Bootstrap approach to the multi-sample test of means with imprecise data

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Abstract

A bootstrap approach to the multi-sample test of means for imprecisely valued sample data is introduced. For this purpose imprecise data are modelled in terms of fuzzy values. Populations are identified with fuzzy-valued random elements, often referred to in the literature as fuzzy random variables. An example illustrates the use of the suggested method. Finally, the adequacy of the bootstrap approach to test the multi-sample hypothesis of means is discussed through a simulation comparative study.

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1. Introduction

The analysis of variance is concerned with analyzing variation in the means of several independent populations. The central point in classical ANOVA is a test (for the significance of the difference among population means), which allows us to conclude whether or not the differences among the sample means are too great to be attributed to the sampling error.

Populations are traditionally identified with the probability distributions of certain variables or vectors involved in random experiments, so that a single real number or vector is assigned to each and every possible experimental outcome. However, there are several real-life populations in which imprecise values can be assigned to each and every possible experimental outcome. In this way, random sets and, more generally, fuzzy random variables (in Puri and Ralescu’s sense, 1986), are suitable models to formalize and handle these populations.

In Montenegro et al. (2004a) an exact one-way ANOVA testing procedure for the case in which the involved fuzzy random variables are normal (as intended by Puri and Ralescu, 1985) has been presented. An introduction to the asymptotic multi-sample testing of means for simple fuzzy random variables is also sketched in the paper. It can be noted that in Cuevas et al. (2004) a one-way ANOVA study has been developed for functional data of a given Hilbert space; nevertheless, although in the present paper data involved are special upper semicontinuous functions, operations between them do not lead to a Hilbert space, but to a cone in it.

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On the other hand, in recent studies (see Montenegro et al., 2004b), the convenience of incorporating bootstrap methods to approximate the asymptotic one-sample tests with fuzzy random variables has been verified.

The aim of this paper is to present a bootstrap approach to the multi-sample test of means for the significance of the difference among (fuzzy-valued) population means on the basis of the evidence supplied by a set of sample fuzzy data (which include interval data as particular instances). The resulting procedure is based on measuring differences between fuzzy-valued means by considering an appropriate generalized metric, and using later bootstrap techniques to obtain an approximation of the asymptotic distribution of a statistic based on a resampling from a reference sample.

The approach will be illustrated by means of an example.

The absence of widely applicable probability distribution models for fuzzy random variables makes the study in this paper more interesting. For purposes of discussing the suitability of the bootstrap approach with fuzzy data in contrast to the case of real-valued data, a simulation study is finally developed.

### 2. Preliminaries

Fuzzy random variables were first introduced by Kvakernaak (1978, 1979) and Kruse and Meyer (1987) have later formalized the notion in greater rigor. More precisely, Kruse and Meyer introduced fuzzy random variables as an extension of random variables and to model the situations in which experiments conducted on a population of individuals assign an imprecise (fuzzy) value to each and every possible experimental outcome. Statistical methods based on Kruse and Meyer’s notion consider either real/vector parameters of the original distribution or some fuzzy parameters defined in terms of an extension principle (see Kruse and Meyer, 1987; Kruse et al., 1999).

On the other hand, fuzzy random variables in Puri and Ralescu’s (1986) sense were introduced as an extension of random sets and to model the situations in which fuzziness arises in either the observation or the report of a classical random variable (which is often referred to as the original). Statistical methods based on Kruse and Meyer’s notion consider either real/vector parameters of the original distribution or some fuzzy parameters defined in terms of an extension principle (see Kruse and Meyer, 1987; Kruse et al., 1999).

Let $\mathcal{X}_c(\mathbb{R})$ be the class of non-empty compact intervals, and let $\mathcal{F}_c(\mathbb{R})$ denote the class of normal and convex upper semicontinuous fuzzy sets of $\mathbb{R}$ with bounded closure of the support, that is, $\mathcal{F}_c(\mathbb{R}) = \{\widetilde{B} : \mathbb{R} \to [0, 1] \mid \widetilde{B}_z \in \mathcal{X}_c(\mathbb{R})\}$ for all $z \in [0, 1]$ where the $z$-level sets are defined as

$$\widetilde{B}_z = \begin{cases} \{x \in \mathbb{R} \mid \widetilde{B}(x) \geq z\} & \text{if } z \in (0, 1], \\ \text{cl} \{x \in \mathbb{R} \mid \widetilde{B}(x) > 0\} & \text{if } z = 0. \end{cases}$$

This class is often referred to in the literature as the class of fuzzy numbers.

$\mathcal{F}_c(\mathbb{R})$ can be endowed with an inner composition law extending the Minkowski addition between sets, and an external one which is the product by a scalar. These laws are compatible with the ones obtained by applying Zadeh’s (1975) extension principle. Thus, for all $\widetilde{B}, \widetilde{C} \in \mathcal{F}_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$ we can define $\widetilde{B} \circ \widetilde{C}$ and $\lambda \widetilde{B}$ so that for all $z \in [0, 1]$

$$\left(\widetilde{B} \circ \widetilde{C}\right)_z = \widetilde{B}_z + \widetilde{C}_z = \{b + c \mid b \in \widetilde{B}_z, c \in \widetilde{C}_z\},$$

$$\left(\lambda \widetilde{B}\right)_z = \lambda \widetilde{B}_z = \{\lambda b \mid b \in \widetilde{B}_z\}.$$

**Definition 2.1.** Given a probability space $(\Omega, \mathcal{A}, P)$, a mapping $X : \Omega \to \mathcal{F}_c(\mathbb{R})$ is said to be an $\mathcal{F}_c(\mathbb{R})$-valued fuzzy random variable (also called random fuzzy set, random fuzzy variable or random upper semicontinuous function) associated with $(\Omega, \mathcal{A})$ if, whatever $z \in [0, 1]$ may be, the $z$-level mapping

$$X_z : \Omega \to \mathcal{X}_c(\mathbb{R}) \quad \text{with} \quad X_z(\omega) = (X(\omega))_z \quad \text{for all } \omega \in \Omega$$

is a random compact convex set (i.e., it is measurable with respect to the Borel $\sigma$-field on $\mathcal{X}_c(\mathbb{R})$ associated with the Hausdorff metric on this space). We will say that an $\mathcal{F}_c(\mathbb{R})$-valued fuzzy random variable is simple if the cardinality of $X(\Omega)$ is finite.
Puri and Ralescu (1986) defined fuzzy random variables in a more general setting by removing convexity and considering the class \( \mathcal{F}(R^p) \) of \([0, 1]\)-valued upper semicontinuous and normalized functions in \( R^p \) with compact 0-level. The study in this paper concerns the simple convex one-dimensional case, which fits properly a large number of real-life situations (see López-Díaz and Ralescu’s, 2006 work in this special issue).

