In this work the strange behavior of an impact oscillator with a one-sided elastic constraint discovered experimentally is compared with the predictions obtained using its mathematical model. Extensive experimental investigations undertaken on the rig developed at the Aberdeen University reveal different bifurcation scenarios under varying excitation frequency near grazing which were recorded for a number of values of the excitation amplitude. In the paper, particular attention is paid to the chaotic oscillations recorded near grazing frequency when a nonimpacting orbit becomes an impacting one under increasing excitation frequency. It was found that the evolution of the attractor is governed by a complex interplay between smooth and nonsmooth bifurcations, and the interactions between a number of coexisting orbits. The occurrence of coexisting attractors is manifested in the experimental results through discontinuous transitions from one orbit to another via boundary crisis. In some cases, the basins of attraction have a fractal structure. Detailed numerical exploration also revealed coexisting unstable periodic orbits. These stable and unstable coexisting orbits are often born far from the parameter values at which they influence the system dynamics. The very rich dynamics of the bilinear oscillator close to grazing is demonstrated and typical mechanisms of the attractors’ appearance and disappearance are explained using stability analysis.

Keywords: Impact oscillator; grazing bifurcations; chaos; coexisting attractors.

1. Introduction

Intermittent contact in dynamical systems has been well studied, both for so-called soft and hard impacts. The simplest of the former is the bilinear oscillator which nevertheless can show a very complex response. Much of the existing work for nonsmooth systems has focused on classification of possible bifurcations [Banerjee & Grebogi, 1999], the existence and stability of the period-1 orbit after grazing [Ma et al., 2006] and normal forms [di Bernardo et al., 2001], and an introduction to all three can be found in [di Bernardo et al., 2007]. While some older work concerned with ship mooring has looked at the global attractors [Thompson et al., 1983], and their domains of attraction, relatively few recently published papers have done this. Notable examples are [Mason et al., 2009], which considers a backlash gear system and plots the slender basins of attraction which result from low damping in the system, and [de Souza et al., 2008], which demonstrates the disappearance of global attractors as a result of interaction between smooth nonlinearity and impact. Nevertheless, it is well known that even when an attractor
remains stable after a grazing event, its basin of attraction may shrink to arbitrarily small size, [Ganguli & Banerjee, 2005]. Additionally, “invisible” grazing events can result in the birth of an unstable periodic orbit which may later become stable, or may otherwise degrade the structural stability of existing attractors [Banerjee et al., 2009]. This means that a global analysis is required to fully uncover the system behavior.

This paper demonstrates the range and complexity of the bilinear oscillator behavior by looking in detail at one of the experimentally obtained bifurcation diagrams [Ing et al., 2008], explaining the observed bifurcations, and determining the coexisting attractors along with their domains of attraction. Its aim is to show the very rich dynamics of the bilinear oscillator close to grazing, as a large number of orbits come into and go out of existence within a small range of the parameters. The typical mechanisms of the attractors’ appearance and disappearance are also explained. The analysis of the bifurcation scenarios close to grazing is done in terms of the nonlinear bimodal maps that result from solving the linear equations in each subspace. It is shown that a number of unstable periodic orbits are born in so-called invisible grazing events [Banerjee et al., 2009]. The range for which one orbit is stable can be small but is nevertheless important for understanding the system response.

2. Experimental Set-Up and Mathematical Modeling

The experimental investigations were carried out on the impact oscillator [Ing et al., 2008; Banerjee et al., 2009] shown in Fig. 1, which consists of a block of mild steel supported by parallel leaf springs providing the primary stiffness and preventing the mass from rotation. The secondary stiffness provided by an elastic beam is mounted on a separate column. Contact between the mass and the beam is made when their relative displacement is equal to zero. In practice, the contact is through a bolt which is attached to the beam. The length

