

Consider the rotating pendulum [Cao et al., 2011] coupled with SD oscillator proposed in [Cao et al., 2006] as shown in Fig. 1, the equation of motion reads

$$m \ddot{x} - mg \sin x + kh \sin x \left(1 - \frac{l}{\sqrt{l^2 + h^2 - 2lh \cos x}} \right) = 0,$$

(1)

![Fig. 1. The model of the rotating pendulum coupled with SD oscillator: (a)-(c) for $h > 0$, $h = 0$ and $h < 0$ respectively.](image-url)
where \( m, L, k \) and \( l \) are the lump mass, pendulum length, the stiffness and relax length of the spring, while \( h \) is the height from A to B, always assumed positive and negative in Fig. 1(a) for \( h > 0 \) and Fig. 1(c) for \( h < 0 \), respectively. Even the stiffness of the spring is linear and the resistance force supplied to the system is strongly irrational nonlinearity due to geometry configuration.

Without loss of generality, it is always assumed that \( L = 1, l = 1 \), and \( \omega_0^2 = \frac{3}{2} \) in the following discussion. The dimensionless form of Eq. (1) can be written in the following form by introducing a triangular transformation,

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= g \sin x \\
\dot{z} &= -\omega_0^2 \sin x \left( 1 - \frac{1}{\sqrt{1 + h^2 - 2h \cos x}} \right)
\end{align*}
\]

(2)

which is a strongly nonlinear system of irrational type exhibiting both smooth for \( |h| \neq 1 \) and discontinuous for \( h = 1 \) at \( x = 0 \) and \( h = -1 \) at \( x = \pi \) (see [Cao et al., 2008b]). In the discontinuous stage, system (2) can be written in the following form by introducing a triangular transformation,

\[
\begin{align*}
\dot{x} &= (\omega_0^2 - g) \cos x \\
\dot{y} &= -\omega_0^2 \cos x \sgn \left( \sin \frac{x}{2} \right) = 0, \quad (h = 1) \\
\dot{z} &= (-\omega_0^2 + g) \sin x \\
&\quad + \omega_0^2 \sin x \left( \cos \frac{x}{2} \right) = 0, \quad (h = -1),
\end{align*}
\]

(3)

where

\[
\text{sgn}(X) = \begin{cases} 
1 & \text{if } X > 0 \\
0 & \text{if } X = 0 \\
-1 & \text{if } X < 0.
\end{cases}
\]

It is worth noticing that the discontinuous dynamics is obtained by changing the parameter \( h \) to \( \pm 1 \) smoothly, which is the limit case as \( h \to \pm 1 \) from the mathematical point of view.

The equilibrium of system (2) can be obtained as

\[
(x_{1,1}, y_1) = (0, 0), \quad (x_{2,3}, y_{2,3}) = (\pm \pi, 0),
\]

\[
(x_{4,5}, y_{4,5}) = \left( \pm \arccos \left( \frac{1 + h^2}{2h} - \frac{\omega_0^2 h}{2(g - \omega_0^2 h^2)} \right), 0 \right),
\]

where \( x_{4,5} \) exist only for \( \left| \frac{1 + h^2}{2h} - \frac{\omega_0^2 h}{2(g - \omega_0^2 h^2)} \right| < 1 \).

The Hamiltonian of system (2) can be obtained and written as

\[
H(x, y) = \frac{q^2}{2} + (g - \omega_0^2 h) \cos x \\
-\omega_0^2 \sqrt{1 + h^2 - 2h \cos x} - (g - \omega_0^2 h) \\
+ \omega_0^2 |1 - b|,
\]

(4)

with the help of which, the trajectories of system (2) can be classified and are plotted in Fig. 2 showing a variety of singular closed orbits for both smooth and discontinuous cases. In the discontinuous case, the homoclinic-like orbits of the second-type connecting the equilibrium \((0, 0)\) in Figs. 2(a)–2(c) and the homoclinic-like orbits of the first-type connecting the equilibria \((\pm \pi, 0)\) in Fig. 2(b) or 2(d), denoted by hom\(2\) and hom\(1\), are marked in solid blue respectively. While for the smooth regime, the homoclinic orbits of the second-type connecting the equilibrium \((0, 0)\) \((\pm \pi, 0)\) and the homoclinic orbits of the first-type connecting equilibrium \((0, 0)\) \((\pm \pi, 0)\) and the double heteroclinic orbits connecting both \((0, 0)\) and \((\pm \pi, 0)\), shown in Figs. 2(d)–2(f) [Figs. 2(h)–2(j)], Fig. 2(b) [Fig. 2(f)] and Fig. 2(g), denoted by hom\(2\), hom\(1\) and het, respectively, all these singular closed orbits are plotted in solid red.

System (2) can be rewritten in the following cylindrical form by letting \( X = \cos x, Y = \sin x, Z = y \).

\[
\begin{align*}
\dot{X} &= -YZ, \\
\dot{Y} &= XZ, \\
\dot{Z} &= gY - \omega_0^2 h Y \left( 1 - \frac{1}{\sqrt{1 + h^2 - 2hX}} \right), \\
X^2 + Y^2 &= 1,
\end{align*}
\]

(5)

of which, the Hamiltonian can be obtained and written as

\[
H(X, Y, Z) = \frac{q^2}{2} + (g - \omega_0^2 h)X - \omega_0^2 \sqrt{1 + h^2 - 2hX} \\
- (g - \omega_0^2 h) + \omega_0^2 |1 - h|, \\
X^2 + Y^2 &= 1.
\]

The typical phase portraits of system (4) are plotted in Fig. 3. Figs. 3(a) and 3(b) for the smooth...
Fig. 2. Graph of transition set, $\Sigma$, and the corresponding phase portraits, (a)–(l).