Fuzzy random variables in Puri and Ralescu’s sense are random elements. More precisely, a fuzzy random variable is a Borel-measurable mapping with respect, for instance, to the Borel σ-field associated with the Skorokhod metric on \( \mathcal{F}(R) \) (cf. Colubi et al., 2001, 2002a; López-Díaz and Ralescu, 1999). As a consequence, there is an induced probability distribution associated with a fuzzy random variable, and notions like independent and identically distributed fuzzy random variables are trivially defined.

If \( \tilde{X} \) is an \( \mathcal{F}(R) \)-valued fuzzy random variable in the above sense (from now on simply referred to as a fuzzy random variable) then, for all \( x \in [0, 1] \), \( \inf \tilde{X}_x : \Omega \to R \) and \( \sup \tilde{X}_x : \Omega \to R \) are real-valued random variables. This assertion indicates that on \( \mathcal{F}(R) \), the Kruse and Meyer and the Puri and Ralescu definitions coincide formally, although they were conceived to handle different situations in practice.

A fuzzy random variable \( \tilde{X} \) is said to be integrally bounded if its magnitude \( \| \tilde{X}_0 \| \in L^1(\Omega, \mathcal{A}, P) \), where \( \| \tilde{X}_0 \| = \sup_{x \in \tilde{X}(\omega)} |x| \).

**Definition 2.2.** If \( \tilde{X} \) is an integrably bounded fuzzy random variable, the expected value (or fuzzy mean) of \( \tilde{X} \) is the unique fuzzy set in \( \mathcal{F}(R) \), \( \tilde{E}(\tilde{X}|P) \), such that

\[
\left( \tilde{E}(\tilde{X}|P) \right)_x = \text{Kudo–Aumann’s integral of } \tilde{X}_x
\]

\[
= \left\{ \int_{\Omega} f(\omega) dP(\omega) \ | \ f : \Omega \to R, \ f \in L^1(\Omega, \mathcal{A}, P), \ f \in \tilde{X}_x \text{ a.s.}[P] \right\},
\]

which in the \( \mathcal{F}(R) \)-valued case equals the compact interval

\[
\left[ E(\inf \tilde{X}_x|P), E(\sup \tilde{X}_x|P) \right]
\]

for all \( x \in [0, 1] \).

The suitability of the expected value above has been supported by some strong laws of large numbers (for instance, Colubi et al., 1999).

In the class \( \mathcal{F}(R) \) several metrics can be defined (see Diamond and Kloeden, 1994; Puri and Ralescu, 1981). For probabilistic aspects of fuzzy random variables, like measurability and limit theorems, some distances based on Hausdorff metric have been used (see, for instance, Colubi et al., 1999, 2002a; Krätschmer, 2002, 2004, 2006; Molchanov, 1999).

For previous statistical studies concerning fuzzy random variables, a generalized metric has been shown to be especially valuable and easy to handle and interpret (see, for instance, Montenegro et al., 2001, 2004b). This metric has been introduced in Bertoluzza et al. (1995), and a generalization on \( \mathcal{F}(R^p) \) has been later established in Körner and Náther (2002).

Bertoluzza et al.’s (1995) metric makes use of two previously specified weight normalized measures \( W \) and \( \varphi \) which can be formalized by means of two probability measures on the measurable space \( ([0, 1], B_{[0,1]}) \) \((B_{[0,1]} \text{ being the Borel } \sigma\text{-field on } [0, 1])\). More precisely, \( W \) is assumed to be associated with a non-degenerate distribution and \( \varphi \) is assumed to have a strictly increasing distribution function on \([0, 1]\) (to guarantee it is in fact a metric). It should be pointed out that although \( W \) and \( \varphi \) are formalized in terms of probability measures, they do not have a real stochastic meaning.

**Definition 2.3.** The \((W, \varphi)\)-distance between \( \tilde{B}, \tilde{C} \in \mathcal{F}(R) \) is defined by

\[
D_W^\varphi (\tilde{B}, \tilde{C}) = \sqrt{\int_{[0,1]} \int_{[0,1]} \left[ f_\tilde{B}(x, \lambda) - f_\tilde{C}(x, \lambda) \right]^2 dW(\lambda) d\varphi(x)}
\]

with \( f_\tilde{B}(x, \lambda) = \lambda \sup \tilde{B}_x + (1 - \lambda) \inf \tilde{B}_x \).
3. Bootstrap approach to multi-sample test of means for fuzzy random variables

Consider a factor which can act at \( J \) possible different levels and having fixed effects, and a response fuzzy random variable determining \( J \) independent populations (independence being intended as usual—see, for instance, Billingsley, 1995).

In this way, let \((\Omega_1, \mathcal{A}_1, P_1), \ldots, (\Omega_J, \mathcal{A}_J, P_J)\) be probability spaces, \( \mathcal{X}_1 : \Omega_1 \to \mathcal{F}_c(\mathbb{R}), \ldots, \mathcal{X}_J : \Omega_J \to \mathcal{F}_c(\mathbb{R})\)-valued fuzzy random variables associated with these spaces, and

\[
\tilde{\mu}_1 = \tilde{E}(\mathcal{X}_1|P_1), \ldots, \tilde{\mu}_J = \tilde{E}(\mathcal{X}_J|P_J)
\]

the corresponding population fuzzy means.

From the \( j \)th population, \( \mathcal{X}_j \), generate a simple random sample, \( \mathcal{X}_{j1}, \ldots, \mathcal{X}_{jn_j} \) (\( j = 1, \ldots, J \)). The goal for this section is testing the null hypothesis

\[ H_0 : \tilde{\mu}_1 = \cdots = \tilde{\mu}_J \]

on the basis of the available sample fuzzy information. This null hypothesis describes a testable partition of an underlying statistical family for the joint distribution of \( \mathcal{X}_1, \ldots, \mathcal{X}_J \).

For purposes of testing \( H_0 \), a crucial point is that of decomposing the total variation among sample fuzzy observations as the sum of the variation among the sample fuzzy means (the between-group variation) and the variation within the \( J \) groups (the within-group variation).

The sample fuzzy mean in the \( j \)th group is given by

\[
\bar{\mathcal{X}}_j = \frac{1}{n_j} \left( \mathcal{X}_{j1} \oplus \cdots \oplus \mathcal{X}_{jn_j} \right),
\]

and the total sample fuzzy mean is given by

\[
\bar{\mathcal{X}} = \frac{1}{n} \left( \mathcal{X}_{11} \oplus \cdots \oplus \mathcal{X}_{1n_1} \oplus \cdots \oplus \mathcal{X}_{J1} \oplus \cdots \oplus \mathcal{X}_{Jn_J} \right) = \frac{n_1}{n} \mathcal{X}_1 \oplus \cdots \oplus \frac{n_J}{n} \mathcal{X}_J
\]

(with \( n = n_1 + \cdots + n_J \)).

