Planar Metric Dimension is NP-complete

Josep Diaz∗ Olli M. Pottonen∗ Erik Jan van Leeuwen†

Abstract
We show that Metric Dimension on planar graphs is NP-complete.

1 Introduction
Given a graph $G = (V, E)$, its metric dimension is the cardinality of a smallest set $W \subseteq V$ such that for every pair $u, v \in V$, there is a $w \in W$ such that the length of a shortest path from $w$ to $u$ is different from the length of a shortest path from $w$ to $v$. In this case we say that $u, v$ are separated by $w$ and $W$. Elements of the set $W$ are called landmarks. The problem of deciding whether, given a graph $G$ and an integer $k > 0$, the metric dimension of $G$ is at most $k$ is called the Metric Dimension problem. This problem was defined independently by Harary and Melter [5] and Slater [9].

The Metric Dimension problem is known to be NP-hard on general graphs [4], by reduction from 3-Dimensional Matching. Khuller et al. [8] gave a linear-time algorithm to compute the metric dimension of a tree. Characterizations of graphs with metric dimension 1 and 2 are given in the same work. The authors also provide a $2 \log n$-approximation algorithm for the metric dimension of any graph. Finally, the authors present an alternative proof of the NP-completeness of Metric Dimension by a reduction from 3-SAT. Caceres et al. [2] study the metric dimension of the cartesian product of graphs, and give bounds for the metric dimension of cartesian products of several specific graphs. Beerliova et al. [1] proved that the metric dimension of a graph cannot be approximated within $o(\ln n)$, unless $P=NP$. Hauptmann et al. [3] further investigate the approximability of the metric dimension. They show that for bounded-degree graphs determining the metric dimension is APX-hard, and that for superdense graphs (graphs for which the complement is a sparse graph) it is APX-complete to approximate the metric dimension. They also provide a greedy constant-factor approximation algorithm for determining the metric dimension of superdense graphs.

In the present work, we show that Metric Dimension on planar graphs, called Planar Metric Dimension, is NP-complete.

∗ Departament de Llenguatges i Sistemes Informatics, Universitat Politecnica de Catalunya, Barcelona, Spain, diaz@lsi.upc.edu, olli.pottonen@iki.fi
† Department of Informatics, University of Bergen, Norway, E.J.van.Leeuwen@ii.uib.no
2 NP-hardness on planar graphs

It is well known that 3-SAT is NP-complete (see [4]). We require a special planar version of 3-SAT to be NP-complete.

**Definition 2.1** Let \( \Psi \) be a boolean formula, which uses the set of variables \( V \) and has the set of clauses \( C \). Then the graph \( G_\Psi = (V \cup C, E) \), where \( E = \{(v, c) \mid v \in V, c \in C, v \in c\} \), is the clause-variable graph of \( \Psi \).

With the notation \( v \in c \) we mean that variable \( v \) (or its negation) occurs in clause \( c \). Observe that \( G_\Psi \) is always bipartite.

**Theorem 2.2** ([3, p. 877]) The problem of deciding whether a boolean formula \( \Psi \) is satisfiable is NP-complete, even if

- every variable occurs in exactly three clauses (twice positive, once negative),
- every clause contains two or three distinct variables, and
- \( G_\Psi \) is planar.

As a corollary of Theorem 2.2 we get the following result, which is the starting point of our work.

**Theorem 2.3** The problem of deciding whether a boolean formula \( \Psi \) is satisfiable is NP-complete, even if

- every variable occurs exactly once negatively and once or twice positively,
- every clause contains at least one negative literal,
- every clause contains two or three distinct variables, and
- \( G_\Psi \) is planar.

We call this decision problem 1\(^+\)-NEGATIVE PLANAR 3-SAT.

**Proof:** Let \( \Psi \) be a boolean formula satisfying the constraints of Theorem 2.2. By modifying \( \Psi \) we will construct a formula \( \Psi' \) that fulfills all the constraints of the theorem statement and is satisfiable if and only if \( \Psi \) is satisfiable.

We only need to eliminate those clauses containing no negative literal. Suppose that \( x \lor y \lor z \) is such a clause of \( \Psi \) with distinct variables \( x, y, z \). Now add a new variable \( x' \), and replace the original clause by the following two clauses:

\[
x \lor x' \quad \neg x' \lor y \lor z.
\]
This completes the construction of $\Psi'$. As this construction replaces some edges of $G_\Psi$ with paths, it preserves planarity.

Given a satisfying truth assignment of $\Psi$, we get a satisfying assignment of $\Psi'$ by setting $x' = \neg x$. A satisfying assignment of $\Psi'$ implies a satisfying assignment of $\Psi$. So $\Psi'$ is satisfiable if and only if $\Psi$ is. The theorem now follows straightforwardly from Theorem 2.2.

