Abstract. When dealing with two-dimensional (2D) discrete state-space models, reachability, controllability and zero-controllability are introduced in two different forms: a local form, which refers to single local states, and a global form, which instead pertains the infinite set of local states lying on a separation set. In this paper, these concepts are investigated in the context of 2D positive systems. Their combinatorial nature suggests a graph theoretic approach to their analysis, as, indeed, to every 2D positive state-space model of dimension $n$ with $m$ inputs one can associate a 2D influence digraph with $n$ vertices and $m$ sources.

For all these properties, necessary and sufficient conditions, which refer to the structure of the digraph, are provided. It turns out that while the global reachability index is bounded by the system dimension $n$, the local reachability index may far exceed the system dimension and even reach $n^2/4$.

Key words. 2D positive systems, reachability, controllability, zero-controllability, influence digraph, finite memory systems.

1. Introduction

The interest in two-dimensional (2D) systems goes back to the early seventies [1, 7, 17], and was initially motivated by the relevance of these models in seismology applications, X-ray image enhancement, image deblurring, digital picture processing, etc. More recently, some contributions dealing with river pollution modelling [6] and the discretization of PDE’s which describe gas absorption and water stream heating [16], naturally introduced a nonnegativity constraint in 2D system equations. Also, two-dimensional models involving only nonnegative variables were successfully adopted for describing the diffusion process of a tracer into a blood vessel [21]. This kind of instances stimulated, in the late nineties, a systematic analysis of “2D positive systems”, i.e. 2D state-space models whose input, state and output variables take positive (or at least nonnegative) values, where the results presented in [6, 16, 21] could be naturally framed.

Research efforts in this context were first oriented to extend positive matrix theory to pairs of matrices [10, 11, 12, 20], thus leading to a satisfactory analysis of the free state evolution of 2D positive systems and a complete characterization of their asymptotic stability [19]. More recently, research efforts in 2D positive systems have concentrated on the analysis of their structural properties, and some preliminary results about reachability and controllability have been presented in [13, 14].

When dealing with 2D systems, the concepts of reachability, controllability and zero-controllability are naturally introduced in two different forms: a weak (local) form, which refers to single “local states”, and a strong (global) form, which pertains the infinite set of local states lying on some “separation set” [3, 7]. In this paper, the aforementioned
concepts are introduced and investigated in the context of 2D positive systems, driven by nonnegative inputs and described by the following state-updating equation [7]:

\[(1.1) \ x(h+1, k+1) = A_1 x(h, k+1) + A_2 x(h+1, k) + B_1 u(h, k+1) + B_2 u(h+1, k),\]

where the \( n \)-dimensional local states \( x(\cdot, \cdot) \) and the \( m \)-dimensional inputs \( u(\cdot, \cdot) \) take nonnegative values, \( A_1 \) and \( A_2 \) are nonnegative \( n \times n \) matrices, \( B_1 \) and \( B_2 \) are nonnegative \( n \times m \) matrices, and the initial conditions are assigned by specifying the (nonnegative) values of the state vectors on the separation set \( \mathcal{C}_0 := \{(h, k) : h, k \in \mathbb{Z}, h + k = 0\} \), namely by assigning all local states of the initial global state \( \mathcal{X}_0 := \{x(h, k) : (h, k) \in \mathcal{C}_0\} \).

As in the 1D case, structural properties of 2D positive systems exhibit a combinatorial nature, which motivates a graph theoretic approach to their analysis. Indeed, to every 2D positive state-space model of dimension \( n \) with \( m \) inputs one can associate a 2D influence digraph [11, 12, 13] with \( n \) vertices, \( m \) sources and two types of arcs interconnecting the sources and the vertices, and every structural property admits both algebraic and graph-theoretic characterizations.

The paper is organized as follows: section 2 introduces some notations and provides both local and global definitions. In section 3, local and global zero-controllability are addressed. Both properties turn out to be equivalent to finite memory, a property which has been investigated in detail in [9] and [10]. Local and global reachability, as well as the corresponding indices, \( I_{LR} \) and \( I_{GR} \), are fully characterized in sections 4 and 5, where it is shown that the global reachability index is bounded by the system dimension \( n \), while the local reachability index may far exceed the system dimension and even reach \( n^2/4 \).