In virtue of the properties of function \( f(\cdot, \cdot) \) in metric \( D_W^\theta \) and properties of this metric, the following decomposition is obtained:

\[
\sum_{j=1}^{J} \sum_{i=1}^{n_j} [D_W^\theta (\mathcal{X}_{ji}, \bar{\mathcal{X}})]^2 = \sum_{j=1}^{J} n_j [D_W^\theta (\bar{\mathcal{X}}_j, \bar{\mathcal{X}})]^2 + \sum_{j=1}^{J} \sum_{i=1}^{n_j} [D_W^\theta (\mathcal{X}_{ji}, \mathcal{X}_j)]^2.
\]

As for the real-valued case, the approach to be considered in this paper will deal with the ratio

\[
\frac{\text{between-group variation}}{\text{within-group variation}} = \frac{\sum_{j=1}^{J} n_j [D_W^\theta (\bar{\mathcal{X}}_j, \bar{\mathcal{X}})]^2}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} [D_W^\theta (\mathcal{X}_{ji}, \mathcal{X}_j)]^2}.
\]

If fuzzy random variables \( \mathcal{X}_1, \ldots, \mathcal{X}_J \) are assumed to be simple, then there are vector-valued parameters satisfying regularity conditions which allow us to apply large sample theory results (more precisely, to apply standard techniques to find the asymptotic distribution of a quadratic form).

Thus, let \( \tilde{x}_{j1}, \ldots, \tilde{x}_{jL_j} \in \mathcal{F}_c(\mathbb{R}) \) be the different values the fuzzy random variable \( \mathcal{X}_j \) takes on the \( j \)th population, and \( p_{j1}, \ldots, p_{jL_j} \) the respective probabilities (i.e., \( P_j \left( \{ \omega_j \in \Omega_j | \mathcal{X}_j(\omega_j) = \tilde{x}_{jl_j} \} \right) = p_{jl_j} > 0, \, l_j = 1, \ldots, L_j, \sum_{l_j=1}^{L_j} p_{jl_j} = 1 \) for \( j = 1, \ldots, J \).

If \( \mathbf{p}_j = (p_{j1}, \ldots, p_{j(L_j-1)}) \in (0, 1)^{L_j-1} \), then the population fuzzy mean can be expressed as

\[
\tilde{\mu}_j = \tilde{E}(\mathcal{X}_j|\mathbf{p}_j) = p_{j1}\tilde{x}_{j1} \oplus \cdots \oplus p_{jL_j}\tilde{x}_{jL_j}.
\]
In this expression, the $\tilde{x}_{ij}$'s are fixed entities (functions), and the unknown parameters are the $p_{ij}$'s. They uniquely determine the unknown population fuzzy expected value $\tilde{\mu}_j$.

Analogously, if $X_{j1}, \ldots, X_{jn_j}$ is a simple random sample from $\mathcal{X}_j$ ($j = 1, \ldots, J$), the $J$ samples being independent, and $f_{n_j}$ denotes the (random) relative frequency of $\tilde{x}_{ij}$ in this sample, and $f_{n_j} = (f_{n_j1}, \ldots, f_{n_j(L_j-1)})$, we have that

$$\overline{X}_j = \tilde{E}(X | f_{n_j}) = f_{n_j1}\tilde{x}_{j1} \oplus \cdots \oplus f_{n_jL_j}\tilde{x}_{jL_j}.$$ 

In this expression, the $\tilde{x}_{ij}$'s are fixed entities (functions), and the $f_{n_j}$'s are random variables. They uniquely determine the sample fuzzy expected value $\overline{X}_j$.

Parameter $p_j$ satisfies regularity conditions allowing us to conclude that the sequence $\{f_{n_j}\}_{n_j}$ of maximum-likelihood estimators of $p_j$ is strongly consistent and it is asymptotically distributed in accordance with a normal $(L_j - 1)$-dimensional distribution $\mathcal{N}(p_j, [I_{\tilde{F}_j}^{(p_j)}]^{-1}/n_j)$ as $n_j \to \infty$, where the inverse of the positive definite Fisher information matrix at $p_j$, $I_{\tilde{F}_j}^{(p_j)}$, is

$$[I_{\tilde{F}_j}^{(p_j)}]^{-1} = [p_{ij}, (\delta_{ij \kappa_j} - p_{j \kappa_j})] \quad \text{with} \quad \delta_{ij \kappa_j} = \text{Kronecker delta}$$

for all $ij, \kappa_j \in \{1, \ldots, L_j - 1\}$. Furthermore, $n_j (f_{n_j} - p_j) I_{\tilde{F}_j}^{(p_j)} (f_{n_j} - p_j)^t$ converges in distribution to a chi-square random variable $\chi^2_{L_j - 1}$ as $n_j \to \infty$, for all $j \in \{1, \ldots, J\}$ (cf. Rao, 1973; Serfling, 1980). As a consequence, the following supporting result is got.

**Lemma 3.1.** For each $J$-tuple $(n_1, \ldots, n_J) \in \mathbb{N}^J$ and each $j \in \{1, \ldots, J\}$, consider $n_j$ fuzzy random variables, $X_{j1}, \ldots, X_{jn_j}$, associated with the probability space $(\Omega_j, \mathcal{F}_j, P_j)$, and being independent and identically distributed as $\mathcal{X}_j$.

Let $P_j\{\{\omega_j \in \Omega_j | \omega_j = \tilde{x}_{jj}\}\} = p_{jj} \in (0, 1)$, $l = 1, \ldots, L_j$, with $\sum_{l=1}^{L_j} p_{jl} = 1$ for all $j = 1, \ldots, J$. Assume the $J$ random samples are independent. Let $p = (p_1, \ldots, p_J)$ and let $(\tau_1, \ldots, \tau_J) \in (0, 1)^J$ so that for all $k \in \{1, \ldots, J\}, n_k/n \to \tau_k$ as $n_j \to \infty$ for all $j \in \{1, \ldots, J\}$.

Then,

(i) Let

$$\tilde{\mu} = \tau_1 \tilde{E}(X_1 | p_1) \oplus \cdots \oplus \tau_J \tilde{E}(X_J | p_J) = \tau_1 \tilde{\mu}_1 \oplus \cdots \oplus \tau_J \tilde{\mu}_J.$$ 

Then,

$$\left[D^{\theta}_W(\overline{X}_j, \overline{X})\right]^{2} \xrightarrow{a.s.} \left[D^{\theta}_W(\tilde{\mu}_j, \tilde{\mu})\right]^{2},$$

as $n_j \to \infty$ for all $j \in \{1, \ldots, J\}$ and $i \in \{1, \ldots, n_j\}$.

