Disturbance decoupling problem for a class of descriptor systems with delay via systems over rings

A. M. Perdon* and M. Anderlucci

Dipartimento di Ingegneria Informatica, Gestionale e dell’Automazione, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

*Corresponding author: perdon@univpm.it andrerlucci@diiga.univpm.it

[Received on 31 January 2010; revised on 8 March 2010; accepted on 6 April 2010]

The disturbance decoupling problem by state feedback for descriptor systems with a finite number of commensurable point delays is formulated and investigated using as models descriptor systems with coefficients in a ring. Necessary and sufficient conditions for its solution are given in geometric terms, as well as algorithmic procedures to test them. Examples are worked out in details.

Keywords: delay systems; systems over a ring; descriptor systems; disturbance decoupling; geometric approach.

1. Introduction

In recent years, a considerable amount of research has been devoted to linear time-invariant descriptor systems (called also singular or differential–algebraic) and to delay differential systems because of their extensive applications. The presence of time delays, in fact, arises in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. and may cause undesirable system transient response or even instability.

Several results in literature are devoted to stability issues for descriptor systems with delays under the hypothesis that the considered system is ‘impulse free’, namely that the solution does not possess dynamical infinite modes nor impulsive behaviour (see, for instance, Liu & Xie, 1996; Fridman, 2002; Xu et al., 2002, 2005; Ma & Cheng, 2005). In this paper, under the same hypotheses, we will investigate the problem of finding a feedback law such that disturbances do not affect the output.

The approach we follow here relies on the possibility of using mathematical models with coefficients in a suitable ring, or systems over a ring, for studying and analysing delay differential dynamical systems (see Conte & Perdon, 1982; Kamen, 1991 and references therein). The use of systems over rings is motivated by the fact that it avoids the necessity of dealing with infinite-dimensional vector spaces, like the state space of classical delay differential systems, and, in place, it allows the use of finite-dimensional modules. Although ring and module algebra is more rich and complicated than linear algebra, this makes possible to extend a number of techniques from the framework of linear dynamical systems, in particular the geometric approach.

The geometric approach for the analysis and synthesis of linear systems with coefficients in a field (see Basile & Marro, 1969, 1992; Wonham, 1985) proved very effective in the solution of decoupling problems. The disturbance decoupling problem (DDP) for linear time-invariant singular systems (without delay) over a field was first formulated and solved by Fletcher & Aasarai (1989) and, more recently, in Wang et al. (2004) by different methods. A geometric theory for dynamical systems over a ring,
similar to that existing over a field, has been developed (see Conte & Perdon, 2000, 2005) and in recent years, it has been extended to neutral and to descriptor systems over a ring (see, for instance, Perdon & Anderlucci, 2006, 2007, 2008).

In this paper, we will present a geometric solution of the DDP for descriptor systems over a ring that can be applied, in particular, to a class of descriptor systems with a finite number of commensurable delays.

Necessary and sufficient conditions for the existence of a feedback such that in the compensated system the output is not influenced by the disturbances will be given and computational aspects will also be discussed.

The example in the paper has been worked out using the package control.cpkg, written in the freely available software CoCoA (see CoCoATeam, n.d.; Perdon et al., 2006). The package has been enriched with new procedures and now it provides effective procedures to check geometric solvability conditions and also to compute solutions for singular and neutral systems over a ring.

2. Descriptor systems with delays and systems over rings

Assume that $\Sigma_d$ is the linear, time-invariant system with a finite number of commensurable point delays, described by the equations

$$
\Sigma_d = \begin{cases} 
\sum_{i=0}^e E_i \dot{x}(t - ih) = \sum_{i=0}^a A_i x(t - ih) + \sum_{i=0}^b B_i u(t - ih), \\
y(t) = \sum_{i=0}^c C_i x(t - ih), \\
x(t) = \varphi(t), \quad t \in [-\alpha h, 0] \quad \alpha > 0,
\end{cases}
$$

where, denoting by $\mathbb{R}$ the field of real numbers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $h \in \mathbb{R}^+$ is the delay, $\alpha = \max(e, a, b, c)$, $\varphi(t)$ is a consistent initial condition, $E_i$, $A_i$, $B_i$ and $C_i$ are matrices of suitable dimensions with entries in $\mathbb{R}$.

Remark that the class of descriptor systems we consider is more general than that usually considered in the literature since delays may appear in the state derivative, in particular it contains delay systems of neutral type.