Fig. 3. Cylindrical phase portraits: (a) a pair of homoclinic orbits of the first and the second-type (red) for $\omega_0 = \sqrt{50}$, $h = -1.1$ with saddles at $(\pm 1, 0, 0)$ and centers at $(\pm \frac{\sqrt{935}}{36}, 0)$; (b) a double heteroclinic orbits (blue) for $\omega_0 = \sqrt{10}$, $h = 2$ with the saddles at $(\pm 1, 0, 0)$ and centers at $(\pm \sqrt{5}, 0)$; (c) a homoclinic orbit of the second-type (red) and a homoclinic-like orbit of the first-type (blue) for $\omega_0 = \sqrt{50}$, $h = -1$ and (d) a homoclinic orbit of the first-type (red) and a homoclinic-like orbit of the second-type (blue) for $\omega_0 = \sqrt{30}$, $h = 1$. 

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estimates are the one homoclinic orbit of the first-type, colored red, coexists with a homoclinic-like orbit of the second-type, marked blue, with the saddle-like equilibrium (1, 0, 0) in Fig. 3(c) and a double heteroclinic orbit, marked by dashed, when \( h > -2 \). Figure 4 plots the graphs of equilibrium bifurcations at \((h, x) = (\pm 4, 0)\) and \((h, x) = (\pm 2, 0)\), respectively, showing the double pitchfork bifurcations at \((h, x) = (\pm 2, 0)\).

The equilibria of system (6) can be obtained as

\[
\begin{align*}
\alpha &= g - \omega_0^2 h^2 - h - 1, \\
\beta &= \frac{\omega_0^2 h}{h^2 - 1},
\end{align*}
\]

where \( g = x - \omega_0^2 h \sin x (1 - \frac{1}{\sqrt{4 + 2\omega_0^2 h^2}}) \). This leads to

\[
\alpha = g - \omega_0^2 h^2 - h - 1, \quad \beta = \frac{\omega_0^2 h}{h^2 - 1}.
\]

The equilibria of system (6) can be obtained as

\[
\begin{align*}
x_1 &= 0, \\
x_{2,3} &= \pm \pi, \\
x_{4,5} &= \pm \arccos \left( \frac{-\alpha}{\beta} \right),
\end{align*}
\]

where \( x_{4,5} \) exist only for \(|\alpha| < 1\) or \(|\alpha^2 - 2\omega_0^2 h^2| < 1\).

In summary, system (6) is an approximated cylindrical system reflecting the nonlinear dynamics properly including the equilibria, stabilities and bifurcations.

### 3.1. Equilibrium Stabilities and the Separatrices

In this subsection, the stability of the equilibria of system (6) is discussed and analyzed by employing the Jacobian.
respectively.

The Jacobian of system (6) can be obtained, written as

$$J(x,y) = \begin{pmatrix} \cos x (\alpha + \beta \cos x) - \beta \sin^2 x & 1 \\ \alpha & \beta \end{pmatrix},$$

which follows the stabilities of the corresponding equilibria.

(i) The Jacobian of equilibria (0, 0)

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix},$$

leads to the characteristic equation

$$\lambda^2 - (\alpha + \beta) = 0,$$

which implies that (0, 0) is a saddle point for $\alpha + \beta > 0$ with the eigenvalues $\lambda_{1,2} = \pm \sqrt{-\alpha - \beta}$, a center for $\alpha + \beta < 0$ with $\lambda_{1,2} = \pm i \sqrt{-\alpha - \beta}$ and a nilpotent center (bifurcation point) for $\alpha + \beta = 0$ with $\lambda_{1,2} = 0$, the details seen in [Holmes & Marsden, 1978].

(ii) The Jacobian of equilibria $(\pm \pi, 0)$

$$J(\pm \pi,0) = \begin{pmatrix} 0 & 1 \\ \beta - \alpha & 0 \end{pmatrix},$$

leads to the characteristic equation

$$\lambda^2 - (\beta - \alpha) = 0,$$

which implies that $(\pm \pi, 0)$ is a saddle point for $\beta - \alpha > 0$ with the eigenvalues $\lambda_{1,2} = \pm \sqrt{\beta - \alpha}$, a center for $\beta - \alpha < 0$ with $\lambda_{1,2} = \pm i \sqrt{\beta - \alpha}$ and a nilpotent center for $\beta - \alpha = 0$ with $\lambda_{1,2} = 0$.

(iii) The Jacobian for equilibria $(\pm \arccos(-\frac{\omega}{5}), 0)$ which only exist for $|\frac{\omega}{5}| < 1$ can be written as

$$J(\pm \arccos(-\frac{\omega}{5}),0) = \begin{pmatrix} 0 & 1 \\ \alpha^2 - \beta^2 & \beta \end{pmatrix},$$

leads to the characteristic equation

$$\lambda^2 - \frac{\alpha^2 - \beta^2}{\beta} = 0,$$

and the eigenvalues are derived as $\lambda_{1,2} = \pm i \sqrt{\frac{\alpha^2 - \beta^2}{\beta}}$, which implies the equilibria are center points.

The transition sets of system (6) can be easily obtained as the following.

$$\Sigma_1 = \Sigma_2 = \Sigma_1 \cup \Sigma_2 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,$$

where $\Sigma_1$ and $\Sigma_2$ correspond to bifurcation set and hysteresis set, respectively, denoted as

$$\Sigma_1 = \{ (\alpha, \beta) | \alpha - \beta = 0, \beta > 0 \}$$

and $\Sigma_3 = \{ (\alpha, \beta) | \alpha + \beta = 0, \beta > 0 \}$

$$= \left\{ (h, \omega) \mid \omega h \left( 1 - \frac{1}{1 + h} \right) - g = 0, \right\},$$

$$\omega \in (0, 2 \sqrt{5}), h \in (1, +\infty),$$

$$= \left\{ (h, \omega) \mid g - \omega \frac{h^2}{h - 1} = 0, \right\},$$

$$\omega \in (0, 2 \sqrt{5}), h \in (1, +\infty).$$
In this section, the solutions of homoclinic and heteroclinic orbits can be obtained by using the Hamiltonian (8).