![Fig. 1. (a) Photographs of experimental set-up. Parallel leaf springs prevent the mass from rotation, ensuring vertical displacement only. Harmonic excitation is provided via the oscillator base by an electrodynamic shaker. Since the oscillator mass is small in comparison with the shaker moving component, it is assumed that the oscillator does not interact with the shaker. (b) Schematic diagram of experimental set-up. Mass displacement, $y_m$, and accelerations of the mass and the base $\ddot{y}_m$, $\ddot{y}_b$ are measured by an eddy current probe and two accelerometers respectively, and then collected by the data acquisition system. Adopted from [Ing et al., 2008].](image)
of the bolt can be adjusted to control the gap, $g$. The oscillator rig was mounted on an electrodynamic shaker which provided harmonic excitation through the base. Displacement of the oscillator was measured using an eddy current probe displacement transducer mounted over one leaf spring. The acceleration of the oscillator was measured using an accelerometer attached directly to the oscillating mass. A Savitzky–Golay algorithm was used to smooth the data, where a second order polynomial fitted to the eight surrounding data points gave the best results. The velocity was obtained by differentiating the smoothed displacement data.

The system can be represented by a simple model of the impact oscillator shown in Fig. 2(a), which has been used in our theoretical studies. A simplifying assumption that the discontinuity surface is not motion or time dependent, and is located at $x-e=0$ has been made, and the equations of motion in the nondimensional form [Shaw & Holmes, 1983; Wiercigroch & Sin, 1998] are given by

$$
\begin{align*}
\dot{x}' &= v' \\
v' &= a\omega_n^2 \sin(\omega \tau) - 2\xi v - x - \beta(x-e)H(x-e)
\end{align*}
$$

(1)

where $x = y/y_0$ is the nondimensionalized vertical displacement of the mass, $v = x'$ is the nondimensionalized velocity, $\tau = \omega_n t$ is the nondimensional time, $\omega_n = \sqrt{k_1/m}$, $\beta = k_2/k_1$ is the stiffness ratio, $\epsilon = g/y_0$ is the nondimensional gap, $a = A/y_0$ and $\omega = \Omega/\omega_n$ are the nondimensional forcing amplitude and frequency, $\xi = c/(2m\omega_n)$ is the damping ratio, $y_0 = 1$ mm, $'$ denotes differentiation with respect to $\tau$, and $H(\cdot)$ is the Heaviside function. Two resulting linear equations (different for contact and no contact modes) can be solved subject to initial conditions, and then maps $P_1$ and $P_2$ of these initial conditions between the two subspaces $X_1$ and $X_2$ produce the global response, see Fig. 2(b). A periodic solution is found either by iterating the map, or, if a particular solution is being sought, via the Newton method (once the Jacobian has been determined). The stability was analyzed by finding the eigenvalues of the global Jacobian, constructed by composition from the local Jacobians in each subspace. Further details, explicit formulae of the solutions in each subspace and Jacobian components can be found in [Ing et al., 2008].

### 3. Numerical and Experimental Results

During the experimental study [Ing et al., 2008], a number of different bifurcation scenarios near grazing were recorded. The most typical one was when a nonimpacting periodic orbit bifurcates into an impacting one via a grazing mechanism. In some cases the resulting orbit is stable, but in most cases it loses stability through grazing. In those cases, the appearance of a narrow band of chaotic behavior was observed [Banerjee et al., 2009], and the example of such bifurcations is shown in Fig. 3(a).
Fig. 3. Bifurcation diagrams obtained (a) experimentally for excitation amplitude equal to 0.44 mm [ing et al., 2008] and (b) numerically for $a = 0.7$. Additional windows demonstrate the trajectories on the phase plane obtained for (a) $\omega = 0.81$ (Poincaré map), 0.842, 0.8849 (Poincaré map), 0.9062 and 0.9542 respectively. (b) $\omega = 0.8023$ (Poincaré map), 0.84, 0.91 (Poincaré map), 0.915 and 0.94 respectively.
To investigate the recorded atypical bifurcation scenarios when close to grazing the nonimpacting period-1 orbit bifurcates into a chaotic regime, extended numerical simulations were conducted using the bilinear oscillator described by Eq. (1). As can be seen from Fig. 3, a very good correspondence between the theoretical predictions and the experimental results was obtained. This figure presents the bifurcation diagrams obtained experimentally [Fig. 3(a)] and calculated numerically [Fig. 3(b)] for one of the values of the excitation amplitude, with the excitation frequency as the bifurcation parameter. Additional windows demonstrate the trajectories and Poincaré maps on the phase plane. As can be seen, an interesting bifurcation structure was recorded. Figure 3(a) demonstrates that close to the parameter value corresponding to grazing of the period-1 orbit, a window of chaotic behavior appears. It persists for a very small range of the parameter, and changes to a period-2 response as the frequency increases. The representative trajectories of these chaotic and period-2 responses on the phase plane are presented for $\omega = 0.81$ and $\omega = 0.8422$, respectively. As the parameter is further increased, a window of chaotic behavior is obtained and a typical Poincaré map of the chaotic attractor is given for $\omega = 0.8849$. This is followed by a reverse period doubling bifurcation, resulting in a period-1 response with one impact per period. A typical phase space trajectory for this condition is shown for $\omega = 0.9542$. Numerical simulation also showed a similar bifurcation scenario, as demonstrated in Fig. 3(b).