(ii) Let

$$\tilde{\mu}_{(n)} = \tilde{E}(X | p) = \frac{n_1}{n} \tilde{E}(X_1 | p_1) \oplus \cdots \oplus \frac{n_J}{n} \tilde{E}(X_J | p_J) = \frac{n_1}{n} \tilde{\mu}_1 \oplus \cdots \oplus \frac{n_J}{n} \tilde{\mu}_J.$$ 

Then, the sequence of random variables

$$\left\{\sqrt{n_1 + \cdots + n_J} \left[ \frac{\sum_{j=1}^{J} n_j [D^{\theta}_W(\overline{X}_j, \overline{X})]^2}{\sum_{j=1}^{J} n_j [D^{\theta}_W(\tilde{\mu}_j, \tilde{\mu})]^2} - \frac{\sum_{j=1}^{J} n_j [D^{\theta}_W(\tilde{\mu}_j, \tilde{\mu}_{(n)})]^2}{\sum_{j=1}^{J} n_j [D^{\theta}_W(\tilde{\mu}_j, \tilde{\mu})]^2} \right] \right\}_{n_1 \ldots n_J}$$
converges in distribution to a (one-dimensional) normal random variable $\mathcal{N}(0, \sigma^2(p))$, as $n_j \to \infty$ for all $j \in \{1, \ldots, J\}$, whenever $\sigma^2(p) > 0$, with

$$\sigma^2(p) = \nabla A^X(p) \frac{1}{B^X(p)} \left( \nabla A^X(p) \right)^{-1},$$

$$A^X(p) = \sum_{j=1}^{J} \tau_j \left[ D^0_W(\tilde{\mu}_j, \tilde{\nu}_j) \right]^2, \quad B^X(p) = \sum_{j=1}^{J} \sum_{i=1}^{L_j} p_{ji} \left[ D^0_W(\tilde{X}_{ji}, \tilde{\mu}_j) \right]^2,$$

$$\nabla A^X(p) / B^X(p) = \left( \frac{\partial}{\partial p_{11}} A^X(p), \ldots, \frac{\partial}{\partial p_{1(L_1-1)}} A^X(p) \right) / B^X(p), \ldots, \left( \frac{\partial}{\partial p_{L_1J-1}} A^X(p) \right) / B^X(p), \left( \frac{\partial}{\partial p_{L_1J}} A^X(p) \right) / B^X(p), \ldots, \left( \frac{\partial}{\partial p_{L_1J}} A^X(p) \right) / B^X(p).$$

\[
\begin{bmatrix}
\frac{I^F_{x_1}(p_1)}{\tau_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{\tau_j} \left[ I^F_{x_j}(p_j) \right]^{-1}
\end{bmatrix}
\]

(iii) If $\sigma^2(p) = 0$ and

$$h_{i,k,j}(p) = \frac{\partial^2}{\partial p_{ji} \partial p_{j'i,j'}} A^X(p) / B^X(p) > 0$$

for some $(i, j, j') \in \{1, \ldots, L_j - 1\} \times \{1, \ldots, L_j' - 1\}$ with $j, j' \in \{1, \ldots, J\}$, then

\[
\left\{ 2(n_1 + \cdots + n_j) \left[ \sum_{j=1}^{J} n_j \left[ D^0_W(\tilde{X}_j, \tilde{\nu}_j) \right]^2 - \sum_{j=1}^{J} \sum_{i=1}^{n_j} \left[ D^0_W(\tilde{X}_{ji}, \tilde{\mu}_j) \right]^2 \right] \right\}_{n_1, \ldots, n_j}
\]

is a sequence of random variables converging in distribution to a linear combination of, at most, $L_1 + \cdots + L_j - J$ independent chi-square variables with 1 degree of freedom as $n_j \to \infty$ for all $j \in \{1, \ldots, J\}$.

The equality $\tilde{E}(\tilde{X}_j|p_j) = \cdots = \tilde{E}(\tilde{X}_j|p_j)$ (i.e., the null hypothesis $H_0$) is equivalent to the condition

$$\sum_{j=1}^{J} n_j \left[ D^0_W(\tilde{X}_j, \tilde{\nu}_j) \right]^2 = 0,$$

and this entails the variance $\sigma^2(p)$ is null. As an implication from the last lemma and properties of the convergence in distribution and the convergence in probability of random variables and vectors (especially Slutsky Theorem), the asymptotic distribution under $H_0$ of the statistic

$$2(n_1 + \cdots + n_j) \frac{\sum_{j=1}^{J} n_j \left[ D^0_W(\tilde{X}_j, \tilde{\nu}_j) \right]^2}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} \left[ D^0_W(\tilde{X}_{ji}, \tilde{\mu}_j) \right]^2}$$

can be easily obtained. On the basis of this distribution, an asymptotic test can be constructed.

The empirical conclusions obtained for the one-sample test for the fuzzy mean in Montenegro et al. (2004b) indicate that the bootstrap approach is usually more powerful and much easier to apply than the asymptotic one. The asymptotic test suggested above is to be approximated by means of bootstrap techniques.
To get bootstrap populations with a common fuzzy mean from the available sample information in this case, one can add to each sample the mean of the other samples. In other words, for each \( j = 1, \ldots, J \) one can define a new fuzzy random variable \( \mathcal{W}_j \) taking on values \( \bar{x}_{j1} \oplus \bar{x}_{j} - j, \ldots, \bar{x}_{jL_j} \oplus \bar{x}_{j} - j \) with sample distribution vector \( f_{nj} = \left( f_{nj1}, \ldots, f_{njL_j} \right) \), where \( \bar{x}_{j} - j \) is the (fuzzy) sum of all the available sample means but the \( j \)th one, that is,

\[
\bar{x}_{j} - j = \bar{x}_{1} \oplus \cdots \oplus \bar{x}_{(j - 1)} \oplus \bar{x}_{(j + 1)} \oplus \cdots \oplus \bar{x}_{J}.
\]

Then, resampling from these new populations is considered, that is, for any \( j \in \{1, \ldots, J \} \) a large number of samples of \( n_j \) independent observations \( \mathcal{W}_{j1}, \ldots, \mathcal{W}_{jn_j} \) from population \( \mathcal{W}_j \) is drawn. The test statistic will be given by

\[
T_{(n_1, \ldots, n_J)} = \frac{\sum_{j=1}^{J} n_j \left[ D_W^0 \left( \mathcal{W}_j, \mathcal{W}^* \right) \right]^2}{\sum_{j=1}^{J} \sum_{i=1}^{n_j} \left[ D_W^0 \left( \mathcal{W}_{ji}, \mathcal{W}_j \right) \right]^2},
\]

where for all \( j \in \{1, \ldots, J \} \)

\[
\mathcal{W}_j^* = \frac{1}{n_j} \left( \mathcal{W}_{j1} \oplus \cdots \oplus \mathcal{W}_{jn_j}^* \right), \quad \mathcal{W}^* = \frac{n_1}{n} \mathcal{W}_1 \oplus \cdots \oplus \frac{n_J}{n} \mathcal{W}_J.
\]

**Theorem 3.2.** To test at the nominal significance level \( \alpha \in [0, 1] \) the null hypothesis

\[ H_0 : \bar{E} \left( \mathcal{X}_1 | \mathbf{p}_1 \right) = \cdots = \bar{E} \left( \mathcal{X}_J | \mathbf{p}_J \right) , \]

\( H_0 \) should be rejected if

\[
T_{(n_1, \ldots, n_J)} = \frac{\sum_{j=1}^{J} n_j \left[ D_W^0 \left( \mathcal{X}_j, \mathcal{X} \right) \right]^2}{\sum_{j=1}^{J} \left( 1/n_j^2 \right) \sum_{i=1}^{n_j} \left[ D_W^0 \left( \mathcal{X}_{ji}, \mathcal{X}_j \right) \right]^2} > z_\alpha,
\]

where \( z_\alpha \) is the 100(1 - \( \alpha \)) fractile of the distribution of \( T_{(n_1, \ldots, n_J)}^* \) (this distribution can be approximated by the Monte Carlo method).