(i) The homoclinic orbits of the second-type denoted by hom2 connecting \((-\pi,0)\) and \((\pi,0)\) located in regions \(\text{III}, \text{IV}\) and the pitchfork bifurcation curve \(\text{III}_2\), \((\alpha < 0, \beta > 0)\) and can be written as

\[
\begin{align*}
\chi_{\text{hom}2}\left(\frac{t}{\alpha}\right), y_{\text{hom}2}\left(\frac{t}{\alpha}\right) \\
 = \begin{pmatrix}
\pm 2\cot^{-1} \left( \sqrt{\frac{\alpha}{\alpha + \beta}} \sinh \left( \sqrt{\frac{\alpha}{\alpha + \beta}} \right) \right) \\
\pm 2\sqrt{\frac{\alpha}{\alpha + \beta}} \cosh \left( \sqrt{\frac{\alpha}{\alpha + \beta}} \right)
\end{pmatrix},
\end{align*}
\]
(iv) The homoclinic orbits of the first-type connecting \((\pm \pi, 0)\) located in region II, can be written as

\[
(x_{\text{hom}1}^{\pm}(t), y_{\text{hom}1}^{\pm}(t)) = \pm 2 \tan^{-1} \left( \sqrt{\alpha} \cosh (\sqrt{\beta - \alpha} t) \right),
\]

\[
\pm \frac{2 \sqrt{\alpha} \sinh (\sqrt{\beta - \alpha} t)}{1 + \frac{\alpha}{\beta - \alpha} \cosh^2 (\sqrt{\beta - \alpha} t)}
\]

and details can be seen in Appendix (A.4);

(v) The double heteroclinic orbits denoted by het connecting \((-\pi, 0), (0, 0)\) and \((\pi, 0)\) located in \( \mathcal{M} \), \((\alpha = 0, \beta > 0)\) and can be written as

\[
(x_{\text{het}}^{\pm}(0), y_{\text{het}}^{\pm}(0)) = \left( \pm \frac{\pi}{2}, \pm \frac{\sqrt{\beta}}{2} \right).
\]

The analytical expressions of unperturbed homoclinic and heteroclinic orbits for the approximate system obtained above enable us to investigate theoretically the chaotic motion of the original system for the smooth regime.
4. Chaotic Thresholds

In this section, the chaotic thresholds for the smooth regime of system (1) perturbed by a viscous damping $\delta L\dot{x}$ and an external harmonic excitation of amplitude $F_0$ and frequency $\omega$ are investigated by means of the approximate system.

If system (1) is perturbed by a viscous damping and an external harmonic excitation of amplitude $F_0$ and frequency $\omega$, this leads to the following system,

$$m\ddot{x} + \delta L\dot{x} = -mg \sin x + F_0 \cos \omega t.$$  

Again system (2) can be made dimensionless by letting $f_0 = \frac{F_0}{mg}$ and $\xi = \frac{x}{l}$.

$$\ddot{x} + \xi \dot{x} - g \sin x + \omega_0^2 \sin x \left(1 - \frac{1}{\sqrt{1 + h^2 - 2h \cos \omega_0 t}}\right) = f_0 \cos \omega t,$$$$

which is very difficult for a direct chaotic analysis to get the chaotic thresholds for the perturbed separatrices. In the following discussions, we will focus our attention on the approximate system obtained above adding a viscous damping and an external harmonic excitation of amplitude and frequency.

$$\dot{x} = y,$$

$$\dot{y} = \sin x(\alpha + \beta \cos x) - \xi y + f_0 \cos \omega t.$$  

Here, we provide the details of Melnikov analysis for system (11), expository discussions of theory can be found in [Melnikov, 1963; Awrejcewicz & Holide, 1999]. Then we introduce the following notation in system (11):

$$F(X) = \begin{pmatrix} y \\ \sin x(\alpha + \beta \cos x) \end{pmatrix},$$

$$G(X, t) = \begin{pmatrix} 0 \\ -\xi y + f_0 \cos \omega t \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\text{Tr}(DF) = \text{Tr} \begin{pmatrix} 0 & 1 \\ \alpha \cos x + \beta \cos 2x & 0 \end{pmatrix} = 0,$$

$$F(X(t), t) \wedge G(X(t), t) = -\xi x_y^2(t) + y(t)f_0 \cos \omega(t + \tau),$$

where $\alpha \wedge b = a_1b_2 - a_2b_1$, for any $a = (a_1, a_2)^T$ and $b = (b_1, b_2)^T$, the corresponding Melnikov function of system (11) is given by

$$M(\tau, f_0, \xi, \omega) = \int_{-\infty}^{\infty} -\xi x_y^2(t)$$

$$+ y(t)f_0 \cos \omega(t + \tau) dt.$$  

If $y_\pm(t)$ is an odd function, system (12) can be rewritten as

$$M(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{\infty} (y_\pm(t))^2 dt$$

$$f_0 \sin \omega t \int_{-\infty}^{\infty} y_\pm(t) \sin \omega t dt;$$

If $y_\pm(t)$ is an even function, system (12) can be rewritten as

$$M(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{\infty} (y_\pm(t))^2 dt$$

$$f_0 \cos \omega t \int_{-\infty}^{\infty} y_\pm(t) \cos \omega t dt.$$

The Melnikov function is proportional to the first variation of the distance function between stable and unstable manifolds of homoclinic or heteroclinic orbits. Therefore, when the function is zero, it predicts the intersection of stable and unstable manifolds. When the stable and unstable manifolds intersect transversely once, they will intersect an infinite number of times. The phase space will have rapid expansion and contraction which will eventually lead to horseshoe dynamics. This is the necessary condition for chaotic dynamics. The Melnikov integrals can be evaluated exactly by the use of theory of residues and the results of [Byrd & Friedman, 1971; Langebartel, 1980; Wan & Li, 1988; Kwek & Li, 1996]. We only give the fundamental formulation without their calculation procedure in the following subsection.