The evolution of the main attractor — the nonimpacting period-1 orbit — was followed as it goes through the grazing condition at $\omega_{gr} = 0.801928$ (which is calculated using the exact solution of the equations for the nonimpacting oscillator), and turns into an impacting orbit. Our detailed numerical simulations reveal that a number of coexisting attractors are present at the frequencies around $\omega = 0.802$. Before grazing two orbits coexist, one being a small nonimpacting period-1 regime and the other one being a large-amplitude impacting period-5 regime. Figure 4 demonstrates the basins of attraction together with the corresponding trajectories of these coexisting regimes found for the excitation amplitude $a = 0.7$ and frequency $\omega = 0.801$ just before the grazing. Here the basins of attraction for the period-1 and period-5 regimes are shown in yellow and orange, respectively.

When, just after the grazing, the stability of the small-amplitude orbit is lost, one would normally expect the orbit to move to the coexisting period-5 attractor. But it was found that, for most initial conditions, it converges on a chaotic attractor. Our numerical investigation revealed that this attractor is situated on the unstable manifold of

![Fig. 4. (a) Basin of attraction calculated for $a = 0.7, \omega = 0.801, g = 1.26, \xi = 0.01, \beta = 29$. (b) Trajectories and Poincaré maps for the coexisting period-1 and period-5 solutions.](image-url)
an unstable period-3 fixed point. For the considered case $a = 0.7$, while diverging away from the period-1 fixed point, the state meets the unstable manifold of a period-3 regime and becomes trapped on this manifold, and thus, the unstable manifold of this period-3 fixed point abruptly becomes a stable attractor. Figure 5 shows the structural similarity between the unstable manifold of the period-3 saddle point (calculated using the Dynamics software [Nusse & Yorke, 1998]), and the chaotic orbit. As has been explained in [Banerjee et al., 2009], this narrow band chaos was possible due to the existence of the unstable period-3 orbits.

It has been shown in [Banerjee et al., 2009] that for the considered parameter set, two period-3 orbits are born at $\omega_1 = 0.7618806$ through a grazing-induced bifurcation. Since the map is smooth for a compliant impact, the bifurcation is a smooth saddle-node bifurcation. The evolution of these period-3 orbits is presented in Fig. 6. The node (branch I shown in Fig. 6) loses stability at $\omega = 0.761958$ (which is very close to $\omega_1$) through a period doubling bifurcation. The other unstable period-3 orbit also born at $\omega_1$ (branch II) approaches the period-1 orbit as the frequency increases and approaches the grazing value $\omega_{gr} = 0.801928$. Another unstable period-3 orbit (branch III), born via a saddle node bifurcation at $\omega_2$, approaches the period-1 fixed point as the frequency decreases. The close-up of the bifurcation.
diagram [Fig. 6(b)] shows two smooth saddle-node bifurcations occurring at very close parameter values, \( \omega_3 \) and \( \omega_4 \), connecting unstable branches II and III with a stable period-3 orbit. Close to the grazing frequency of \( \omega_{gr} \), the unstable period-3 orbit that forms the basin boundary comes very close to the period-1 orbit. Thus, while in the nonsmooth (rigid impact) approximation the distance between the fixed point and the unstable period-3 orbit is ideally zero at the bifurcation point [Banerjee et al., 2009], in the actual system there exists a very small but finite distance. If the ambient noise in the system (which is always present in a realistic situation) can perturb the state across the basin boundary, the state diverges away from the fixed point. Thus, even though the system is smooth from the analytical point of view, a condition similar to that in dangerous border collision bifurcation [Ganguli & Banerjee, 2005] is created.