The literature about systems over rings is quite large. Example of problems and applications of systems over rings are found, in particular, in Kamen (1975), Sontag (1981), Brewer & Vleck (1986), Conte & Perdon (2005), Perdon & Anderlucci (2008) and in the references therein.

By introducing the delay operator $\delta$ defined, for any time function $f(t)$, by $\delta f(t) := f(t - h)$, we can write

$$
\begin{cases} 
\sum_{i=0}^e E_i \delta \dot{x}(t) = \sum_{i=0}^a A_i \delta x(t) + \sum_{i=0}^b B_i \delta u(t), \\
y(t) = \sum_{i=0}^c C_i \delta x(t).
\end{cases}
$$

By formally replacing the delay operator $\delta$ with the algebraic indeterminate $\Delta$, we define

$$E = \sum_{i=0}^e E_i \Delta^i, \quad A = \sum_{i=0}^a A_i \Delta^i, \quad B = \sum_{i=0}^b B_i \Delta^i, \quad C = \sum_{i=0}^c C_i \Delta^i$$

$E$, $A$, $B$ and $C$ are matrices of suitable dimensions with entries in the ring $\mathcal{R} = \mathbb{R}[\Delta]$ of real polynomials in one indeterminate.
In order to help the intuition, it is convenient to associate to the abstract system $\Sigma = (E, A, B, C)$ over the ring $R = \mathbb{R}[\Delta]$ a set of equations of the form

$$\Sigma = \begin{cases} 
Ex(t+1) = Ax(t) + Bu(t), \\
y(t) = Cx(t),
\end{cases} \quad (2.2)$$

where $t$ represents an integer variable, $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ belong, respectively, to the free modules $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$.

For systems over rings, free modules play the role of vector spaces in the description of classical dynamical systems and, although a physical interpretation of (2.2) is generally missing, referring to them we can speak, in an obvious way, of states, inputs and outputs and of a discrete time dynamics for $\Sigma$.

Remark that, by abuse of notations, we used the same letters to represent corresponding variables in (2.1) and (2.2), although they have different meaning, and that the systems $\Sigma_d$ and $\Sigma$ are quite different objects from a dynamical point of view. The key point, however, is that $\Sigma_d$ and $\Sigma$ present the same signal flow graph, so that control problems concerning the input/output behaviour of $\Sigma_d$ can be naturally formulated in terms of the input/output behaviour of $\Sigma$ and possibly solved in the framework of systems over rings. In turn, solutions found in that framework can be brought back to the original delay differential framework.

Although from a technical point of view the study of systems over rings may be more difficult, they are quite a kind of linear systems with real coefficients and, for this reason, the basic ideas of control theory find a natural extension to their framework. This makes them a versatile tool for approaching a number of physically meaningful problems and motivates a concrete interest in them.

In dealing with systems over rings, one has to pay attention to the fact that ring and module algebra is more rich and complicated than linear algebra. Since non-zero elements in a ring are not necessarily invertible, a linear dependency relation like $\sum_{i=1}^{n} a_i x_i = 0$ between elements of a free module over a ring $R$ does not imply that each $x_i$ is a linear combination of the remaining ones.

So, differently from what happens in the case of vector spaces, we can find sets of generators of a free module from which no basis can be extracted, as well as sets of linearly independent elements of maximal cardinality which are not sets of generators. In particular, we may have submodules of a free module which are not direct summands.

Dealing with descriptor systems over $\mathbb{R}$, usually the pencil $\lambda E - A$ is assumed to be ‘regular’, i.e. $\det(\lambda E - A) \neq 0$. It should be remarked that regularity is not invariant with respect to feedback. For a discussion on this topic, see Özçaldiran & Lewis (1990). The regularity assures that the pencil has a ‘standard canonical form’ also called ‘Weierstrass decomposition’ (see Gantmacher, 1966). Namely, there exist non-singular matrices $P$ and $Q$ such that

$$P(\lambda E - A)Q = \lambda \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix}, \quad (2.3)$$

where $J$ is strictly upper triangular, i.e. nilpotent, and $E$ and $A$ are identically partitioned. The ‘index’ of the pencil is the index of nilpotency of $J$, namely the minimum non-negative integer $\nu$ such that $J^\nu = 0$, with $J^{\nu-1} \neq 0$.