4.1. Melnikov analysis for the homoclinic orbit of the second-type

When $(\alpha, \beta)$ or $(\omega_0, h) \in I \cup \mathcal{F}_1 \cup IV \cup \mathcal{F}_2$, the original system (2) and the approximate system (6) exhibit similar behaviors with the inverted pendulum in region I or bifurcation set $\mathcal{F}_1$ and the single pendulum in region IV or bifurcation set $\mathcal{F}_2$ including the homoclinic orbit of the second-type.
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The Melnikov’s function of system (11) for the homoclinic orbit of the second-type, as discussed in (12), can be written as

\[
M(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{+\infty} (y_{\text{hom}2}^{+}(t))^2 dt + f_0 \cos \omega \int_{-\infty}^{+\infty} y_{\text{hom}2}^{+}(t) \cos \omega dt.
\]

(13)

It can be seen that \(M(\tau, f_0, \xi, \omega) = 0\) has simple zero for \(\tau\) if and only if the following inequality holds:

\[
\frac{f_0}{\xi} = \frac{\int_{-\infty}^{+\infty} (y_{\text{hom}2}^{+}(t))^2 dt}{\int_{-\infty}^{+\infty} y_{\text{hom}2}^{+}(t) \cos \omega dt} = R_{\text{hom}2}^{IV}(\omega),
\]

(14)

where

\[
\int_{-\infty}^{+\infty} (y_{\text{hom}2}^{+}(t))^2 dt = 4\sqrt{3} \alpha + 4\alpha \ln \left(\sqrt{\frac{1}{\alpha}} + 1 + \sqrt{1 + \frac{1}{\alpha}}\right);
\]

\[
\int_{-\infty}^{+\infty} y_{\text{hom}2}^{+}(t) \cos \omega dt \int_{-\infty}^{+\infty} \cos \omega dt
\]

\[
= \pm 2\pi \cos \left(\frac{\omega}{\sqrt{\alpha}} \sin^{-1} \left\{\sqrt{\frac{1}{\alpha}}\right\}\right) \times \text{sech} \left(\frac{\pi \omega}{2\sqrt{3} \alpha}\right);
\]

which follows the analytically chaotic criteria for the homoclinic orbit of the second-type if \(f_0/\xi > \frac{\text{R}_{\text{hom}2}^{IV}(\omega)}{\text{R}_{\text{hom}2}^{IV}(\omega)}\). This criterion implies that the roots of \(M(\tau, f_0, \xi, \omega) = 0\) are simple, which enable the transverse intersections between the stable and the unstable manifolds of center-saddles and system (11) might be chaotic.

When \((h, \omega) = \left(\frac{d\theta}{d\tau}, \frac{\omega d\theta}{d\tau}\right)\) located in region IV, the phase portraits of the original and approximate system exhibit similar dynamics with the homoclinic orbit of the second-type connecting \((-\pi, 0)\) and \((\pi, 0)\), as shown in Fig. 5(g). The approximate system (11) can be written as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \sin x \left(1 + \frac{\cos x}{2}\right) - \xi x + f_0 \cos \omega t.
\end{align*}
\]

(15)

The analytical solution of the homoclinic orbit of the second-type connecting \((-\pi, 0)\) and \((\pi, 0)\) is given by

\[
(x_{\text{hom}2}^{\pm}(t), y_{\text{hom}2}^{\pm}(t)) = \left\{ \pm \tan^{-1} \left(\sqrt{\frac{1}{6}} \sinh \left(\sqrt{\frac{5}{6}}\right)\right), \right. \\
2 \cosh \left(\sqrt{\frac{5}{6}}\right) \left(1 + \frac{2}{3} \sinh^2 \left(\sqrt{\frac{5}{6}}\right)\right) \left\}
\]

where

\[
\text{R}_{\text{hom}2}^{IV}(\omega) = \frac{2\sqrt{6} + 2\sqrt{2} \ln(2 + \sqrt{3})}{2\pi \cos \left(\frac{\alpha \omega}{\sqrt{6}} \sin^{-1} \left(\frac{1}{3}\right) \text{sech} \left(\frac{\pi \omega}{3}\right)\right)}.
\]

(16)

The Melnikovian detected chaotic boundary for the approximate system is plotted by letting \(f_0/\xi = \text{R}_{\text{hom}2}^{IV}(\omega)\) in Fig. 6(a). Numerical simulations are carried out to verify the efficiency of the criteria in the following. Figure 6(b) shows the bifurcation diagram for \(f_0\) versus \(x\) for parameter
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\[ \omega_0^2 = \frac{41 + \sqrt{93}}{2 \pi^2}, \quad h = \frac{1 + \sqrt{93}}{\pi^2}, \quad \xi = 0.04, \quad \omega = 0.5 \]

When \((\alpha, \beta) \in \Pi \cup \Xi\), the original system (2) and the approximate system (6) exhibit similar behaviors with a pair of homoclinic orbits of the first and second-type, as shown in Figs. 5(c) and 5(e). The Melnikov function of system (11) for the homoclinic orbits of the first-type, can be obtained as

\[
M_1(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{+\infty} (y_{\text{hom}}^{(1)}(t))^2 dt \\
- f_0 \sin \omega \tau \int_{-\infty}^{+\infty} y_{\text{hom}}^{(1)}(t) \sin \omega t dt.
\]  

4.2. Melnikov analysis for a pair of homoclinic orbits

As can be seen in Figs. 6(c) and 6(d), a good degree of correlation is demonstrated in both attractor structures and the Lyapunov characteristics, the details can be seen in the corresponding captions.
It can be seen that \( M(\tau, f_0, \xi, \omega) = 0 \) has simple zero for \( \tau \) if and only if the following inequality holds:

\[
\frac{f_0}{\xi} > \frac{\int_{-\infty}^{+\infty} (y_{\pm}^{\text{hom}1}(t))^2 dt}{\int_{-\infty}^{+\infty} y_{\pm}^{\text{hom}1}(t) \sin \omega t dt}
= R_{\text{hom1}}(\omega),
\]

where

\[
\int_{-\infty}^{+\infty} (y_{\pm}^{\text{hom}1}(t))^2 dt = 4N_{\pm} \frac{\sinh 2\beta}{2\sqrt{\beta + \alpha}} \ln \left( \sqrt{\frac{\beta + \alpha}{\alpha}} + \sqrt{1 + \frac{\beta}{\alpha}} \right);
\]

\[
\int_{-\infty}^{+\infty} y_{\pm}^{\text{hom}1}(t) \sin \omega t dt = 2\pi \sin \left( \frac{\omega}{2\sqrt{\beta + \alpha}} \sinh^{-1} \sqrt{1 + \frac{\beta}{\alpha}} \right) \times \text{sech} \left( \frac{\pi \omega}{2\sqrt{\beta + \alpha}} \right).
\]