2. Notations and preliminary definitions

Before proceeding, it is convenient to introduce some basic definitions and preliminary concepts that will be used in the paper. The Hurwitz products of two \( n \times n \) matrices \( A_1 \) and \( A_2 \) are inductively defined [7] as

\[
A_1^i u^j A_2 = 0, \quad \text{when either } i \text{ or } j \text{ is negative},
\]

\[
A_1^i u^j A_2 = A_1^i, \quad \text{for } i \geq 0, \quad A_1^i u^j A_2 = A_2^j, \quad \text{for } j \geq 0,
\]

\[
A_1^i u^j A_2 = A_1 (A_1^{i-1} u^j A_2) + A_2 (A_1^i u^{j-1} A_2), \quad \text{for } i, j > 0.
\]

Notice that \( \sum_{i+j=\ell} A_1^i u^j A_2 = (A_1 + A_2)^\ell \).

A 2D influence digraph \( D^{(2)} \) is a direct graph which exhibits two types of arcs and input flows [11, 12, 13]. In detail, it is a sextuple \((S, V, A_1, A_2, B_1, B_2)\), where \( S = \{s_1, s_2, \ldots, s_m\} \) is the set of sources, \( V = \{v_1, v_2, \ldots, v_n\} \) is the set of vertices, \( A_1 \) and \( A_2 \) are subsets of \( V \times V \) whose elements are called \( A_1\)-arcs and \( A_2\)-arcs, respectively, meanwhile \( B_1 \) and \( B_2 \) are subsets of \( S \times V \) whose elements are called \( B_1\)-arcs and \( B_2\)-arcs, respectively.

To every 2D positive system (1.1), we associate a 2D influence digraph \( D^{(2)}(A_1, A_2, B_1, B_2) \) with \( n \) vertices, \( v_1, v_2, \ldots, v_n \) and \( m \) sources \( s_1, s_2, \ldots, s_m \). There is an \( A_1\)-arc (an \( A_2\)-arc) from \( v_j \) to \( v_i \) if and only if the \((i, j)\)th entry of \( A_1 \) (of \( A_2 \)) is nonzero. There is a \( B_1\)-arc (a \( B_2\)-arc) from \( s_j \) to \( v_i \) if and only if the \((i, j)\)th entry of \( B_1 \) (of \( B_2 \)) is nonzero. For instance, the positive system with a single input described by the following matrices

\[
(2.1) \quad (A_1, A_2, B_1, B_2) = \begin{pmatrix}
0 & 5 & 0 \\
0 & 2 & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 4 \\
2 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
10 \\
0 \\
0
\end{pmatrix}
\]
corresponds to the 2D digraph, with 3 vertices and a single source, of Fig. 1.1. We have represented $A_1$-arcs and $B_1$-arcs by means of thick lines, while $A_2$-arcs and $B_2$-arcs by means of thin lines. This will be a steady assumption within the paper. A path $p$ in $D^{(2)}(A_1, A_2, B_1, B_2)$ is a sequence of adjacent arcs and, in particular, an $s_j$-path is a path which originates from the source $s_j$. A path $p$ is specified by assigning its vertices and the type of arcs they are connected by.

Fig. 1.1 2D influence digraph corresponding to (2.1)

If we denote by $|p|_1$ the number of $A_1$-arcs and $B_1$-arcs, and by $|p|_2$ the number of $A_2$-arcs and $B_2$-arcs occurring in $p$, then $||p|_1 - |p|_2|$ is the composition of $p$ and $|p| = |p|_1 + |p|_2$ its length. A path whose extreme vertices coincide is a cycle. In particular, if each vertex appears exactly once as the first vertex of an arc, the cycle is a circuit.

A vector $v$ is said to be an $i$th monomial vector if it can be expressed as $\alpha_i e_i$, where $e_i$ denotes the $i$th canonical vector and $\alpha_i$ is some positive real coefficient. A monomial matrix is a nonsingular (square) matrix whose columns are monomial vectors. If $M$ is a $p \times q$ matrix, in particular a square positive matrix, by Cone $M$ we mean the set of nonnegative combinations of the columns of $M$, namely the (polyhedral) cone generated in $\mathbb{R}^p$ by the columns of $M$. $1_n$ denotes the positive vector of size $n$ with all entries equal to 1, while the symbol $*$ represents the Hadamard product (entry by entry) of two matrices.

According to what has been announced in the Introduction, two distinct definitions of reachability are usually considered [7] for 2D positive state-space models (1.1): local reachability and global reachability. Local reachability refers to the possibility of “reaching” an arbitrary local state $x^* \in \mathbb{R}^n$ in some point $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, starting from zero initial conditions, meanwhile global reachability amounts to the possibility (starting, again, from zero initial conditions) of obtaining an arbitrary sequence of local states $x(h, k)$ on some separation set 

$$C_t := \{(h, k) : h, k \in \mathbb{Z}, h + k = t\}.$$

Of course, all nonnegative input sequences involved have supports included in the half-plane $\{(h, k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0\}$. Similarly, controllability and zero-controllability properties can be given in a local and in a global form, depending on whether one considers a single local state on the (final) separation set $C_t$ or the entire global state 

$$X_t := \{x(h, k) : (h, k) \in C_t\}.$$

We first introduce the local versions of the aforementioned structural properties.