By convention, if $E$ is non-singular, namely $\det(E)$ is a non-zero constant, the pencil $(E, A)$ is said to be of index $\nu = 0$.

Any regular descriptor system can be decomposed into a dynamic part (differential equations) and a possibly anticipative part (see, for instance, Malabre, 1989; Lewis, 1990).
If a descriptor system over $\mathbb{R}$ has index larger than one, then it has classical continuous solutions only if the input $u(t)$ has a certain smoothness, otherwise impulses may arise in the response of the system. The system is ‘impulse free’, namely a smooth response is assured for every continuous $u(t)$, only if $J = 0$, i.e. the system is regular and of index at most one.

2.1 Canonical form over the ring $\mathcal{R}$

If $\det(E)$ is an invertible element of the ring $\mathcal{R}$, trivially $E^{-1}$ is again a matrix with entries in $\mathcal{R}$ and the system can be written as the state space system $(I, E^{-1}A, E^{-1}B, C)$ over $\mathcal{R}$. Recall that every non-zero real constant is invertible in the polynomial ring $\mathbb{R}[\Delta]$. Therefore, we will assume that $\det(E) = 0$ or $\det(E)$ is a non-invertible element of the ring, i.e. in case of the polynomial ring $\mathbb{R}[\Delta]$, $\det(E)$ is a non-constant polynomial.

For a descriptor system $\Sigma$ defined by (2.2) over a ring $\mathcal{R}$, the fact that $\det(\lambda E - A) \neq 0$, does not imply that the system has a canonical form.

This problem has been investigated over Hermite rings in Cobb (2006) introducing the notion of ‘algebraic solvability’. Since principal ideal domains (PIDs), such as the polynomial ring $\mathbb{R}[\Delta]$ we are interested in, are Hermite rings, these results apply in our case.

**Definition 2.1** A pair of matrices $(E, A)$ over an Hermite ring $\mathcal{R}$, is algebraically solvable if there exist unimodular matrices $P$ and $Q$ in $\mathbb{R}^{n \times n}$ such that the pencil $P(\lambda E - A)Q$ is in canonical form (2.3).

It is ‘unit index’ if it can be put in canonical form (2.3) with $J = 0$.

Remark that left multiplication and right multiplication by unimodular matrices $P$ and $Q$, respectively, correspond to row operations on the system’s equations (2.2) and to a change of coordinates in the state module, respectively.

Algebraic solvability may be difficult to establish, but necessary conditions, easier to check, have been established in Cobb (2006).

**Definition 2.2** A pair of matrices $(E, A)$ over an Hermite ring $\mathcal{R}$, is said ‘pre-solvable’ if any of the following conditions holds:

\begin{align*}
PS1 & : \text{Im}E + A \ker E = \mathcal{R}^n, \\
PS2 & : \text{Im}E \cap A \ker E \neq \{0\}, \\
PS3 & : \ker E \cap \ker A \neq \{0\}.
\end{align*}

**Proposition 2.1** (Cobb, 2006) Pre-solvability of a pencil $(E, A)$ is a necessary condition for algebraic solvability.

**Definition 2.3** A system (2.2) over an Hermite ring is ‘impulse free’ if it can be put in canonical form with $J = 0$. It is ‘impulse controllable’ if and only if can be made algebraically solvable and impulse free by a feedback of the form $u(t) = Fx(t)$.

**Proposition 2.2** (Cobb, 2006) Let $\mathcal{R}$ be an Hermite ring. The pair $(E, A)$ is impulse controllable (by proportional feedback) if and only if

(i) $\text{rank}(E) = \rho(E),$

(ii) $\text{Im}E + A \ker E + \text{Im}B = \mathcal{R}^n,$

(iii) $(E, A)$ is presolvable.
We recall that the above notations relate to the following definitions:

$$\text{rank}(E) = \max\{k | E \text{ has a non-zero } k\text{-th order minor}\}$$

and

$$\rho(E) = \max\{k | \text{the } k\text{-th order minors of } E \text{ satisfy a Bézout identity}\},$$

where, if \(x_1, \ldots, x_k \in \mathbb{R}\), a Bézout identity is an equation of the form \(\sum_{i=1}^{k} a_i x_i = 1 (a_i \in \mathbb{R})\).

An integral domain in which Bézout’s identity holds is called a Bézout domain. The PIDs, such as the polynomial ring \(\mathbb{R}[\Delta]\), are Bézout domains.