To prove that Planar Metric Dimension is NP-hard, we will give a reduction from $1^+\text{-Negative Planar 3-SAT}$. The idea behind the graph constructed in this reduction is the following. Given an instance $\Psi$ of $1^+\text{-Negative Planar 3-SAT}$, we first find a planar embedding of its clause-variable graph $G_\Psi$. We then replace each variable vertex of $G_\Psi$ by a variable gadget (see Figure 1), and each clause vertex of $G_\Psi$ by a clause gadget (see Figure 2). By identifying vertices of variable gadgets and vertices of clause gadgets in an appropriate way (see Figure 3), we obtain a planar graph $H_\Psi$ that will be our instance of Planar Metric Dimension.
We now describe our construction in detail. Consider a planar embedding of $G_\Psi$, which can be found in linear time \cite{7}. We first replace each variable vertex of $G_\Psi$ by a variable gadget. In Figure 1 a yellow color is used to highlight those vertices that will be identified with vertices from a clause gadget later on. There are three groups (connected components) of yellow vertices in the figure. The groups containing vertices $(t_1, f_1)$ and $(t_3, f_3)$ will be identified with vertices in clause gadgets where this variable appears positively in the corresponding clause; the group containing $(t_2, f_2)$ will be identified with vertices in clause gadgets where this variable appears negatively. By rotating and contorting the variable gadget appropriately, we can ensure that the three groups point into the right direction (i.e. the negative-appearance group faces the clause vertex where the variable appears negatively).

Next, we replace the clause vertices by clause gadgets. The exact gadget we use depends on whether the clause contains two or three variables (see Figure 2). We restrict our description to the three-variable case, as the two-variable case is similar and simpler. In Figure 2 a yellow color is used to highlight those vertices that will be identified with vertices from a variable gadget. There are again three groups of yellow vertices, one for each variable occurring in the clause.

Obviously, we will identify the $t$-vertex of a variable group with the $t$-vertex of a clause group, and the same for the $f$-vertices. We call this matching. It is not entirely straightforward to do this matching in a manner that preserves planarity. Consider the way in which the groups and the $t$ and $f$ vertices appear on the boundary of the clause gadget. In Figure 2 the pairs appear in order $(t, f), (t, f), (f, t)$ clockwise starting from the top. As illustrated in Figure 4 $(t, f), (f, t), (f, t)$ is also possible. The remaining two
alternatives, \((t, f), (t, f), (t, f)\) and \((f, t), (f, t), (f, t)\) are to be avoided. This is accomplished by choosing a variable appearing negatively in the clause and mirroring the corresponding variable gadget around the axis \(T_1—F\) (see Figure 1). This does not affect our ability to connect the variable to other clauses.

This completes the construction. Call the resulting graph \(H_\Psi\), which is planar by construction. We remark that each variable appears once negatively in \(\Psi\), and once or twice positively. So if the variable appears only twice, then \((t_1, f_1)\) or \((t_3, f_3)\) in the corresponding variable gadget will not be identified with a group of vertices in a clause gadget.

In Figure 5 we can see an example of the reduction and the resulting planar graph from the specific instance of 1\(^+\)-NEGATIVE PLANAR 3-SAT \((\neg x_1 \lor x_2) \land (x_1 \lor \neg x_2 \lor x_3) \land (x_2 \lor x_4 \lor \neg x_3) \land (\neg x_4 \lor x_3)\). The yellow (light) dotted lines indicate the vertices of the clause and variable gadgets that will be identified.

We now make several observations about the graph \(H_\Psi\) and the way landmarks need to be positioned on it.

Each \(f\)-vertex is contained in a triangle, say with vertices \(r, s\). Observe that \(r\) and \(s\) can only be separated by a landmark on \(r\) or \(s\). We call these \textit{forced landmarks}. In fact, in any smallest set of landmarks exactly one of \(r, s\) will be a landmark. Then it follows by construction that \(H_\Psi\) requires exactly \(3n\) forced landmarks, where \(n\) is the number of variables of \(\Psi\).

Using the forced landmarks, we can separate most pairs of vertices, as shown by the following lemma. We say that \(T_1, T_2, N_1, N_2\) and \(F\) are \textit{strictly inside} the variable gadget.

\textbf{Lemma 2.4} Let \(x, y\) be a pair of vertices, which is not \(\{w_1, w_2\}, \{T_1, T_2\}, \{T_1, N_1\}\) or \(\{T_2, N_1\}\) from the same gadget. Then the pair is separated by forced landmarks.

\textbf{Proof:} It is straightforward to check the cases where both \(x\) and \(y\) are in
Figure 5: The planar graph obtained for the
\((\neg x_1 x_2)(x_1 \neg x_2 x_3)(x_2 x_4 \neg x_3)(\neg x_4 x_3)\)
the same clause or variable gadget. There are two remaining cases: either $x,y$ are in different clause gadgets, or $x$ is strictly inside a variable gadget and $y$ outside that gadget.