The unstable period-3 orbit (branch I) subsequently becomes stable through a reverse period doubling bifurcation (see the black portion in Fig. 6(a)). This orbit and the unstable period-3 orbit created at \( \omega_3 \) (branch III) merge and vanish at a smooth saddle-node bifurcation at the parameter value \( \omega_2 \). The presented analysis allows one to explain the appearance of the narrow band of chaos in the vicinity of the grazing. However, the dynamics of the system in this frequency range is even more complex as other coexisting regimes were found as the frequency increases. A detailed numerical analysis of the system behavior under varying excitation frequency has been carried out and the results are depicted in Figs. 7, 8, 10–12 and 15. Figure 7(a) shows the same bifurcation diagram as one presented in Fig. 6(b), where frequency ranges with different coexisting regimes are highlighted by different shading and marked by numbers from 1 to 4. Some of these ranges overlap, which means that more than two regimes coexist for some values of frequency.

Figure 7(b) presents the bifurcations of two different regimes coexisting at \( \omega \in (0.791, 0.808) \); the range marked by number 1 in Fig. 7(a). As can be seen from this graph, the nonimpacting period-1 orbit shown in black coexists with the period-5 orbit with three impacts shown in blue for most of this range. In the direction of decreasing frequency the period-5 orbit bifurcates into a period-10 orbit at \( \omega = 0.79424 \) through a smooth period doubling which initiates a period doubling cascade leading to chaos, which exists for a small frequency range. For increasing frequency the period-5 orbit disappears around \( \omega = 0.80562 \) through a saddle-node bifurcation. The nonimpacting period-1 orbit bifurcates into an impacting one through grazing and soon after, at \( \omega = 0.80535 \), it loses stability. At this frequency the system moves to a chaotic attractor as shown in Fig. 7(a). This is accompanied by a period doubling bifurcation but the obtained period-2 orbit is also unstable for \( \omega \in (0.802035, 0.80322) \) and this part of period-2 regime is colored in red in Fig. 7(b).

Figure 7(c) presents the bifurcations in the range marked by the number 2 in Fig. 7(a), and this range partly overlaps the range marked by 1. Here again only two coexisting regimes are shown and they are in orange and green. The period-2 regime with one impact is stable in this frequency range and as the frequency increases the amplitude of the oscillations grows. It coexists with a period-5 orbit with three impacts, which is different from the one discussed earlier and shown in Fig. 7(b) for \( \omega \in (0.80568, 0.8062) \). As the frequency increases the period-5 orbit with two impacts is found at \( \omega = 0.80658 \) and persists up to \( \omega = 0.8296 \) where this regime ceases to exist. For decreasing frequency at \( \omega = 0.80568 \), a smooth period doubling bifurcation is observed resulting in a period-10 orbit, which loses stability at \( \omega = 0.80552 \).

The bifurcations in the range marked by number 3 are shown in Fig. 7(d). Here the period-2 regime with one impact (colored in green) coexists with another period-5 regime with three impacts (colored in black) for \( \omega \in (0.8491, 0.87925) \). For the frequencies \( \omega < 0.8491 \) the period-5 orbit was not found. At \( \omega = 0.87925 \) there is a smooth period doubling bifurcation resulting in a period-10 orbit with six impacts. Soon after, the state moves to a chaotic attractor which undergoes crisis at \( \omega = 0.88042 \). Thereafter only chaotic transients are observed.

Figure 7(e) presents a period-1 impacting orbit and a coexisting period-3 regime with two impacts. This period-3 regime was not found either for \( \omega < 0.9274 \) or for \( \omega > 1.0487 \). Finally Fig. 7(f) demonstrates two of the coexisting attractors for \( \omega \in (0.8038, 0.8048) \). As can be seen from this graph, a period-2 regime with one impact shown in green coexists with a period-8 regime with four impacts shown in magenta at \( \omega \in (0.8041, 0.8046) \). For decreasing frequency, a smooth period doubling bifurcation occurs at \( \omega = 0.8041 \) resulting in a period-16 regime with eight impacts. This is followed by a period doubling cascade leading to chaos...
Fig. 7. Bifurcation diagrams obtained numerically for the mass displacement under varying frequency, $\omega$ at $\xi = 0.01, \beta = 29, e = 1.26$ and excitation amplitude $a = 0.7$. Coexisting attractors (two per plot to avoid confusion) are shown by different colors in the frequency ranges they were found and some of these ranges are highlighted by different shadows. As these ranges overlap, more than two regimes coexist at some values of frequency. The range presented in part (f) is very narrow and cannot be seen in part (a).
Fig. 8. Bifurcation diagram and corresponding coexisting attractors calculated for four values of $\omega \in (0.8, 0.808)$. The number of the coexisting attractors varies from 2 to 4.
and the resulting chaotic attractor ceases to exist at \( \omega \approx 0.80391 \).