Moreover, the probability of rejecting \( H_0 \) under the alternative hypothesis \( H_a \) (which assumes that there is not coincidence of all the fuzzy population means) converges to 1 as \( n_j \to \infty \) for all \( j = 1, \ldots, J \).

**Remark 3.1.** In practice, and especially when samples have quite different sizes, the replacement of the statistic in Theorem 3.2 and its bootstrap version by

\[
\frac{\sum_{j=1}^{J} n_j \left[ D_W^0 \left( \mathcal{X}_j, \mathcal{X} \right) \right]^2}{\sum_{j=1}^{J} \left( 1/n_j^2 \right) \sum_{i=1}^{n_j} \left[ D_W^0 \left( \mathcal{X}_{ji}, \mathcal{X}_j \right) \right]^2}
\]

usually leads to more stable results (see Babu and Singh, 1983; Efron and Tibshirani, 1998).

On the basis of the simple random sample \( \mathcal{X}_{j1}, \ldots, \mathcal{X}_{jn_j} \), from \( \mathcal{X}_j \) for each \( j \in \{1, \ldots, J \} \), the application of the test in Theorem 3.2 can be easily developed by using the following:

**Algorithm 3.3.**

1. **Step 1.** Compute the sample fuzzy mean for the \( j \)th group,

\[
\bar{\mathcal{X}}_j = \frac{1}{n_j} \left( \mathcal{X}_{j1} \oplus \cdots \oplus \mathcal{X}_{jn_j} \right),
\]

the total sample fuzzy mean

\[
\bar{\mathcal{X}} = \frac{n_1}{n_1 + \cdots + n_j} \bar{\mathcal{X}}_1 \oplus \cdots \oplus \frac{n_J}{n_1 + \cdots + n_J} \bar{\mathcal{X}}_J.
\]
and the value of the statistic

\[
\theta = \frac{\sum_{j=1}^{J} n_j \left[ D_W^\alpha (\mathcal{X}_j, \mathcal{X}) \right]^2}{\sum_{j=1}^{J} \left( 1/n_j^2 \right) \sum_{i=1}^{n_j} \left[ D_W^\alpha (\mathcal{X}_{ji}, \mathcal{X}_j) \right]^2}.
\]

**Step 2.** For each \( j \in \{1, \ldots, J\} \), compute the bootstrap population \( \mathcal{Y}_j \) taking on values (some of them may be coinciding)

\[
\mathcal{X}_{j1} \oplus \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_{(j-1)} \oplus \mathcal{X}_{(j+1)} \oplus \cdots \oplus \mathcal{X}_J,
\]

+ ........................................

\[
\mathcal{X}_{jn_j} \oplus \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_{(j-1)} \oplus \mathcal{X}_{(j+1)} \oplus \cdots \oplus \mathcal{X}_J
\]

chosen at random (i.e., each of them having frequency \( 1/n_j \)).

**Step 3.** For each \( j \in \{1, \ldots, J\} \), obtain a sample of independent and identically distributed fuzzy random variables \( \mathcal{Y}_{j1}^*, \ldots, \mathcal{Y}_{jn_j}^* \) from the \( j \)th bootstrap population \( \mathcal{Y}_j \).

**Step 4.** Compute the sample fuzzy mean for the \( j \)th ‘bootstrap group’,

\[
\bar{\mathcal{Y}}_j^* = \frac{1}{n_j} \left( \mathcal{Y}_{j1}^* \oplus \cdots \oplus \mathcal{Y}_{jn_j}^* \right),
\]

the total sample fuzzy mean

\[
\bar{\mathcal{Y}}^* = \frac{n_1}{n_1 + \cdots + n_j} \bar{\mathcal{Y}}_1^* \oplus \cdots \oplus \frac{n_j}{n_1 + \cdots + n_j} \bar{\mathcal{Y}}_j^*,
\]

and the value of the statistic

\[
\theta^* = \frac{\sum_{j=1}^{J} n_j \left[ D_W^\alpha (\bar{\mathcal{Y}}_j^*, \bar{\mathcal{Y}}^*) \right]^2}{\sum_{j=1}^{J} \left( 1/n_j^2 \right) \sum_{i=1}^{n_j} \left[ D_W^\alpha (\mathcal{Y}_{ji}^*, \bar{\mathcal{Y}}_j^*) \right]^2}.
\]

**Step 5.** Steps 3 and 4 should be repeated a large number \( B \) of times to get a set of \( B \) estimators, denoted by \( \{\theta_1^*, \ldots, \theta_B^*\} \).

**Step 6.** Compute the bootstrap \( p \)-value as the proportion of values in \( \{\theta_1^*, \ldots, \theta_B^*\} \) being greater than \( \theta \).

**Remark 3.2.** It should be pointed out that the choice of the test-statistic \( T \) has been based on the natural extension from the real-valued case, whose rationale is well established in the classical linear model theory. Of course, other choices would also be possible, like the one inspired in Cuevas et al. (2004), although expected inferential results are similar and the one suggested in this paper is suitable for the empirical comparisons with the real-valued case which will be later developed in Section 5.

4. **Illustrative example**

To illustrate the application of the introduced method the following example is considered.