The Melnikov function of system (11) for the second-type of homoclinic orbits, can be obtained as

\[
M_2(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{+\infty} (y_{\pm}^{\text{hom}2}(t))^2 dt + f_0 \cos \omega \tau \int_{-\infty}^{+\infty} y_{\pm}^{\text{hom}2}(t) \cos \omega t dt.
\]

Similarly, \( M(\tau, f_0, \xi, \omega) = 0 \) has simple zero for \( \tau \) if and only if the following inequality holds:

\[
\frac{f_0}{\xi} > \frac{\int_{-\infty}^{+\infty} (y_{\pm}^{\text{hom}2}(t))^2 dt}{\int_{-\infty}^{+\infty} y_{\pm}^{\text{hom}2}(t) \cos \omega t dt}
= R_{\text{hom2}}(\omega),
\]

where

\[
\int_{-\infty}^{+\infty} (y_{\pm}^{\text{hom}2}(t))^2 dt = 4N_{\pm} \frac{\sinh 2\beta}{2\sqrt{\beta + \alpha}} \ln \left( \sqrt{\frac{\beta + \alpha}{\alpha}} + \sqrt{1 + \frac{\beta}{\alpha}} \right);
\]

\[
\int_{-\infty}^{+\infty} y_{\pm}^{\text{hom}2}(t) \cos \omega t dt = \pm 2\pi \cos \left( \frac{\omega}{\sqrt{\beta + \alpha}} \sinh^{-1} \sqrt{1 + \frac{\beta}{\alpha}} \right) \times \text{sech} \left( \frac{\pi \omega}{2\sqrt{\beta + \alpha}} \right).
\]

When \( (b, \omega) = (\frac{1}{4}, \frac{\pi}{2}) \) or \( (\alpha, \beta) = (-\frac{1}{4}, 1) \) is located in region III, the phase portrait of two systems exhibit similar dynamics with a pair of homoclinic orbits of the first-type connecting \((0, 0)\) and the second-type connecting \((-\pi, 0)\) and \((\pi, 0)\), as shown in Fig. 5(e). The approximate system (11) can be written as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \sin x \left( -\frac{1}{2} + \cos x \right) - \xi y + f_0 \cos \omega t.
\end{align*}
\]

(21)

We focus our attention on the homoclinic orbit of the first-type, the corresponding Melnikov integral can be obtained as system (17). The solution of the homoclinic orbits of the first-type of system (21) connecting \((0, 0)\) is given by

\[
(y_{\pm}^{\text{hom}1}(t), y_{\pm}^{\text{hom}1}(t)) = \left\{ \begin{array}{l}
\pm 2 \cot^{-1} \left( \coth \left( \frac{\sqrt{2}}{2} \right) \right), \\
\mp \frac{\sqrt{2} \sinh \left( \frac{\sqrt{2}}{2} \right)}{1 + \cosh^2 \left( \frac{\sqrt{2}}{2} \right)} \end{array} \right.
\]
The Melnikov function of system (21) for the homoclinic orbit of the second-type, as described in (18), can be obtained as

\[ R_{\text{hom2}}(\omega) = \frac{2\sqrt{2} - \ln(3 + 2\sqrt{2})}{2\pi \sin(\sqrt{2}\omega) \sinh^{-1}(1) \text{sech} \left( \frac{\pi \omega}{\sqrt{2}} \right)} \]

where

\[ \int_{-\infty}^{+\infty} (y_{\text{hom2}}(t))^2 dt \]

\[ = 2\sqrt{2} - \ln(3 + 2\sqrt{2}); \]

\[ \int_{-\infty}^{+\infty} y_{\text{hom2}}(t) \sin \omega t dt \]

\[ = \pm 2\pi \sin(\sqrt{2}\omega) \sinh^{-1}(1) \text{sech} \left( \frac{\pi \omega}{\sqrt{2}} \right) \]

Next we focus our attention on the homoclinic orbit of the second-type, the corresponding Melnikov integral can be obtained as system (19). The solution of the homoclinic orbit of the second-type of system (21) connecting \((-\pi, 0)\) and \((\pi, 0)\) is given by

\[ (x_{\text{hom2}}(t), y_{\text{hom2}}(t)) \]

\[ = \left( \pm 2\tan^{-1} \left( \frac{\sqrt{2}}{\pi} \sinh \left( \frac{\sqrt{2}}{\pi} \right) \right), \right.

\[ \sqrt{2} \cosh \left( \frac{\sqrt{2}}{\pi} \right) \]

\[ \pm \frac{1}{1 + \frac{3}{\pi^2} \sinh^2 \left( \frac{\sqrt{2}}{\pi} \right)} \]

The Melnikov function of system (21) for the homoclinic orbit of the second-type, as described in (18), can be obtained as

\[ R_{\text{hom2}}(\omega) = \frac{2\sqrt{2} + \ln(5 + 6\sqrt{6})}{2\pi \cos \left( \frac{2\omega}{\sqrt{6}} \sinh^{-1}(\sqrt{2}) \text{sech} \left( \frac{\omega}{\sqrt{6}} \right) \right)} \]

where

\[ \int_{-\infty}^{+\infty} (y_{\text{hom2}}(t))^2 dt = 2\sqrt{6} + \ln(5 + 2\sqrt{6}); \]

\[ \int_{-\infty}^{+\infty} y_{\text{hom2}}(t) \cos \omega t dt \]

\[ = \pm 2\pi \cos \left( \frac{2\omega}{\sqrt{6}} \sinh^{-1}(\sqrt{2}) \text{sech} \left( \frac{\omega}{\sqrt{6}} \right) \right) \]