**Definition 2.1.** A 2D state-space model (1.1) is

- **(positively) locally reachable** [7] if, upon assuming $X_0 = 0$, for every $x^* \in \mathbb{R}^n_+$ there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a nonnegative input sequence $u(\cdot, \cdot)$ such that $x(h, k) = x^*$. When so, we will say that $x^*$ is reachable in $h + k$ steps and the smallest number of steps which allows to reach every nonnegative local state represents the **local reachability index** $I_{LR}$ of the 2D positive system;
• (positively) locally controllable if, corresponding to any nonnegative $X_0$ and any $x^* \in \mathbb{R}_+^n$, there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a nonnegative input sequence $u(\cdot, \cdot)$ such that $x(h, k) = x^*$;

• (positively) locally zero-controllable if, corresponding to any nonnegative $X_0$, there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a nonnegative input sequence $u(\cdot, \cdot)$ such that $x(h, k) = 0$.

In the following, as structural properties will steadily refer to the case of nonnegative/positive states and nonnegative inputs, the specification “positively” will be omitted. The global versions of the previous properties are introduced in Definition 2.2, below.

Definition 2.2. A 2D state-space model (1.1) is

• globally reachable [7, 8] if, upon assuming $X_0 = 0$, for every global state $X^*$ with entries in $\mathbb{R}_+^n$ and a nonnegative input sequence $u(\cdot, \cdot)$ such that the global state $X_N := \{x(h, k) : h, k \in \mathbb{Z}, h + k = N\}$ coincides with $X^*$. When so, we will say that $X^*$ is reachable in $N$ steps. The smallest number of steps which allows to reach every nonnegative global state represents the global reachability index $I_{GR}$ of the system;

• globally controllable if, corresponding to any nonnegative initial global state $X_0$ and any nonnegative $X^*$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $u(\cdot, \cdot)$ such that the global state $X_N$ coincides with $X^*$;

• globally zero-controllable if, corresponding to any nonnegative initial global state $X_0$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $u(\cdot, \cdot)$ such that the global state $X_N$ is identically zero.

Clearly, each of the above global properties ensures the corresponding local one, and a 2D positive system is locally (globally) controllable if and only if it is both locally (globally) reachable and locally (globally) zero-controllable. These results are consistent with the analogous ones for standard 2D systems.

3. Local/global zero-controllability

As a first step, we aim at showing that, when dealing with 2D positive systems, local zero-controllability and global zero-controllability are equivalent properties and they both coincide with the finite memory property [3, 10].

A standard (i.e., not necessarily positive) 2D system is said to be finite memory if for every initial global state $X_0$ there exists $N \in \mathbb{Z}_+$ such that the corresponding free state evolution goes to zero within $N$ separation sets, namely $X_N = 0$.

Finite memory definition for 2D positive systems is obtained by simply introducing the positivity constraint on the initial global state $X_0$. Several characterizations of finite memory positive systems have been provided in [10]. In particular, the finite memory property for 2D positive systems corresponds to the lack of cycles in the associated 2D graph.

It is immediately apparent that, when dealing with positive systems, both local and global zero-controllability are properties which just pertain the free state evolution, as nonnegative inputs could not make the task of obtaining a zero local or global state easier!

Basing on this simple remark, which holds true also for 1D positive systems, the proof of the following proposition becomes almost straightforward.

Proposition 3.1. Given a 2D positive system (1.1), of dimension $n$, the following facts are equivalent:
i) the system is locally zero-controllable;

ii) the system is finite memory;

iii) the system is globally zero-controllable.

Proof. i) ⇒ ii) Suppose that the system is locally zero-controllable and choose as \( X_0 \) the positive global state whose local states \( x(i, -i), i \in \mathbb{Z} \), are all equal to the vector \( 1_n \). For every \((h, k) \in \mathbb{Z} \times \mathbb{Z}, \) with \( h + k > 0 \), we have \( x(h, k) = (A_1 + A_2)^{h+k}1_n \). Since there exists \((h, k) \) such that \( x(h, k) = 0 \), we have also \( (A_1 + A_2)^{h+k} = 0 \), which ensures [10] the finite memory property of the 2D system described by the positive matrix pair \((A_1, A_2)\).

ii) ⇒ iii) For every nonnegative \( X_0 \), just leave the system evolve freely.

iii) ⇒ ii) For every nonnegative \( X_0 \), just leave the system evolve freely.