**Example 4.1.** Clients of a savings bank are classified in accordance with their ‘degree of aversion’ to investment. Because of the labels assigned to clients by the bank managers, the classification process can be viewed as a fuzzy random variable \( \mathcal{X} \) which takes on values

\[
\tilde{x}_1 = ‘\text{very low degree of aversion}', \quad \tilde{x}_2 = ‘\text{low degree of aversion}',
\]

\[
\tilde{x}_3 = ‘\text{medium degree of aversion}', \quad \tilde{x}_4 = ‘\text{high degree of aversion}',
\]

\[
\tilde{x}_5 = ‘\text{almost total aversion}',
\]

with these five values being identified with fuzzy ones described, for instance, by the fuzzy numbers in Fig. 1 (which are defined in terms of \( S \)-curves, see for instance Cox, 1994).
More precisely,

\[
\bar{x}_1 = \begin{cases} 
1 & \text{in } [0, 6.25], \\
1 - S(6.25, 18.75) & \text{in } [6.25, 18.75], \\
0 & \text{otherwise},
\end{cases}
\]

\[
\bar{x}_i = \begin{cases} 
S(a_i, a_i + 12.5) & \text{in } [a_i, a_i + 12.5], \\
1 & \text{in } [a_i + 12.5, a_i + 25], \\
1 - S(a_i + 25, a_i + 37.5) & \text{in } [a_i + 25, a_i + 37.5], \\
0 & \text{otherwise}
\end{cases}
\]

for \(i = 2, 3, 4\), \(a_2 = 6.25\), \(a_3 = 31.25\), \(a_4 = 56.25\),

\[
\bar{x}_5 = \begin{cases} 
S(75, 100) & \text{in } [75, 100], \\
0 & \text{otherwise}
\end{cases}
\]

with

\[
S(a, b)(t) = \begin{cases} 
0 & \text{if } t \leq a, \\
2 \left( \frac{t - a}{b - a} \right)^2 & \text{if } t \in \left[ a, \frac{a + b}{2} \right], \\
1 - 2 \left( \frac{t - b}{b - a} \right)^2 & \text{if } t \in \left[ \frac{a + b}{2}, b \right], \\
1 & \text{otherwise}.
\end{cases}
\]

Bank managers are interested in testing the equality of the ‘mean degrees of aversion’ in the six main offices of the savings bank in a given area, which will be denoted by \(\Omega_1\), \(\Omega_2\), \(\Omega_3\), \(\Omega_4\), \(\Omega_5\), and \(\Omega_6\). For this purpose, assume they consider a whole sample of \(n = 80\) clients, and observe fuzzy random variable \(\bar{x}\) on the subsamples of sizes \(n_1 = 11\), \(n_2 = 14\), \(n_3 = 12\), \(n_4 = 12\), \(n_5 = 16\), and \(n_6 = 15\), from the six populations, in which the fuzzy data obtained are those in Table 1.

If \(W\) and \(\varphi\) are both the Lebesgue measure on \([0, 1]\), by following the steps in Algorithm 3.3 with \(B = 1000\) bootstrap replications, one can conclude that the \(p\)-value equals 0.029, so that at nominal level 0.05 the equality of the fuzzy means in the six offices should be rejected, but it could be accepted at lower levels such as 0.025 or 0.01.

**Remark 4.1.** It is worth remarking that although the classical chi-square test for homogeneity could also be applied to the example above, the null hypothesis to be tested is different from the one we have considered. In fact, as for the real-valued case, the rejection of the homogeneity hypothesis would not necessarily imply that of the equality of (fuzzy) means. Furthermore, the chi-square statistic test would not be always consistent with the null hypothesis tested in this paper, since deviations from this null hypothesis do not necessarily mean deviations from the null hypothesis.
for homogeneity; more precisely, in analyzing the power of the test (as made in the simulation comparative studies in Section 5) fuzzy variable values can be modified without modifying probabilities so that the power of the chi-square test would remain invariant.

5. Simulation studies on the bootstrap approach

As commented above, the empirical conclusions obtained for one-sample tests for the fuzzy mean (see Montenegro et al., 2004b) indicate that the bootstrap technique is usually better and much easier to apply than the asymptotic one (see Körner, 2000, and Montenegro et al., 2004b) although it requires a greater computational cost. For this reason the discussion on the validity of the bootstrap is skipped.

Simulations are considered to justify the validity of the statistical treatment of fuzzy data suggested in the preceding sections. More precisely, we are going to present simulation studies to compare the test in Algorithm 3.3 (\(A_1\)) with the analogous (\(A_2\)) for the real-valued case when values of variables are defuzzified by means of the 0.5-average function (Yager, 1981; Gil and López-Díaz, 1996; Gil et al., 1998 for other statistical applications).

Simulations have been based on the ideas in Colubi et al. (2002b). They have been reduced to the case \(J = 3\), and for weighted measures \(W\) and \(\varphi\) coinciding with the Lebesgue measure on \([0, 1]\). Each simulation corresponds to 40,000 iterations and the number of bootstrap replications was 1000.

First, a comparison concerning the probabilities of type I error has been developed. In this way, Table 2 assembles the obtained outputs for the percentage of rejections at the nominal significance level 0.05, for some different sample size allocations (\(n_j\)'s being equal to either 30 or 100). On the basis of these simulations, one can notice that tests \(A_1\) and \(A_2\) show a quite similar behavior, and they improve for samples of the same size (as happens in the real-valued case).

On the other hand, Table 3 shows the power of tests \(A_1\) and \(A_2\) (more precisely, the percentage of rejections at the nominal significance level 0.05) which has been analyzed by considering the three sample sizes being equal to 30, and

<table>
<thead>
<tr>
<th>(n_1, n_2, n_3)</th>
<th>(30, 30, 30)</th>
<th>(30, 30, 100)</th>
<th>(30, 100, 100)</th>
<th>(100, 100, 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1) (fuzzy-valued)</td>
<td>5.11</td>
<td>4.74</td>
<td>4.49</td>
<td>4.97</td>
</tr>
<tr>
<td>(A_2) (real-valued)</td>
<td>5.09</td>
<td>4.69</td>
<td>4.48</td>
<td>5.01</td>
</tr>
</tbody>
</table>
Table 3
Simulations for the power of the bootstrap multi-sample test of means

<table>
<thead>
<tr>
<th>Fuzzy deviation</th>
<th>$\tilde{d}_0$</th>
<th>$\tilde{d}_1$</th>
<th>$\tilde{d}_2$</th>
<th>$\tilde{d}_3$</th>
<th>$\tilde{d}_4$</th>
<th>$\tilde{d}_5$</th>
<th>$\tilde{d}_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ (fuzzy-valued)</td>
<td>4.77</td>
<td>6.77</td>
<td>13.18</td>
<td>29.94</td>
<td>57.24</td>
<td>88.99</td>
<td>99.85</td>
</tr>
<tr>
<td>$A_2$ (real-valued)</td>
<td>4.77</td>
<td>5.99</td>
<td>8.96</td>
<td>15.28</td>
<td>24.30</td>
<td>37.28</td>
<td>50.81</td>
</tr>
</tbody>
</table>

In Table 3 a clear loss of power due to the defuzzification process can be appreciated.

6. Concluding remarks and future directions

The bootstrap approach in this paper can be applied to testing the equality of distributions of real-valued random variables by converting sample data into fuzzy ones in accordance with a characterizing fuzzification like the one by González-Rodríguez et al. (2006b) in this special issue.

The results in Section 3 could be extended to non-simple fuzzy random variables and even for $F_c (\mathbb{R}^p)$-valued ones by considering Körner and Näther’s (2002) metric along with bootstrapped $J$-sample results for Banach space-valued random elements, and the embedding of $F_c (\mathbb{R}^p)$ into a Banach space through the support function (by following the ideas in González-Rodríguez et al., 2006a).