The greatest common divisor of \(x_1, \ldots, x_k \in \mathbb{R}\) is in fact the smallest positive integer that can be written as a linear combination of \(x_1, \ldots, x_k \in \mathbb{R}\), so that \(\rho(E)\) is the maximum order of co-prime minors of \(E\).

In Perdon & Anderlucci (2009) was considered the problem of impulse elimination by proportional plus derivative feedback (DF) of the form \(u(t) = Fx(t) + Kx(t + 1)\) for singular systems over a PID and constructive procedures were given to achieve this goal. The use of the term ‘derivative’, improper in this context, is due to the fact that, when the system over the PID \(\mathbb{R}[\Delta]\) is a model for a delay system over \(\mathbb{R}\), the action of this kind of feedback corresponds to the action of a DF \(u(t) = Fx(t) + K\dot{x}(t)\) on the original delay system.

**Definition 2.4** The descriptor system \(\Sigma = (E, A, B)\) over a ring \(\mathbb{R}\) is ‘proportional-derivative algebraically solvable’ if there exist suitable matrices \(K \in \mathbb{R}^{n \times m}\) and \(F \in \mathbb{R}^{m \times m}\) such that the pair \(((E + BK), (A + BF))\) is algebraically solvable, namely, has a canonical form for suitable unimodular matrices \(P\) and \(Q\) in \(\mathbb{R}^{m \times n}\).

Remark that, the fact that \(K\) and \(F\) are matrices with elements in the ring, means that, going back to the original delay system we will have feedbacks possibly with delays.

**Definition 2.5** (Perdon & Anderlucci, 2009) The descriptor system \(\Sigma = (E, A, B)\) over a ring \(\mathbb{R}\) is ‘DF impulse controllable’ if there exists a matrix \(K \in \mathbb{R}^{m \times n}\) such that \((E + BK, A)\) is unit index, namely algebraically solvable with \(J = 0\).

**Proposition 2.3** Assume that for the descriptor systems \(\Sigma\) described by equations of the form (2.2) over the ring \(\mathbb{R}\)

\[
\text{Im } E + \text{Im } B = \mathbb{R}^t, \quad t \leq n,
\]

i.e. the columns of \(E\) and \(B\) span a submodule isomorphic to \(\mathbb{R}^t\). Then there exist matrices \(P\), \(Q\) and \(K\) of suitable dimensions such that \(E + BK = \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}\), i.e. the singular system reduces to a state space system plus \(n - t\) algebraic equations.

### 3. Disturbance decoupling problem

Assume that in the descriptor system with delay

\[
\begin{align*}
\sum_{i=0}^{\epsilon} E_i \dot{x}(t - ih) &= \sum_{i=0}^{\alpha} A_i x(t - ih) + \sum_{i=0}^{\beta} B_i u(t - ih) + \sum_{i=0}^{\delta} D_i q(t - ih), \\
y(t) &= \sum_{i=0}^{\epsilon} C_i x(t - ih), \\
x(t) &= \varphi(t), \quad t \in [-\alpha h, 0] \quad \alpha > 0
\end{align*}
\]
a disturbance \( q(t) \in \mathbb{R}^q \) is present. The ‘disturbance decoupling problem with measurable disturbances’ (DDPMs) consists in finding an integer \( r \geq 0 \) and a dynamic feedback law of the form

\[
\begin{align*}
\dot{x}_a(t) &= \sum_{i=0}^{l} L_i x_a(t - i h) + \sum_{i=0}^{g} G_{1i} q(t - i h), \\
u(t) &= \sum_{i=0}^{f} F_i x(t - i h) + \sum_{i=0}^{h} H_i x_a(t - i h) + \sum_{i=0}^{g} G_{2i} q(t - i h),
\end{align*}
\]

(3.2)

where \( x_a \in \mathbb{R}^r \), \( L_i \), \( F_i \), \( H_i \), \( G_{1i} \) and \( G_{2i} \) are matrices of suitable dimensions with entries in \( \mathbb{R} \), such that the compensated system is regular and its output is not affected by the disturbance \( q \).