The Melnikovian detected chaotic boundaries for the approximate system (21) are plotted by letting \(f_0/\xi = R_{\text{hom1}}(\omega)\) and \(f_0/\xi = R_{\text{hom2}}(\omega)\) in Fig. 7(a). The curve indicates the chaotic boundary of the homoclinic orbit of the first-type marked solid and the homoclinic orbit of the second-type marked thin, which divide the parameters space \((\omega, f_0/\xi)\) into four regions marked (1)–(4). The bifurcation diagram of the original system (10) for \(f_0\) versus \(x\) is plotted with the threshold values \(f_0 = 0.0588\) (dashed line) and \(f_0 = 0.2623\) (dotted line) in Fig. 7(b) for \(\omega_0^2 = 22\sqrt{3}/\pi\), \(h = 1 + \sqrt{2}/6\), \(\xi = 0.07, \omega = 1\). As can be seen in Fig. 7(b), there are the periodic windows and the chaotic areas, in which a periodic-1 in Fig. 7(c) for \(f_0 = 0.15\) and chaotic attractors are plotted in Figs. 7(d)–7(f) for \(f_0 = 0.38, 0.67, 2\) showing the transition from an SD-type to the mixture and then a pendulum-type with their Lyapunov characteristics \((\lambda_1, \lambda_2) = (0.130042, -0.200178), D_L = 1.65022, (\lambda_1, \lambda_2) = (0.115655, -0.185478), D_L = 1.62865\) and \((\lambda_1, \lambda_2) = (0.168609, -0.239683), D_L = 1.70683\), respectively.

Figure 8 shows the comparison between the original system (10) and its approximation (21) with the bifurcation diagrams, the attractors and periodic-1 solutions. The bifurcation diagrams of the original system and the approximate system for \(f_0\) versus \(x\) are plotted in Figs. 8(a) and 8(b) for \(\omega_0^2 = 22\sqrt{3}/\pi\), \(h = 1 + \sqrt{2}/6\), \(\xi = 0.07, \omega = 1\), with the threshold values \(f_0 = 0.0588\) (blue dashed line) and \(f_0 = 0.2623\) (red dashed line). SD-type attractors can be predicted in both the original and the approximate system at \(f_0 = 0.3\), shown in Figs. 8(c) and 8(d), with their Lyapunov characteristics \((\lambda_1, \lambda_2) = (0.13442, -0.2044), D_L = 1.6575\) and \((\lambda_1, \lambda_2) = (0.12048, -0.1904), D_L = 1.6325\). Periodic-1 solutions of the original and the approximate system are plotted in Figs. 8(c) and 8(f) for \(f_0 = 1.5\). While \(f_0 = 2.2\), pendulum-type attractors are obtained for the original system and the approximate system in Figs. 8(g) and 8(h) with their Lyapunov characteristics \((\lambda_1, \lambda_2) = (0.168958, -0.238959), D_L = 1.70706\) and \((\lambda_1, \lambda_2) = (0.185716, -0.255742), D_L = 1.70683\) respectively.
Fig. 7. (a) Chaotic boundaries for system (21) detected by the Melnikov method; (b) bifurcation diagram for \( f_0 \) versus \( x \) with the threshold values \( f_0 = 0.0588 \) (dashed line) and \( f_0 = 0.2623 \) (dotted line) at \( \xi = 0.07, \omega = 1 \); (c) periodic-1 solution for \( f_0 = 0.15 \); (d)-(f) the transition of the attractors from SD-type to the mixture and then the pendulum-type for \( f_0 = 0.38, 0.67, 2 \), respectively.
Fig. 8. The comparison between the original system (10) and its approximation (21) with (a) and (b) the bifurcation diagrams, (c) and (d) the SD-type attractors, (e) and (f) the periodic-1 solutions and (g) and (h) the pendulum-type attractors, the details can be seen in the context.
D_L = 1.72618, respectively. As can be seen in Fig. 8, a good degree of correlation is demonstrated in both attractor structures and the Lyapunov characteristics.

4.3. Melnikov analysis for double heteroclinic orbits
($\alpha = 0, \beta > 0$)

The Melnikov function of system (11) for the double heteroclinic orbits, can be obtained as

\[
M(\tau, f_0, \xi, \omega) = -\xi \int_{-\infty}^{\infty} (y_{het}^+(t))^2 dt + f_0 \cos \omega \int_{-\infty}^{\infty} y_{het}^+(t) \cos \omega dt. \tag{22}
\]

It can be seen that \(M(\tau, f_0, \xi, \omega) = 0\) has simple zero for \(\tau\) if and only if the following inequality holds:

\[
f_0 > \frac{\int_{-\infty}^{\infty} (y_{het}^+(t))^2 dt}{\int_{-\infty}^{\infty} y_{het}^+(t) \cos \omega dt}. \tag{23}
\]

where

\[
\int_{-\infty}^{\infty} (y_{het}^+(t))^2 dt = \int_{-\infty}^{\infty} (\pm \sqrt{\beta} \text{sech} \sqrt{\beta} t)^2 dt = 2\sqrt{\beta};
\]

\[
\int_{-\infty}^{\infty} y_{het}^+(t) \cos \omega dt = \int_{-\infty}^{\infty} \pm \sqrt{\beta} \text{sech} \sqrt{\beta} t \cos \omega dt = \pm \pi \text{sech} \frac{\pi \omega}{2\sqrt{\beta}}.
\]

We take \((\alpha, \beta) = (0, \frac{12}{35})\) or \((h, \omega) = (2, 6)\) as an example that is located on bifurcation curves \(H\) and \(H'\). The phase portraits of the approximate and original systems exhibit similar dynamics with double heteroclinic orbits connecting \((0, 0)\) and \((\pm \pi, 0)\), as shown in Fig. 5(d). The approximate system becomes

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \frac{12}{35} \sin x - \xi \dot{x} + f_0 \cos \omega t.
\end{align*} \tag{24}
\]

The solution of the double heteroclinic orbits of system (24) connecting \((0, 0)\) and \((\pi, 0)\) is given by

\[
(x_{het}^+(t), y_{het}^+(t)) = \left( \frac{\pi}{2} \pm \tan^{-1} \left( \sinh \left( \frac{12}{35} t \right) \right), \frac{12}{35} \text{sech} \left( \sqrt{\frac{12}{35}} t \right) \right).
\]