Multi-sample tests of means for fuzzy random variables should be complemented, on one hand, with posterior tests for multiple comparison and, on the other hand, with other types of inferences (namely, fuzzy estimates, confidence regions, and so on).

Finally, it would be also interesting to develop in the near future some robustness studies concerning the effects of the choice for weighted measures $W$ and $\varphi$ as well as the choice of the fuzzy numbers considered to describe the available imprecise data.

Acknowledgments

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Appendix A

Proof of Lemma 3.1. (i) The vector-valued parameters $\mathbf{p}_j$ (and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_J)$) satisfies the regularity conditions allowing us to apply large sample theory results guaranteeing that, under the assumed conditions, $\left\{ \mathbf{f}_{nj} \right\}_{nj}$ is the sequence of the maximum-likelihood estimators of $\mathbf{p}_j$ (respectively, $\left\{ \mathbf{f}_n \right\}_n$ is the sequence of the maximum-likelihood estimators of $\mathbf{p}$) and, hence, it is strongly consistent. Consequently, as $n_j \to \infty$ for all $j \in \{1, \ldots, J\}$

$$\left[ D^0_W (\tilde{E}(\mathcal{X} | \mathbf{f}_{nj}), \tilde{E}(\mathcal{X} | \mathbf{f}_n)) \right]^2 \xrightarrow{a.s.} \left[ D^0_W (\tilde{E}(\mathcal{X} | \mathbf{p}_j), \tilde{E}(\mathcal{X} | \mathbf{p})) \right]^2,$$

$$\left[ D^0_W (\tilde{E}(\mathcal{X}_{ji} | \mathbf{f}_{nj}), \tilde{E}(\mathcal{X}_{ji} | \mathbf{f}_n)) \right]^2 \xrightarrow{a.s.} \left[ D^0_W (\tilde{E}(\mathcal{X}_{ji} | \mathbf{p}_j), \tilde{E}(\mathcal{X}_{ji} | \mathbf{p})) \right]^2.$$
(ii) On the other hand, since \{\hat{f}_{nj}\}_{nj} is the sequence of the maximum-likelihood estimators of \(p_j\) (respectively, \{f_n\}_n is the sequence of the maximum-likelihood estimators of \(p\)), it is asymptotically normal.

More precisely, \(n_j^{-1/2} (f_{nj} - p_j)\) is asymptotically \(\mathcal{N}(0, \left[ I_{\mathcal{X}}(p_j) \right]^{-1})\) as \(n_j \to \infty\) (respectively, \(n^{-1/2} (f_n - p)\) is asymptotically \(\mathcal{N}(0, \left[ I_{\mathcal{X}}(p) \right]^{-1})\) as \(n \to \infty\) for all \(j \in \{1, \ldots, J\}\)).

Let

\[
A_n^{\mathcal{X}}(p) = \sum_{j=1}^{J} \frac{n_j}{n} [D_W^\varphi (E (\mathcal{X}_j | p_j), E(\mathcal{X} | p))]^2,
\]

\[
B_n^{\mathcal{X}}(p) = \sum_{j=1}^{J} \frac{n_j}{n} \sum_{i=1}^{L_j} p_{ji} [D_W^\varphi (\mathcal{X}_{ji}, \tilde{E}(\mathcal{X}_j | p_j))]^2.
\]

If the first order Taylor expansion of

\[
\frac{A_n^{\mathcal{X}}(f_n)}{B_n^{\mathcal{X}}(f_n)} = \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} + \nabla \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} (f_n - p)^t + \frac{1}{2} (f_n - p) H \left( \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} \right) (f_n - p)^t,
\]

where

\[
H \left( \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} \right) = \left[ h_{ijk}^{\mathcal{X}}(p) \right]
\]

is the Hessian matrix and \(p_n^* = (p_{n1}^*, \ldots, p_{nJ}^*)\) is such that \(\|p_{n_j} - p_j\| \leq \|f_{nj} - p_j\|\) for all \(j \in \{1, \ldots, J\}\). Therefore,

\[
\sqrt{n} \left[ \frac{A_n^{\mathcal{X}}(f_n)}{B_n^{\mathcal{X}}(f_n)} - \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} \right] = \nabla \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} (\sqrt{n} (f_n - p))^t + \frac{1}{2} (f_n - p) H \left( \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} \right) (\sqrt{n} (f_n - p))^t.
\]

Taking into account that \(\nabla (A_n^{\mathcal{X}}(p)/B_n^{\mathcal{X}}(p))\) converges almost sure to \(\nabla (A^{\mathcal{X}}(p)/B^{\mathcal{X}}(p))\) as \(n_j \to \infty\) for all \(j \in \{1, \ldots, J\}\), if \(\sigma^2(p) > 0\) the sequence

\[
\left\{ \nabla \frac{A_n^{\mathcal{X}}(p)}{B_n^{\mathcal{X}}(p)} (\sqrt{n} (f_n - p))^t \right\}_n
\]

converges in distribution to the normal distribution

\[
\mathcal{N} \left( 0, \nabla \frac{A^{\mathcal{X}}(p)}{B^{\mathcal{X}}(p)} \left[ I_{\mathcal{X}}^{\mathcal{X}}(p) \right]^{-1} \left( \nabla \frac{A^{\mathcal{X}}(p)}{B^{\mathcal{X}}(p)} \right)^t \right)
\]

as \(n_j \to \infty\) for all \(j \in \{1, \ldots, J\}\), and from Slutsky Theorem the convergence in probability of the residual term in the Taylor expansion to zero is derived, whence the result is immediately concluded.

(iii) Since \(I_{\mathcal{X}}^{\mathcal{X}}(p)^{-1}\) is positive definite, the quadratic forms associated with \(I_{\mathcal{X}}^{\mathcal{X}}(p)\) and \(I_{\mathcal{X}}^{\mathcal{X}}(p)^{-1}\) are also positive definite, whence \(\sigma^2(p) = 0\) implies that \(\nabla (A^{\mathcal{X}}(p)/B^{\mathcal{X}}(p)) = 0\).
If the second order Taylor expansion of \( A_n^{\mathcal{F}} (f_n) / B_n^{\mathcal{F}} (f_n) \) is considered, we have that
\[
\frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} = \frac{A_n^{\mathcal{F}} (p)}{B_n^{\mathcal{F}} (p)} + \nabla \frac{A_n^{\mathcal{F}} (p)}{B_n^{\mathcal{F}} (p)} (f_n - p) + \frac{1}{2} (f_n - p)^t \frac{1}{B_n^{\mathcal{F}} (p)} (f_n - p) t + \frac{1}{6} \sum_{j=1}^{J} \sum_{i,j=1}^{L_j-1} \sum_{j'=1}^{L_j'} \sum_{k,j'=1}^{L_{j''}-1} \sum_{l,j''=1}^{L_{j'''}} \left( \frac{\partial^3 (A_n^{\mathcal{F}} (p)/B_n^{\mathcal{F}} (p))}{\partial p_{j,j'} \partial p_{j',k,j'} \partial p_{j'',l,j''}} \right)_{p=p_n^{\circ\circ}} (f_{n,j,i,j} - f_{n,j,i}) (f_{n,j',k,j'} - f_{n,j',k'}) (f_{n,j'',l,j''} - f_{n,j'',l})
\]
with \( \| p_{n,j,i}^{\circ\circ} - f_{n,j} \| \leq \| f_{n,j} - p_{n,j} \| \) for all \( j \in \{1, \ldots, J\} \).