To provide an overview of the system behavior, all the coexisting regimes are given in Fig. 8 for the frequencies near grazing. Five different colors are used to mark different attractors. The period-5 regime with three impacts observed for \( \omega \in (0.800, 0.8062) \) and appearing in Fig. 7(b) is shown in blue. The nonimpacting period-1 regime bifurcating into chaos and then into period-3 with two impacts through smooth cascade of period doubling bifurcations is depicted in black. A period-2 regime with one impact appearing in Figs. 7(b)–7(d) and 7(f) is shown in green. A period-8 regime with four impacts and its bifurcations featured in Fig. 7(f) are given in magenta, and finally the period-5 response with three and two impacts shown in Fig. 7(c) is presented in orange.

The trajectories and Poincaré maps located below the bifurcation diagram have the same color code as the bifurcation diagram. Four representative values of the frequency were selected to demonstrate the trajectories of the coexisting responses. As can be seen from this figure, for \( \omega = 0.8023 \) the chaotic attractor (shown as Poincaré map in black) and period-5 orbit (trajectory is in blue and Poincaré map is marked by red points) are presented. For \( \omega = 0.8044 \) the period-2 orbit (in green), period-5 orbit (in blue with red Poincaré map points), period-3 orbit (in black) and period-8 (in magenta with red Poincaré map points) are given. For \( \omega = 0.80455 \) three regimes coexist and they are a period-2 orbit (shown in green), period-5 orbit (given in blue with red Poincaré map points) and period-8 orbit (shown in magenta with red Poincaré map points). Finally for \( \omega = 0.806 \) a period-2 orbit is shown in green, a period-5 orbit is given in blue with red Poincaré map points and another period-5 orbit is shown in orange with with red Poincaré map points.

To investigate how those coexisting attractors are born, stability analysis has been carried out. Figure 9 shows the result for the period-5 attractor appearing in Fig. 7(b) and the period-8 attractor

![Fig. 9. Bifurcation diagrams showing (a) stable (blue) and unstable (light and dark grey) period-5 regimes at \( \omega \in (0.752, 0.807) \) and (b) stable (magenta) and unstable (light and dark grey) period-8 regimes at \( \omega \in (0.763, 0.8046) \). Additional windows demonstrate the phase portraits on the displacement–velocity planes at the saddle node bifurcation points for (a) \( \omega = 0.7521 \) and 0.8066; (b) \( \omega = 0.7638 \) and 0.8042.](image-url)
Fig. 10. Evolution of the basins of attraction computed for $\omega \in (0.8, 0.808)$. Here a period-1 orbit, chaos, period-6 and period-3 orbits shown in green have light yellow basins; the period-5 orbit in black has an orange basin; the period-8 orbit in blue has a light brown basin; the period-2 orbit in yellow has a dark brown basin and the period-5 orbit in red has a pink basin.
appearing in Fig. 7(f). Both the attractors are born via saddle-node bifurcations at the frequencies well below grazing and in both cases the nodes become unstable very close to those frequencies. The births take place through grazing-induced bifurcations. Since the map for compliant impact is smooth, we see smooth saddle-node bifurcations at the grazing points. As can be seen from this figure, the two unstable attractors shown by dark and light grey branches coexist before one of the attractors becomes stable at the higher frequencies ($\omega = 0.8067$ for period-5 regime and $\omega = 0.8043$ for period-8 regime). At $\omega = 0.8058$, one of the unstable period-5 branches is involved in another grazing-induced bifurcation. In case of the period-8 orbits, the similar event occurs at $\omega = 0.803534$. Finally for the period-5 and period-8 attractors another saddle-node bifurcation takes place at $\omega = 0.8066$ and $\omega = 0.80452$ respectively, where stable and unstable branches collide and disappear.