At this point, it is clear that a 2D positive system is locally (globally) controllable if and only if it is both finite memory and locally (globally) reachable. Since finite memory property is very easy to check, either by algebraic means or via graph inspection, our interest will focus on local and global reachability properties. Characterizations of such properties will immediately lead to characterizations of local and global controllability.

4. Local reachability

When dealing with standard 2D systems, local reachability is easily tested by evaluating the column span of the reachability matrix in \( k \) steps [7], i.e.

\[
\mathcal{R}_k = \begin{bmatrix}
B_1 & B_2 & A_1B_1 & A_1B_2 + A_2B_1 & A_2B_2 & A_1^2B_1 & (A_1^i)A_1^jA_2B_1 + A_1^iA_2B_2 & \ldots & A_2^{k-1}B_2
\end{bmatrix}
\]

as \( k \) varies over the set \( \mathbb{N} \) of positive integers. Indeed, reachable states in \( k \) steps, i.e. local states that can be reached in any assigned position of the separation set \( C_k \), starting from \( X_0 = 0 \), constitute a linear subspace \( X_k \subseteq \mathbb{R}^n \), spanned by the columns of \( \mathcal{R}_k \). Clearly, the ascending chain \( X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots \) eventually reaches stationarity and this necessarily happens, by the 2D Cayley-Hamilton theorem [9], in no more than \( n \) steps. As a consequence, if the system is locally reachable, the point \((h, k)\) where \( x(h, k) \) can reach the desired value \( x^* \) (see Definition 2.1) can always be chosen on the separation set \( C_n \).

Once we constrain the input sequence to be nonnegative, the reachability subspaces \( X_k, k \in \mathbb{N} \), are replaced by the reachability cones \( X_k^+ \), \( k \in \mathbb{N} \). In fact, the set \( X_k^+ \) of all local states that can be reached in any assigned position of the separation set \( C_k \), by means of nonnegative inputs and starting from initial zero conditions \((X_0 = 0)\), obviously coincides with the set of all nonnegative combinations of the columns of \( \mathcal{R}_k \), namely \( X_k^+ = \text{Cone} \mathcal{R}_k \). Consequently, a system is locally reachable if and only if there exists \( k \in \mathbb{N} \) such that \( \text{Cone} \mathcal{R}_k = \mathbb{R}^n_+ \). When so, the smallest such \( k \) represents the reachability index \( I_{LR} \) of the (locally reachable) 2D positive system. It is worth remarking, however, that as in the case of 1D positive systems [15], the chain of reachability cones does not necessarily reach stationarity and, indeed, certain positive states can be reached only asymptotically.

Positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the canonical basis in \( \mathbb{R}^n \) by means of nonnegative inputs, which in turn amounts to saying that there exists some \( k \in \mathbb{N} \) such
that the reachability matrix in \( k \) steps, \( \mathcal{R}_k \), includes an \( n \times n \) monomial submatrix \([2, 5]\). Keeping in mind the structure of the columns of \( \mathcal{R}_k \), the previous condition can be equivalently stated by saying that a 2D system is locally reachable if and only if there exist \( n \) pairs \((h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n\), and \( n \) indices \( j = j(i) \in \{1, 2, \ldots, m\} \) such that \((A_1^{h_i-1} \mathbf{w}^{k_i} A_2)B_1 e_j + (A_1^{h_i} \mathbf{w}^{k_i-1} A_2)B_2 e_j\) is an \( i \)th monomial vector. If so,

\[
I_{LR} = \max_{i \in \{1, 2, \ldots, n\}} \min_{h_i, k_i} \{h_i + k_i: \exists j = j(i) \text{ s.t. } (A_1^{h_i-1} \mathbf{w}^{k_i} A_2)B_1 e_j + (A_1^{h_i} \mathbf{w}^{k_i-1} A_2)B_2 e_j \text{ is an } i \text{th monomial vector}\}.
\]

Furthermore, when dealing with 2D systems with scalar inputs, the aforementioned condition simply becomes: there exist \( n \) pairs \((h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n\), such that \((A_1^{h_i-1} \mathbf{w}^{k_i} A_2)B_1 + (A_1^{h_i} \mathbf{w}^{k_i-1} A_2)B_2\) is an \( i \)th monomial vector. Consequently,

\[
I_{LR} = \max_{i \in \{1, 2, \ldots, n\}} \min_{h_i, k_i} \{(h_i + k_i): (A_1^{h_i-1} \mathbf{w}^{k_i} A_2)B_1 + (A_1^{h_i} \mathbf{w}^{k_i-1} A_2)B_2\} \text{ is an } i \text{th monomial vector}\}.
\]

Notice, finally, that all pairs \((h_i, k_i)\) are necessarily distinct, but the case may occur that \( h_i + k_i = h_j + k_j \) for \( i \neq j \).