By arguing like for the one-sample case in Montenegro et al. (2004b), one can conclude that
\[
\left\{ (\sqrt{n} (f_n - p)) \cdot H \left( \frac{A_n^{\mathcal{F}} (p)}{B_n^{\mathcal{F}} (p)} \right) \cdot (\sqrt{n} (f_n - p))^t \right\}_n
\]
converges in distribution to
\[
\lambda_1 U_1^2 + \cdots + \lambda_q U_q^2,
\]
where \( \lambda_1, \ldots, \lambda_q \) \( (q \leq L_1 + \cdots + L_J - J) \) are the non-null eigenvalues of the matrix
\[
DH \left( \frac{A_n^{\mathcal{F}} (p)}{B_n^{\mathcal{F}} (p)} \right) D^t,
\]
\[
DD^t = \left[ I_d^{\mathcal{F}} (p) \right]^{-1} \text{ and } U_1, \ldots, U_q \text{ are independent and identically distributed as a normal } \mathcal{N}(0, 1). \]

**Proof of Theorem 3.2.** To prove this result we are going to check that \( T_n^* \) has an asymptotic distribution and, under the null hypothesis, this coincides with the asymptotic one for \( T_n \).

In this sense, first of all note that
\[
T_n^* = A_n^{\mathcal{F}} (f_n^*) / B_n^{\mathcal{F}} (f_n^*) .
\]

If the second order Taylor expansion of \( A_n^{\mathcal{F}} (f_n^*) / B_n^{\mathcal{F}} (f_n^*) \) in a neighborhood of \( f_n \) is considered, we have that
\[
\frac{A_n^{\mathcal{F}} (f_n^*)}{B_n^{\mathcal{F}} (f_n^*)} = \frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} + \nabla \frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} (f_n^* - f_n) t + \frac{1}{2} (f_n^* - f_n) \cdot H \left( \frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} \right) (f_n^* - f_n)^t + \frac{1}{6} \sum_{j=1}^{J} \sum_{i,j=1}^{L_j-1} \sum_{j'=1}^{L_j'} \sum_{k,j'=1}^{L_{j''}-1} \sum_{l,j''=1}^{L_{j'''}} \left( \frac{\partial^3 (A_n^{\mathcal{F}} (p)/B_n^{\mathcal{F}} (p))}{\partial p_{j,j'} \partial p_{j',k,j'} \partial p_{j'',l,j''}} \right)_{p=p_n^{\circ\circ}} (f_{n,j,i,j}^* - f_{n,j,i}) (f_{n,j',k,j'}^* - f_{n,j',k'}) (f_{n,j'',l,j''}^* - f_{n,j'',l})
\]
with \( \| p_{n,j,i}^{\circ\circ} - f_{n,j} \| \leq \| f_{n,j}^* - f_{n,j} \| \) for all \( j \in \{1, \ldots, J\} \).

It can be verified that
\[
\frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} = 0, \quad \nabla \frac{A_n^{\mathcal{F}} (f_n)}{B_n^{\mathcal{F}} (f_n)} (f_n^* - f_n)^t = 0.
\]
Since \( B_n^y(f_n) = B_n^x(f_n) \), we obtain that for all \((i_k, l_{k'}) \in \{1, \ldots, L_k - 1\} \times \{1, \ldots, L_{k'} - 1\}\) with \(k, k' \in \{1, \ldots, J\}\),

\[
H \left( \frac{A_n^y(f_n)}{B_n^y(f_n)} \right)_{i_k,l_{k'}} = \left( \frac{A_n^y(f_n)}{B_n^y(f_n)} \right)_{i_k,l_{k'}}^{-2} \left( \frac{\partial^2 A_n^y(p)}{\partial p_{j_k} \partial p_{j_{k'}}} \right)_{p=f_n} = 2 \int_{[0,1]^2} \left( f_{X_{ik}} - f_{X_{ik}'} \right) \left( f_{X_{i'k'}} - f_{X_{i'k'}}' \right) dW \text{d} \phi \sum_{j=1}^J \frac{j}{n} \left( \delta_{j_k} - n_k/n \right) \left( \delta_{j_{k'}} - n_{k'}/n \right)
\]

which converges almost surely to \(h_{i_k,l_{k'}}\) (in Lemma 3.1(iii)) as \(n_j \to \infty\) for all \(j \in \{1, \ldots, J\}\).

In addition, classical bootstrap results (see, for instance, Shao and Tu, 1995) guarantee that \(\sqrt{n} (f_n^* - f_n)\) is asymptotically normal \(\mathcal{N} \left( 0, \left[ \int f^2_x(p) \right]^{-1} \right)\) as \(n_j \to \infty\) for all \(j \in \{1, \ldots, J\}\). Consequently, by arguing like for the one-sample case in Montenegro et al. (2004b) the sequence

\[
\left\{ \left( f_n^* - f_n \right) H \left( \frac{A_n^y(f_n)}{B_n^y(f_n)} \right) \left( f_n^* - f_n \right) \right\}
\]

converges in distribution to \(\lambda_1 U_1^2 + \cdots + \lambda_q U_q^2\), where \(\lambda_1, \ldots, \lambda_q\) (\(q \leq L_1 + \cdots + L_J - J\)) are the non-null eigenvalues of the matrix

\[
DH \left( \frac{A_x^x(p)}{B^x_x(p)} \right) D^t,
\]

with \(DD^t = \left[ \int f^2_x(p) \right]^{-1}\) and \(U_1, \ldots, U_q\) are independent and identically distributed as a normal \(\mathcal{N}(0, 1)\).

Finally, regarding to the remainder term, we can prove that

\[
n^{-1/4} \left( \frac{\partial^3 (A_n^y(p)/B_n^y(p))}{\partial p_{j_i} \partial p_{j_{k'}} \partial p_{j_{k'}}'} \right)_{p=f_n^y} \xrightarrow{a.s.} 0
\]

as \(n_j \to \infty\) for all \(j \in \{1, \ldots, J\}\) and consequently from Slutsky Theorem the convergence in probability of the residual term in the Taylor expansion to zero is derived, whence the result is immediately concluded. \(\Box\)

References