Consider the first case, that is, $x,y$ are in different clause gadgets. Denote the gadget containing $x$ by $g_x$, and the gadget containing $y$ by $g_y$. Let $z_x$ be a forced landmark that is closest to $x$, and let $z_y$ be a forced landmark that is closest to $y$. Without loss of generality, $d(x,z_x) \leq d(y,z_y)$. We will show that $d(z_x,x) < d(z_x,y)$. Since $x,y$ are in distinct clause gadgets, the shortest path $P$ from $y$ to $z_x$ crosses at least one variable gadget $g$, and enters it through a group $(t,f)$. Let $w$ be the first vertex of $\{t,f\}$ that occurs on $P$, and let $z_f$ be the forced landmark in the triangle connected to $f$. If $w=t$, then $P$ contains at least two edges in $g$, and if $w=f$, $P$ contains at least one edge in $g$. In either case $P$ also contains at least one edge in $g_u$. Since $d(z_f,f) = 1$, $d(z_f,t) = 2$, in both cases the inequality $d(w,z_f) < d(w,z_x)$ holds. Hence $d(x,z_x) \leq d(y,z_y) \leq d(y,z_f) \leq d(y,w) + d(w,z_f) < d(y,w) + d(w,z_x) = d(y,z_x)$.

Now consider the second case, and assume that $x$ is strictly inside the variable gadget in Figure 6. If $y$ is in the picture, it can be readily verified that $x$ and $y$ are separated by the forced landmarks of the variable gadget. We claim that if $y$ is outside of the picture, then $d(z_1,y) + d(z_3,y) \geq 7$, where $z_1$ and $z_3$ are the forced landmarks in the triangles attached to $f_1$ and $f_3$ respectively. This implies that $z_1$ or $z_3$ is at distance at least four from $y$, whereas the distance of $z_1$ and $z_3$ to $x$ is at most three, implying that $x$ and $y$ are separated.

To prove the claim, note that if shortest paths from $y$ to $z_1$ and $z_3$ both contain $f_1$ (or equivalently $f_3$), then $d(z_1,y) + d(z_3,y) = d(z_1,f_1) + d(z_3,f_1) + 2d(f_1,y) = 4 + 2d(f_1,y) > 6$. Now consider the case where shortest paths from $y$ to $z_1,z_3$ include $f_1,f_3$ respectively. Any path from $f_1$ to $f_3$ that is not contained in the figure must cross at least two clause gadgets and one variable gadget, so it has length at least 5. This gives $d(y,f_1) + d(y,f_3) \geq 5$, and $d(z_1,y) + d(z_3,y) \geq 7$. \[\Box\]

It remains to analyze how the pairs excluded in Lemma 2.4 can be separated. This will rely on the satisfiability of $\Psi$ of course (as described below), but the following auxiliary lemma is crucial.

**Lemma 2.5** All pairs of vertices that are strictly inside a variable gadget are separated if and only if there is a landmark strictly inside the variable gadget.

**Proof:** It is easy to check that a landmark that is strictly inside a variable gadget together with the forced landmarks separates all pairs of vertices that are strictly inside the gadget. If no landmark is strictly inside the variable gadget, then a shortest path from any landmark $z$ to $T_1$ or $T_2$ contains $t_1$ or $t_3$. But then $d(z,T_1) = d(z,T_2)$. \[\Box\]
This lemma and the forced landmarks together imply that $H_\Psi$ has metric dimension at least $4n$. With this fact in mind, we present the proof of our main result.

**Theorem 2.6** Planar Metric Dimension is NP-complete.

**Proof:** Let $\Psi$ be an instance of 1⁺-NEGATIVE PLANAR 3-SAT with $n$ variables. Construct the graph $H_\Psi$ in the manner described before. Constructing $H_\Psi$ clearly takes time polynomial in the number of variables and clauses of $\Psi$. We now claim that $H_\Psi$ has metric dimension $4n$ if and only if $\Psi$ is satisfiable.

Assume that a satisfying truth assignment for $\Psi$ is given. Place $3n$ forced landmarks. If a variable has value true, place a landmark on $T_i$ in the corresponding gadget; otherwise, place a landmark on $F$. After applying Lemmata 2.4 and 2.5 we only need to check that pairs $w_1, w_2$ are separated. But each such pair is separated by the landmark strictly inside the variable that satisfies the corresponding clause, and we are done.

Now assume that the metric dimension is $4n$. We will construct a satisfying assignment for $\Psi$. Each variable gadget contains exactly one landmark, which is on $T_i$, $N_i$, or $F$. If the landmark is on $T_i$, set the variable to true. If the landmark is on $F$, set it to false. Otherwise we can choose arbitrarily. It remains to show that because the pairs $w_1, w_2$ are separated, the truth
assignment is satisfying. Note that a landmark $z$ separates pair $w_1, w_2$ if a shortest path from a landmark to either of them enters the clause gadget through some $t_i$. If a shortest path from landmark $z$ to $w_1$ or $w_2$ intersects more than one clause gadget, it leaves the first clause through an $f$-vertex, after which it enters all subsequent ones through an $f$-vertex. But then $w_1, w_2$ in the final clause are not separated. It follows that a landmark $z$ separates $w_1$ and $w_2$ only if it is in an adjacent variable gadget and the corresponding variable satisfies the corresponding clause.

Following the claim, the reduction should construct $H_{\Psi}$ as the graph for the instance of Planar Metric Dimension and set $k$ to $4n$. □

Acknowledgment: The authors thank David Johnson for sending a scanned copy of the NP-completeness proof of Metric Dimension of [4].

References