This bifurcation study was complemented by investigating the evolution of the basins of attraction, which is shown in Fig. 10. These basins of attraction confirm that the regimes identified and discussed earlier coexist and show how the phase space is divided between them. They also demonstrate that no dynamical behavior other than the ones reported above exists in this frequency range.

Basins of the period-5 attractor with three impacts featured in Fig. 7(b) are given in orange in Figs. 10(a)–10(i) as it exists for $\omega \in (0.79424, 0.80662)$. As can be seen from this sequence of graphs, the basin of the period-5 motion is gradually shrinking as the frequency increases before disappearing completely at $\omega = 0.80662$, so it is not present in Figs. 10(j)–10(l). This period-5 attractor is marked in black points in all the graphs.

The yellow basins in Figs. 10(a)–10(d) correspond to a nonimpacting period-1 orbit [Fig. 10(a)] and its evolution (as was shown in Fig. 8 and explained above) into chaos [Fig. 10(b)] and then through a reverse period doubling cascade to period-6 [Fig. 10(c)] and period-3 orbits [Fig. 10(d)] which are all marked by green points. As can be seen from these pictures, the yellow basin reduces as the frequency increases and becomes more and more fractal, and for $\omega = 0.8043$ it is hardly seen. Basins of the period-2 attractor with one impact featured in Figs. 7(b)–7(d), 7(f) are given in dark brown in Figs. 10(d)–10(l), whereas the attractor is shown by yellow points. Another attractor appearing at $\omega = 0.8043$ and $\omega = 0.8045$ is period-8 featured in Fig. 7(f). It is shown by light blue points and its basin has a light brown color. Again the basin is very small and highly fractal.
As the frequency increases above $\omega = 0.8046$, both period-3 and period-8 attractors disappear, but a new period-5 orbit with three impacts featured in Fig. 7(c) emerges [see Figs. 10(f)–10(h)]. As explained earlier it disappears at $\omega = 0.8062$ and re-emerges as period-5 regime with two impacts at $\omega = 0.8068$ [see Figs. 10(k)–10(l)]. Both of these period-5 attractors are marked by red points and their basins are pink.

Fig. 12. Bifurcation diagram showing coexisting attractors under varying frequency, $\omega$ for $\omega \in (0.92, 0.95)$ and their representative phase portraits at $\omega = 0.928$ (marked by dash line) obtained theoretically for the mass displacement at $\xi = 0.01, \beta = 29, \epsilon = 1.26$ and $a = 0.7$. 

It should be noted that all these attractors appear and disappear within a very narrow frequency range. As can be seen from Figs. 10(d)–10(h), the basins for period-2, period-3 and period-8 regimes are small and very fractal, and it is obvious that these attractors would not be robust in the presence of even small noise, which makes behavior of the real system for those initial conditions highly unpredictable and likely chaotic.

Our calculations reveal another frequency range, \( \omega \in (0.8491, 0.8042) \), having coexisting attractors, which is shown in Fig. 7(d). Here two attractors were found and Fig. 11 presents their evolution showing the basins for period-2 (dark brown) and period-5 (white) attractors for \( \omega = 0.85, 0.86 \) and 0.87. As can be seen the basin of the period-5 attractor gradually grows as the frequency increases and it becomes very fractal close to the upper band of the considered range. As explained above at \( \omega = 0.87925 \) the period-5 orbit undergoes a period doubling bifurcation and soon after the state reaches a chaotic attractor, which then goes through crisis at \( \omega = 0.88042 \). For the frequencies \( \omega > 0.88042 \) only one attractor was found.