The Melnikov function of system (24) for the double heteroclinic orbits, as described in (23), can be obtained as

\[
R_{het}^M(\omega) = \left| \frac{\sqrt{35}}{2\sqrt{35}} \pm \pi \text{sech} \left( \sqrt{\frac{35}{48}} \pi \omega \right) \right|. \tag{25}
\]
where
\[
\int_{-\infty}^{+\infty} (y_x^h(t))^2 dt = \int_{-\infty}^{+\infty} \left( \pm \sqrt{\frac{12}{35}} \tanh \left( \sqrt{\frac{12}{35}} t \right) \right)^2 dt = \sqrt{\frac{48}{35}}
\]
\[
\int_{-\infty}^{+\infty} y_x^h(t) \cos \omega t dt = \int_{-\infty}^{+\infty} \left( \pm \sqrt{\frac{12}{35}} \tanh \left( \sqrt{\frac{12}{35}} t \right) \cos \omega t \right) dt = \pm \tanh \left( \sqrt{\frac{35}{12}} \omega \right).
\]

The Melnikovian detected chaotic boundary for the approximate system (24) is plotted by letting \( f_0/\xi = H_{het}(\omega) \) in Fig. 9(a). Numerical simulations are carried out to verify the efficiency of the criteria in the following. Bifurcation diagram of the original system for \( f_0 \) versus \( x \) is plotted for parameter \( \omega_0^2 = 2, h = 6, \xi = 0.02, \omega = 0.5 \) taken fixed with the threshold \( f_0 = 0.01988, \) marked dash line, in Fig. 9(b). When \( f_0 \) increases beyond the above threshold value reaching \( f_0 = 0.1, \) the original system jumps to the chaotic motion. Chaotic attractors are displayed for the original system [Fig. 9(c)] and the approximate system [Fig. 9(d)] for the same parameters \( \omega_0^2 = 2, h = 6, f_0 = 0.43, \xi = 0.02, \omega = 0.5. \) As can be seen in Figs. 9(c) and 9(d), a good degree of correlation is demonstrated in both attractor structures and the Lyapunov characteristics \( (\lambda_1, \lambda_2) = \left( 0.121777, -0.141175 \right), \) \( D_L = 1.85833 \) and \( (\lambda_1, \lambda_2) = \left( 0.122814, -0.142832 \right), \) \( D_L = 1.85984, \) respectively.

Fig. 9. (a) Chaotic boundary of system (24) detected by Melnikov method; (b) bifurcation diagram for \( f_0 \) versus \( x \) with the threshold \( f_0 = 0.01988, \) marked dash line; at \( \xi = 0.02, \omega = 0.5, (c) \) and (d) the attractors for the original system (10) and approximate system (24) when \( \xi = 0.02, \omega = 0.5, f_0 = 0.43 \) with their Lyapunov characteristics \( (\lambda_1, \lambda_2) = \left( 0.121777, -0.141175 \right), \) \( D_L = 1.85833 \) and \( (\lambda_1, \lambda_2) = \left( 0.122814, -0.142832 \right), \) \( D_L = 1.85984, \) respectively.
characteristics, the details can be seen in the corresponding captions.

5. Conclusions
In this paper, we have investigated the nonlinear dynamics of the recently proposed rotating pendulum coupled with SD oscillator. The chaotic boundary of this system has been obtained by constructing an approximate system successfully avoiding the barrier of the associated irrational nonlinearity which completely reflects the natural dynamics of the original system. The efficiency of the proposed method has been presented by using numerical simulations, which clearly demonstrates the predicted chaotic attractors [Yagasaki & Uozumi, 1997; Szemplinska-Stupnicka et al., 2000; Szemplinska-Stupnicka & Tyrkiel, 2002] of pendulum-type, SD-type and their mixture depending on the coupling of the nonlinearities. The results obtained herein provide a typical example to reveal the cylindrical dynamics phenomena. The future study on the complicated nonlinear dynamics of this cylindrical pendulum is being carried out by the current authors in two aspects: the first is the chaotic behaviors of discontinuous regime (see [Wiggins, 1988, 1990; Chen & Leung, 1998; Formalskii, 2006] “An inverted pendulum on a fixed and a moving base,” J. Appl. Math. Mech. 70, 56–64). The second is the Hopf bifurcations [Hopf, 1942; Tian et al., 2010; Guckenheimer & Holmes, 1983].

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Appendix A

The Expressions of Homoclinic Orbits

In this Appendix, we will focus our attention on the analytical solutions of system (8) for the singular closed orbits including the homoclinic and heteroclinic orbits by a series of tedious integrations, as described in the following cases.
The calculation procedure for the homoclinic orbits of the second-type connecting \((-\pi, 0)\) and \((\pi, 0)\), as shown in Figs. 5(e)–5(g), can be displayed in the following. The theoretical expression for Hamiltonian (8), connecting \((-\pi, 0)\) and \((\pi, 0)\), can be written as

\[
\frac{y^2}{2} + \alpha \cos x - \frac{\beta}{2} \sin^2 x + \alpha = 0.
\]

It turns out that

\[
dt = \pm \frac{dx}{\sqrt{-2\alpha \cos x - 2\alpha + \beta \sin^2 x}}
\]

Integrating both sides, it follows that

\[
t = \pm \int_0^t \frac{1}{\sqrt{-2\alpha \cos x - 2\alpha + \beta \sin^2 x}} \, dx = \pm \int_0^t \frac{1}{\sqrt{-4\alpha \cos^2 \frac{x}{2} + 4\beta \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}} \, dx
\]

\[
= \pm \int_0^t \frac{1}{\sqrt{4\alpha \cos^2 \frac{x}{2} \left( \beta \sin^2 \frac{x}{2} - \alpha \right)}} \, dx = \pm \int_0^t \frac{1}{\sqrt{4(\beta - \alpha) \tan^2 \frac{x}{2} - 4\alpha}} \, dx
\]

\[
= \pm 2 \int_0^\epsilon \frac{1}{\sqrt{4(\beta - \alpha) \tan^2 \frac{x}{2} - 4\alpha}} \, \tan \frac{x}{2} \, d(\alpha < 0)
\]

\[
= \pm \frac{2}{\sqrt{\beta - \alpha}} \int_0^t \frac{1}{\sqrt{\beta - \alpha} \tan^2 \frac{x}{2} + 1} \, tan \frac{x}{2} = \pm \frac{1}{\sqrt{\beta - \alpha}} \int_0^t \frac{1}{\sqrt{\beta - \alpha} \tan^2 \frac{x}{2} + 1} \, d(\beta - \alpha)\tan \frac{x}{2}
\]