### 3.1 DDP over \( \mathbb{R} \)

For the descriptor system over the ring \( \mathbb{R} = \mathbb{R}[\Delta] \), associated to the system (3.1) as described in Section 3,

\[
\begin{align*}
Ex(t + 1) &= Ax(t) + Bu(t) + Dq(t), \\
y(t) &= Cx(t),
\end{align*}
\]

(3.3)

where \( D = \sum_{i=0}^{d} D_i \Delta^i \), the DDPMs consists in finding an integer \( r \geq 0 \) and a dynamic feedback law of the form

\[
\begin{align*}
x_a(t + 1) &= L x_a(t) + G_1 q(t), \\
u(t) &= F x(t) + H x_a(t) + G_2 q(t),
\end{align*}
\]

(3.4)

where \( x_a \in \mathbb{R}^r \), \( A_1 \), \( F \), \( H \), \( G_1 \) and \( G_2 \) are matrices of suitable dimensions with entries in the ring \( \mathbb{R} \), such that the compensated system is regular and its output is not affected by the disturbance \( q \).

If the disturbances are not measurable, the DDP consists in finding an integer \( r \geq 0 \) and a dynamic feedback law of the form (3.4), where \( G_1 = G_2 = 0 \), such that the compensated system is regular and its output is not affected by the disturbance \( q \).

If \( r = 0 \), (3.4) reduces to a static feedback law.

A key tool in the geometric approach is the notion of controlled invariance.

**Definition 3.1 (Hautus, 1982)** Given a linear system defined over a ring \( \mathbb{R} \) by equations of the form

\[
\begin{align*}
x(t + 1) &= Ax(t) + Bu(t) + Dq(t), \\
y(t) &= Cx(t),
\end{align*}
\]

(3.5)

a submodule \( \mathcal{V} \subset \mathcal{X} = \mathbb{R}^n \) is ‘\((A, B)\)-invariant’ or ‘controlled invariant’ if \( A \mathcal{V} \subset \mathcal{V} + \text{Im}B \) and ‘\((A + B F)\)-invariant’ or controlled invariant of feedback type, shortly ‘feedback invariant’, if there exists a linear map \( F: \mathcal{X} \to \mathcal{V} \) such that \( (A + B F)\mathcal{V} \subset \mathcal{V} \). Any such \( F \) is called a ‘friend’ of \( \mathcal{V} \).

The family of controlled invariant submodules contained in a given submodule \( \mathcal{K} \subset \mathbb{R}^n \) is closed with respect to the sum, then over a Noetherian ring, in particular a PID, it has a maximum element denoted by \( \mathcal{V}^* (\mathcal{K}) \) or simply \( \mathcal{V}^* \) if no confusion may arise.

A procedure to practically compute \( \mathcal{V}^* \) over a PID, for instance, when \( \mathbb{R} = \mathbb{R}[\Delta] \), can be found in Assan et al. (1999) and Perdon et al. (2006).

Necessary and sufficient conditions for the solution of DDP and DDPM for systems over a ring have been given in geometric terms for retarded systems over a PID in Conte & Perdon (1995, 2000) and in the case of neutral systems over a PID in Perdon & Anderlucci (2008).

For lack of space, we recall here only one of these results.
Proposition 3.1 (Conte & Perdon, 1995) Given the system (3.5), denote by $\mathcal{V}^*$ the maximum controlled invariant submodule in Ker $C$. Then, the DDPM is solvable for the system (3.5) if and only if

$$\text{Im } D \subseteq \mathcal{V}^* + \text{Im } B.$$  \hfill (3.6)

If $\mathcal{V}^*$ is a controlled invariant submodule of feedback type, a static solution of the form $u(t) = Fx(t) + Gq(t)$ is given by any friend $F$ of $\mathcal{V}^*$ and a suitable $G$ since the feedback action of $F$ causes $\mathcal{V}^*$ to become invariant with respect to the closed-loop dynamics and forces the image of the disturbance to evolve in Ker $C$. Obviously, any other feedback invariant submodule $\mathcal{V} \subseteq \text{Ker } C$ such that $\text{Im } D \subseteq \mathcal{V} + \text{Im } B$ can be used to define, by any friend, a solution.

If maps take values in vector spaces and matrices have entries in a field such as $\mathbb{R}$, then controlled invariance and feedback invariance are equivalent; but over a ring this is no longer true. If the hypotheses of Proposition 3.1 are satisfied but the controlled invariant $\mathcal{V}^*$ is not of feedback type, then a dynamic feedback solving the problem can be obtained by the following procedure.