As for 1D positive systems, local reachability of 2D positive systems is a structural property, by this meaning that it only depends on the nonzero patterns of the system matrices and not on the specific values of their nonzero elements. However, differently from the 1D case and the standard 2D case, the reachability index \( I_{LR} \) of a (locally reachable) 2D positive system is not bounded by the system dimension.

**Example 1**  The positive system described by the following matrices

\[
(A_1, A_2, B_1, B_2) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),
\]

corresponds to the 2D digraph of Fig. 4.1.

![Fig. 4.1 2D influence digraph corresponding to Example 1](image)

It is easy to verify that the system is locally reachable and \( I_{LR} = 3 > 2 = n \), as

\[
\mathcal{R}_1 = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\mathcal{R}_2 = \begin{bmatrix} B_1 & B_2 & A_1 B_1 & A_2 B_1 + A_1 B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\]

\[
\mathcal{R}_2 = \begin{bmatrix} B_1 & B_2 & A_1 B_1 & A_2 B_1 + A_1 B_2 & A_2 B_2 & A_1^2 B_1 & (A_1^1 \mathbf{w} A_2) B_1 + A_1^2 B_2 \\ B_2 & A_2 B_1 + A_1 B_2 & A_2 B_2 & A_1^2 B_1 & (A_1^1 \mathbf{w} A_2) B_1 + A_1^2 B_2 & A_2^2 B_2 + (A_1^1 \mathbf{w} A_2) B_2 & A_2^2 B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.
\]

**Example 2** In the following 2D positive system

\[
(A_1, A_2, B_1, B_2) = ([0 \ e_3 \ e_1 \ 0 \ e_6 \ e_7 \ e_4], [e_2 \ 0 \ 0 \ e_5 \ 0 \ 0 \ 0], [e_1 \ e_4], [0]),
\]
which corresponds to the 2D digraph of Fig. 4.2,

![Fig. 4.2 2D influence digraph corresponding to Example 2](image)

the local reachability index is 13 while the system dimension is \( n = 7 \). The above example can be generalized to 2D influence digraphs consisting of two loops including \( n \) and \( n + 1 \) vertices, respectively, connected by arcs of type 1 and 2 as indicated in Fig. 4.3. The reachability index turns out to be of the same order as \( n \cdot (n + 1) \), namely of the same order as \( n^2 / 4 \), since \( n = n + (n + 1) \).

![Fig. 4.3 2D influence digraph generalizing Example 2](image)

It seems reasonable to conjecture that \( n^2 / 4 \) represents an upper bound for the index \( I_{LR} \) of every 2D positive system, but a formal proof of this result is not available, yet.

A necessary condition for local reachability is the following one, which extends a similar result obtained for 1D positive systems [18].

**Proposition 4.1.** If the positive system (1.1) is locally reachable then the matrix

\[
\begin{bmatrix}
A_1 & A_2 & B_1 & B_2
\end{bmatrix}
\]

includes an \( n \times n \) monomial submatrix.

**Proof.** If the system is locally reachable, then there exist \( n \) nonnegative pairs \((h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n\), and \( n \) indices \( j = j(i) \in \{1, 2, \ldots, m\} \) such that

\[
(A_1^{h_i-1}w^{k_i}A_2)B_1e_j + (A_1^{h_i}w^{k_i-1}A_2)B_2e_j
\]

is an \( i \)th monomial vector. If \( h_i + k_i = 1 \) then the \( i \)th monomial vector is a column either of \( B_1 \) or of \( B_2 \). If \( h_i + k_i > 1 \), the \( i \)th monomial vector is a column either of \( A_1 \) or of \( A_2 \) (possibly both). \( \square \)

As for 1D positive systems, local reachability property admits an interesting and useful characterization in terms of the 2D influence digraph associated with the system. Indeed, saying that \((A_1^{h_i-1}w^{k_i}A_2)B_1e_j + (A_1^{h_i}w^{k_i-1}A_2)B_2e_j\) is an \( i \)th monomial vector just means that the set of \( s_j \)-paths \( p \) of composition \([|p|_1, |p|_2] = [h_i, k_i]\) is not empty and each of them
reaches the vertex \( v_i \) alone. If so, we will say that the vertex \( v_i \) is **deterministically reached** by all \( s_j \)-paths of composition \([h_i \ k_i]\). As a consequence, the 2D system (1.1) is locally reachable if and only if for every \( i \in \{1, 2, \ldots, n\} \) there exists \( j = j(i) \) such that the vertex \( v_i \) is deterministically reached by all \( s_j \)-paths of some composition \([h_i \ k_i]\). Moreover,

\[
I_{LR} = \max_i \min_{h_i, k_i} \{h_i + k_i \mid \exists j = j(i) \text{ s.t. all } s_j \text{-paths of composition } [h_i \ k_i] \text{ deterministically reach } v_i\}.
\]