It turns out that

\[
t = \pm \frac{1}{\sqrt{\beta - \alpha}} \sinh^{-1} \left( \sqrt{\beta - \alpha} \tan \frac{x}{2} \right)
\]

The above function for \(t \) of \(x \) can be inverted in the following

\[
x_{+}^{\text{hom2}}(t) = \pm 2 \tan^{-1} \left( \sqrt{\beta - \alpha} \sinh \sqrt{\beta - \alpha} \right)
\]

Letting \(y(t) = \frac{\partial x(t)}{\partial t} \), we can obtain the \(y(t) \) in the follow form.

\[
y_{+}^{\text{hom2}}(t) = \pm \frac{2\sqrt{-\alpha} \cosh \left( \sqrt{\beta - \alpha} \right)}{1 + \frac{\alpha}{\beta - \alpha} \sinh^2 \left( \sqrt{\beta - \alpha} \right)}
\]
The homoclinic orbits of the second-type connecting \((-\pi, 0)\) and \((\pi, 0)\) can be written as

\[
(x_{\text{hom}2}(t), y_{\text{hom}2}(t)) = \left( \pm 2 \tan^{-1} \left( \sqrt{\frac{\alpha}{\beta - \alpha}} \sinh \sqrt{\beta - \alpha t} \right), \pm \frac{2 \sqrt{\alpha} \cosh(\sqrt{\beta - \alpha t})}{1 + \frac{\alpha}{\beta - \alpha} \sinh^2(\sqrt{\beta - \alpha t})} \right),
\]

(A.1)

where the base points are \((x_{\text{hom}2}(0), y_{\text{hom}2}(0)) = (0, \pm 2\sqrt{\alpha})\).

(ii) With the help of Hamiltonian (8), the theoretical expression for the homoclinic orbit of the first-type, connecting \((0, 0)\) and crossing \((\mp \cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}}, 0)\), as shown in Fig. 5(e), can be written as

\[
\frac{\beta}{2} + \alpha \cos x - \frac{\beta}{2} \sin^2 x - \alpha = 0.
\]

It turns out that

\[
dt = \pm \frac{dx}{\sqrt{2\alpha - 2\alpha \cos x + \beta \sin^2 x}}
\]

Integrating both sides, it follows that

\[
t = \pm \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{1}{\sqrt{4\alpha \sin^2 \frac{x}{2} + 4\beta \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}} \, dx
\]

\[
= \pm \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{1}{\sqrt{4\alpha \sin^2 \frac{x}{2} \left(\alpha + \beta \sin^2 \frac{x}{2}\right)}} \, dx
\]

\[
= \pm \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{1}{\sqrt{4\alpha \sin^2 \frac{x}{2} \left(\alpha + \beta \cos^2 \frac{x}{2} + \alpha \sin^2 \frac{x}{2}\right)}} \, dx
\]

\[
= \pm \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{\csc^2 \frac{x}{2}}{\sqrt{4(\alpha + \beta) \cot^2 \frac{x}{2} + 4\alpha}} \, dx
\]

\[
= \mp 2 \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{1}{\sqrt{4(\alpha + \beta) \cot^2 \frac{x}{2} + 4\alpha}} \, d\cot \frac{x}{2} \quad (\alpha < 0)
\]

\[
= \mp 2 \int_{\cot^{-1}(\sqrt{\frac{\alpha}{\beta - \alpha}})}^{\pm \infty} \frac{1}{\sqrt{\left(\frac{\alpha + \beta}{\alpha - \beta} \cot \frac{x}{2}\right)^2 - 1}} \, d\cot \frac{x}{2}
\]
In the same way, the solution of the homoclinic orbit of the second-type connecting \((0, 0)\) can be written as
\[
\begin{align*}
(x(t, \text{hom}1), y(t, \text{hom}1)) &= \pm 2\cot^{-1} \left( \sqrt{\frac{\alpha + \beta}{\alpha \beta}} \cosh \frac{\alpha + \beta t}{\sqrt{\alpha \beta}} \right),
\end{align*}
\]
where the base points are \((x(\text{hom}1), y(\text{hom}1)) = (\pm 2\cot^{-1} \left( \sqrt{\frac{\alpha + \beta}{\alpha \beta}} \right), 0)\).

(iii) In the same way, the solution of the homoclinic orbit of the first-type connecting \((0, 0)\) shown in Figs. 5(a) - 5(c), can be written as
\[
\begin{align*}
(x(t, \text{hom}1), y(t, \text{hom}1)) &= \pm 2\cot^{-1} \left( \sqrt{\frac{\alpha + \beta}{\alpha \beta}} \cosh \frac{\alpha + \beta t}{\sqrt{\alpha \beta}} \right),
\end{align*}
\]
where the base points are \((x(\text{hom}1), y(\text{hom}1)) = (\pm 2\cot^{-1} \left( \sqrt{\frac{\alpha + \beta}{\alpha \beta}} \right), 0)\).

(iv) In the same way, the solution of the homoclinic orbit of the first-type connected with \((-\pi, 0)\) and \((\pi, 0)\) seen in Fig. 5(c), can be written as
\[
\begin{align*}
(x(t, \text{hom}1), y(t, \text{hom}1)) &= \pm 2\tan^{-1} \left( \sqrt{\frac{\beta - \alpha}{\beta + \alpha}} \sinh \frac{\beta - \alpha t}{\sqrt{\beta + \alpha}} \right),
\end{align*}
\]
where the base points are \((x(\text{hom}1), y(\text{hom}1)) = (\pm 2\tan^{-1} \left( \sqrt{\frac{\beta - \alpha}{\beta + \alpha}} \right), 0)\).