The last frequency range where a few coexisting attractors were found is shown in Fig. 7(e). However the true behavior is more complex as demonstrated in Fig. 12. In addition to period-1 (in green) and period-3 (in black) regimes presented in Fig. 7(e), period-5 (in blue) and another period-3 (in magenta) were discovered through a detailed numerical study. Stability of these regimes was investigated and the results are shown in Figs. 13 and 14. Here two stable branches of period-3 attractors are given in black and magenta (the same colors as in Fig. 12) and unstable branches are in red and in green (see Fig. 13). It was found that at \( \omega = 0.92431 \) two period-3 orbits are born via a normal

![Figure 13: Bifurcation diagram showing stable (black and magenta) and unstable (red and green) period-3 regimes at \( \omega \in (0.924, 1.05) \). Additional windows demonstrate the phase portraits on the displacement — velocity planes at the saddle-node bifurcation points for \( \omega = 0.924304, 0.9269, 0.9366 \), and 1.05.](image-url)
saddle node bifurcation. The unstable branch shown in red exists up to $\omega = 1.05024$ where a grazing-induced saddle-node bifurcation takes place. The grazing induced saddle-node bifurcation at $\omega = 0.92694$ and the normal saddle-node bifurcation at $\omega = 0.93672$ connect the stable black and magenta branches through the unstable green branch. For the period-5 attractor in blue shown in Fig. 12,

![Bifurcation diagram showing stable (blue) and unstable (light grey) period-5 regimes at $\omega \in (0.9243, 1.0503)$. Additional windows demonstrate the phase portraits on the displacement–velocity planes at the saddle-node bifurcation points for $\omega = 0.9248$, and 0.93864.](image)

![Evolution of the basins of attraction for $\omega \in (0.9245, 1.049)$. Here the period-1 regime shown in green has a light yellow basin; the period-5 orbit in blue has light grey basin; the period-3 orbit in black has a light green basin; and the period-3 orbit in pink has a light pink basin.](image)
the picture is simpler: it appears and disappears through saddle-node bifurcations observed at $\omega = 0.92476$ and $\omega = 0.93862$ which connect stable (in blue) and unstable (in light grey) branches (see Fig. 14).

Figure 15 presents the evolution of the basins of attraction for these regimes. Here the basin of the period-1 attractor is given in yellow, the basin of the period-5 attractor is in light grey, the basin of the black period-3 attractor is in green, and finally the basin of the magenta period-3 attractor is in pink. As can be seen from this figure, the basin of the period-5 attractor featured in Figs. 15(b)–15(e) remains very small and fractal for the whole range of the frequency where this attractor exists. The yellow basin of the period-1 attractor is dominant at $\omega < 0.928$ [see Figs. 15(a)–15(c)], but for the higher frequencies the pink basin of the magenta period-3 attractor first becomes larger [see Figs. 15(d)–15(j)] and then smaller [Figs. 15(k)–15(l)]. As the frequency increases, the fractality of the basins increase, and just before the magenta period-3 attractor disappears, its basin is very small [see Fig. 15(l)].
4. Conclusions

The detailed bifurcation analysis of an impact oscillator with one-sided elastic constraint has revealed a highly complex dynamical behavior. It has been shown that as the main period-1 orbit of the system goes through grazing and enters the impacting regime, a complex sequence of events unfolds. As the period-1 orbit loses stability, the system enters a chaotic behavior which occurs for a small range of the parameter. But our investigation revealed that there are a large number of coexisting attractors, and over a parameter range the system could display any of those behaviors depending on the initial condition. For example, two different period-5 regimes and a period-2 regime occur for \( \omega \in (0.8056, 0.8062) \); various period-5 regimes (one at a time) and a period-2 regime coexist at \( \omega \in (0.8062, 0.8066) \), \( \omega \in (0.8067, 0.8296) \) and \( \omega \in (0.8488, 0.8793) \); period-1 and two period-3 regimes coexist at \( \omega \in (0.9269, 0.9366) \); period-1 and period-3 regimes coexist at \( \omega \in (0.9243, 0.9269) \) and \( \omega \in (0.9366, 1.05) \), etc.

Following these coexisting periodic orbits over large parameter ranges, we have found that these are often born at grazing-induced bifurcations, occurring far from the parameter values at which they are observed. The evolution of these orbits is then governed by a complex interplay between the usual smooth bifurcations and grazing-induced bifurcations. These orbits are not stable for long enough to be observed, yet play a very important role in determining the dynamics by degrading the structural stability of coexisting attractors. Rapid cascades of period doubling bifurcations and transitions to chaos result in the responses being unstable until another cascade brings them back into play. There are also discontinuous transition from one orbit to another via boundary crisis. The basins of attraction of the coexisting attractors often have fractal structure, adding to the dynamical complexity in this apparently simple system.

References