5. Global reachability

When addressing global reachability, it suffices to focus on the reachability of global states consisting of all zero (local) states except for one of them, which coincides with the monomial vector \( e_i \), \( i \in \{1, 2, \ldots, n\} \). Indeed, if the system is globally reachable then, in particular, all such global states must be reachable. On the other hand, if all such global states are reachable, each of them can be reached by means of a suitable finite support nonnegative input sequence. So, by superposing nonnegative combinations of such finite support input sequences, one can reach every nonnegative global state.

Obviously, if we denote by \( N_i \) the minimum number of steps required to reach any global state consisting of all zero local states except for one, which coincides with \( e_i \), then it is easily seen that the global reachability index, \( I_{GR} \), coincides with \( \max_i N_i \).

As for the local case, global reachability may be characterized in terms of the columns of the reachability matrix.

**Proposition 5.1.** A 2D system (1.1) is globally reachable if and only if there exist \( n \) pairs \( (h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n \), and \( n \) indices \( j = j(i) \in \{1, 2, \ldots, m\} \) such that

\[
\begin{align*}
(1) & \quad (A_1^{h_1-1}w^{k_1}A_2)B_1e_j + (A_2^{h_1-w^{k_1}}A_2)B_2e_j = \text{an } i\text{-th monomial vector,} \\
(2) & \quad (A_1^{h-1}w^{k}A_2)B_1e_j + (A_1^{h-w^{k}}A_2)B_2e_j = 0, \forall (h, k) \neq (h_i, k_i) \text{ with } h + k = h_i + k_i.
\end{align*}
\]

**Proof.** Let \( \mathcal{X}(i) \) denote a global state consisting of all zero local states except for one of them, located in \( (\tilde{h}_i, \tilde{k}_i) \), which coincides with \( e_i \), and suppose that \( \mathcal{X}(i) \) is globally reachable after \( N_i = \tilde{h}_i + \tilde{k}_i \) steps. By the system nonnegativity, the support of any input sequence that allows reaching \( \mathcal{X}(i) \) can be restricted to the triangular region

\[
T_{(h_i, k_i)} := \{(h, k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0, h \leq \tilde{h}_i, k \leq \tilde{k}_i\}.
\]

In particular, there exists at least one input sequence consisting of a nonzero, say \( j\text{-th, monomial vector, located in some specific point } (\tilde{h}_i, \tilde{k}_i) \text{ of } T_{(h_i, k_i)}, \text{ and zero elsewhere.}
\]

But then, it is immediately seen that the value of each local state \( x(h, k) \) generated by such input on the separation set \( C_N \), coincides with the (nonnegative multiple of the) \( j\text{-th column of the block matrix } (A_1^{h-h_i-1}w^{k-k_i}A_2)B_1 + (A_1^{h-w^{k}}A_2)B_2. \) As a consequence, the global state \( \mathcal{X}(i) \) can be reached if and only if there exists an integer pair \( (h_i, k_i) = (\tilde{h}_i - \tilde{h}_i, \tilde{k}_i - \tilde{k}_i) \) and some \( j = j(i) \) such that (5.1)-(5.2) hold. Since this condition must be verified for every \( i \in \{1, 2, \ldots, n\} \), the proposition is proved.

When dealing with systems with scalar inputs,

i) (5.1) and (5.2) simply become

\[
\begin{align*}
(3) & \quad (A_1^{h-1}w^{k}A_2)B_1 + (A_1^{h-w^{k}}A_2)B_2 = \text{an } i\text{-th monomial vector,} \\
(4) & \quad (A_1^{h-1}w^{k}A_2)B_1 + (A_1^{h-w^{k}}A_2)B_2 = 0, \forall (h, k) \neq (h_i, k_i) \text{ with } h + k = h_i + k_i;
\end{align*}
\]
ii) If the system is globally reachable, then \( h_i + k_i \neq h_\ell + k_\ell \) when \( i \neq \ell \).

Not unexpectedly, even when dealing with positive systems, global reachability is stronger than local reachability. This clearly arises by comparing the results of section 4 with Proposition 5.1, but it is also shown by means of a simple example.

**Example 3** Consider the following 2D positive system:

\[
(A_1, A_2, B_1, B_2) = ([e_2 \ 0 \ 0 \ 0], [e_3 \ e_4 \ 0], [e_1], [e_2]).
\]

![Fig. 5.1 2D influence digraph corresponding to Example 3](image)

It is just a matter of computation showing that such a 2D positive system is locally, but not globally, reachable.

Proposition 5.1 can be interpreted in graph theoretic terms: the 2D system (1.1) is globally reachable if and only if for every \( i \in \{1, 2, \ldots, n\} \) there exists \( j = j(i) \in \{1, 2, \ldots, m\} \) such that the vertex \( v_i \) is deterministically reached by all \( s_j \)-paths of a given composition \([h_i \ k_i]\), and there exist no \( s_j \)-path of the same length \( h_i + k_i \) and different composition. Moreover,

\[
I_{GR} = \max_i \min_{h_i, k_i} \{ h_i + k_i \colon \exists j = j(i) \text{ s.t. all } s_j \text{-paths of composition } [h_i \ k_i] \text{ deterministically reach } v_i \text{ and there is no } s_j \text{-path of length } h_i + k_i \text{ and different composition}\}.
\]

The following lemma leads the way to further characterizations of global reachability.

**Lemma 5.2.** If the 2D system (1.1) is globally reachable then the 1D positive system described by the pair \((A_1 + A_2, B_1 + B_2)\) is (positively) reachable.

**Proof.** From Proposition 5.1 it follows that if the 2D system (1.1) is globally reachable then there exist \( n \) pairs \((h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n\), and \( n \) indices \( j = j(i) \in \{1, 2, \ldots, m\} \) such that

\[
\alpha_i e_i = \sum_{h, k \in \mathbb{Z}_+} (A_1^{h-1} A_2^k)B_1 e_j + (A_1^h A_2^{k-1})B_2 e_j = (A_1 + A_2)^{h_i + k_i - 1}(B_1 + B_2)e_j,
\]

for some \( \alpha_i > 0 \). This ensures [5] that the pair \((A_1 + A_2, B_1 + B_2)\) is (positively) reachable. \( \square \)

For positive systems with scalar inputs, Lemma 5.2 leads to some sort of “canonical” global reachability form.

**Proposition 5.3.** For a 2D positive system (1.1) with scalar inputs the following facts are equivalent:
i) the system is globally reachable;

ii) there exists a permutation matrix $P$ such that

\begin{align}
(5.5) \quad P^T(A_1 + A_2)P &= \begin{bmatrix}
* & + & 0 \\
* & 0 & + & 0 \\
* & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & + \\
* & 0 & \cdots & 0 & 0
\end{bmatrix}, \\
& \quad P^T(B_1 + B_2) = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
+ 
\end{bmatrix},
\end{align}

where $*$ and $+$ represent a nonnegative and a positive entry, respectively, and

\begin{align}
(5.6) \quad P^T(A_1 \ast A_2)P &= \begin{bmatrix}
* & \vdots \\
* & \ast \\
\ast & \ast \\
\ast & \ast \\
0_{n \times (n-1)}
\end{bmatrix}, \\
& \quad P^T(B_1 \ast B_2) = 0.
\end{align}

Proof. For the sake of simplicity, as positive (either 1D or 2D global) reachability does not depend on the values of the nonzero entries of all matrices involved, within the proof all nonzero entries will be assumed unitary.

i) ⇒ ii) If the system is globally stable then, by Lemma 5.2, the pair $(A_1 + A_2, B_1 + B_2)$ is (positively) reachable. This ensures [5] that there exists a permutation matrix $P$ such that (5.5) holds. Moreover, conditions (5.3)-(5.4) imply that only one among $P^T B_1$ and $P^T B_2$ is nonzero, which gives the second identity in (5.6). Suppose, without loss of generality, $P^T B_1 = e_n$ and $P^T B_2 = 0$. From

$$P^T [(A_1 1 \mathbf{w}^0 A_2) B_1 + (A_1 0 \mathbf{w}^1 A_2) B_1] = P^T (A_1 + A_2) B_1 = P^T (A_1 + A_2) P P^T (B_1 + B_2) = e_{n-1}$$

and conditions (5.3)-(5.4), it follows that only one among $P^T [(A_1 1 \mathbf{w}^0 A_2) B_1]$ and $P^T [(A_1 0 \mathbf{w}^1 A_2) B_1]$ coincides with $e_{n-1}$, while the other is zero. But this means that only one among $(P^T A_1 P) e_n$ and $(P^T A_2 P) e_n$ is $e_{n-1}$, and hence an $(n-1)$th monomial vector, while the other is zero. By proceeding in this way, we show that only one among $(P^T A_1 P) e_i$ and $(P^T A_2 P) e_i$ is an $(i-1)$th monomial vector, $i = 2, \ldots, n-1, n$, while the other is zero. This proves the first identity in (5.6).

ii) ⇒ i) Conditions (5.5) and (5.6) easily imply that there exist $n$ pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n$, such that (5.3) and (5.4) hold. So, the 2D system is globally reachable.

**Remarks** As a consequence of the previous proposition, all pairs $(h_i, k_i)$, that make (5.3) and (5.4) satisfied, sum up to $n$ distinct integers $h_i + k_i$ and none of them exceeds $n$. This means that the set of all such $h_i + k_i, i = 1, 2, \ldots, n$, coincides with the set $\{1, 2, \ldots, n\}$ and hence, in particular, the global reachability index for 2D (globally reachable) systems with scalar inputs coincides with $n$. This situation is quite different from the one arising when local reachability is concerned, since the local reachability index can far exceed the system dimension.

Even more, for systems with scalar inputs, Proposition 5.1 can be restated in terms of the reachability matrices. Indeed, the reachability matrix (in $n$ steps), $R := R_n$, can always be block-partitioned as

$$R = [R_1 \mid R_2 \mid \ldots \mid R_n].$$
where $R_\ell$ represents the block matrix including all columns $(A_1^{i-1}A_2)B_1 + (A_1^{j-1}A_2)B_2$ with $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, and $i + j = \ell$. By referring to this expression of $R$, equations (5.1) and (5.2) (and hence global reachability) hold if and only if, for every $i = 1, 2, \ldots, n$, there exists $\ell_i \in \{1, 2, \ldots, n\}$ such that $R_{\ell_i}$ consists of all zero columns except for one which is an $i$th monomial vector.

As a corollary of the previous proposition, and basing on the analysis carried on in section 3, a “canonical” global controllability form may also be derived.

**Corollary 5.4.** For a 2D positive system (1.1) with scalar inputs the following facts are equivalent:

i) the system is globally controllable;

ii) there exists a permutation matrix $P$ such that

$$
(5.7) \quad P^T(A_1 + A_2)P = \begin{bmatrix}
0 & + & 0 \\
0 & 0 & + & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \ddots & + & 0 \\
0 & \ldots & 0 & 0
\end{bmatrix},
$$

$$
(5.8) \quad P^T(A_1 * A_2)P = 0 \quad P^T(B_1 * B_2) = 0.
$$

When dealing with systems with several inputs, Lemma 5.2 leads to a characterization of global reachability (controllability) similar to the one given in Proposition 5.3 (in Corollary 5.4, respectively). This requires, however, to consider the canonical forms available for reachable 1D positive systems with several inputs [4, 18]. As such forms are rather complicate, except when the 1D system matrix $(A_1 + A_2$, in this case) is devoid of zero columns, we restrict ourselves to this special case. The proof of the following proposition can be easily obtained by resorting to the canonical form given in [18] and the same reasonings adopted within the proof of Proposition 5.3.

**Proposition 5.5.** For a 2D positive system (1.1) with $m$ inputs and $A_1 + A_2$ devoid of zero columns, the following facts are equivalent:

i) the system is globally reachable;

ii) there exist $r \in \mathbb{N}$, with $r \leq m$, and permutation matrices, $P$ and $Q$, of suitable dimensions, such that

$$
[P^T(A_1 + A_2)P \mid P^T(B_1 + B_2)Q] = \begin{bmatrix}
F_{11} & F_{12} & \ldots & F_{1r} & e_{n_1} & 0 & \ldots & 0 \\
F_{21} & F_{22} & \ldots & F_{2r} & 0 & e_{n_2} & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
F_{r1} & F_{r2} & \ldots & F_{rr} & 0 & 0 & \ldots & e_{n_r}
\end{bmatrix}_{G_{rem}},
$$

where
\[ F_{ii} = \begin{bmatrix} * & + & 0 \\ * & 0 & + & 0 \\ * & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & + \\ * & 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}, \quad F_{ij} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ * & 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_i \times n_j}, \text{ for } i \neq j, \]

* and + represent a nonnegative and a positive entry, respectively, and \( G_{rem} \) collects the “unnecessary” column of \( G \). Moreover, \( P^T (A_1 \ast A_2) P \) has all zero columns except, possibly, for those corresponding to the first columns of the blocks (namely the columns of indices \( 1, n_1 + 1, n_1 + n_2 + 1, \ldots \)) and \( P^T (B_1 \ast B_2) \begin{bmatrix} I_r \\ 0 \end{bmatrix} = 0. \)

REFERENCES