Assume that $\mathcal{V}^*$ has dimension $r$ and denote by $V \in \mathbb{R}^{m \times r}$ a matrix whose columns span $\mathcal{V}^*$, then the properties of $\mathcal{V}^*$ and (3.6) imply that there exist matrices $L \in \mathbb{R}^{m \times r}, M \in \mathbb{R}^{m \times q}, G_1 \in \mathbb{R}^{r \times q}$ and $G_2 \in \mathbb{R}^{m \times d}$ such that

$$CV = 0, \quad AV = VL + BM, \quad D = VG_1 + BG_2.$$  \hfill (3.7)

Define

$$x_e(t) = \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix}, \quad u_e(t) = \begin{bmatrix} u(t) \\ u_a(t) \end{bmatrix},$$

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0_{r \times r} \end{bmatrix}, \quad B_e = \begin{bmatrix} B & 0 \\ 0 & I_{r \times r} \end{bmatrix}, \quad D_e = \begin{bmatrix} D \\ G_1 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0_{r \times r} \end{bmatrix}$$

and consider the extended system $\Sigma_e$ given by the equations

$$\Sigma_e = \begin{cases} x_e(t + 1) = A_e x_e(t) + B_e u_e(t) + D_e q(t), \\
                     y(t) = C_e x_e(t). \end{cases}$$  \hfill (3.8)

The submodule $\mathcal{V}_e \subseteq \mathcal{X}_e = \mathcal{X} \oplus \mathbb{R}^r$ generated by the columns of the matrix $V_e = \begin{bmatrix} V & I_{r \times r} \end{bmatrix}$ is, by construction, a direct summand of $\mathcal{X}_e$ and a controlled invariant submodule contained in Ker $C_e$ since

$$A_e V_e = \begin{bmatrix} AV \\ 0_{r \times r} \end{bmatrix} = \begin{bmatrix} V \\ I_{r \times r} \end{bmatrix} L + \begin{bmatrix} B & 0 \\ 0 & I_{r \times r} \end{bmatrix} \begin{bmatrix} M \\ -L \end{bmatrix} = V_e L + B_e M \begin{bmatrix} M \\ -L \end{bmatrix}. \hfill (3.9)$$

The dimension of the dynamic extension can be reduced if $\mathcal{V}^*$ contains a submodule $\mathcal{I}$ which is a direct summand of the state space $\mathcal{X}$. In such case, assume that the columns of the matrix $S$ span the submodule $\mathcal{I}$ and that that they can be completed to a set of generators for $\mathcal{V}^*$. Assume that the columns of the matrix $\begin{bmatrix} S & V_2 \end{bmatrix}$ span $\mathcal{V}^*$ and define $V_e = \begin{bmatrix} S & V_2 \\ 0 & I_{r \times r_2} \end{bmatrix}$. Then $\mathcal{V}_e = \text{span} \{ V_e \}$ is a direct summand of $\mathcal{X} \oplus \mathbb{R}^{r_2}$ so that the dimension of the extension will be $r_2 = r - \dim(\mathcal{I})$. 


Over a commutative Noetherian ring, a controlled invariant submodule which is a direct summand of the state space is feedback invariant (see Conte & Perdon, 1998, Proposition 1), then $\mathcal{V}_e$ is feedback type and it is easy to check that

$$F_e = \begin{bmatrix} 0_{m \times n} & -M \\ 0_{r \times n} & L \end{bmatrix}$$

(3.10)
is a friend of $\mathcal{V}_e$ and the dynamic feedback

$$\begin{cases} x_a(t + 1) = Lx_a(t) + G_1q(t), \\ u(t) = -Mx_a(t) - G_2q(t) \end{cases}$$

(3.11)
solves the DDPM for the system (3.5).

An algorithm for computing $\mathcal{V}^*$ is available only over a PID (see Assan et al., 1999). A different result based on the notion of pre-controllability submodule provides a computable solution when $\mathcal{R}$ is a more general Noetherian ring (see Conte & Perdon, 1995, 2000, for details).

4. Problem solution

The DDP for descriptor systems over a field has been studied and solved assuming that the system is ‘regularizable’, namely that it can be transformed into a closed-loop system that is regular and of index at most one by proportional and/or DF and changes of bases (see Özçaldiran, 1987) or that the system was ‘impulse controllable’ (see, for instance, Wang et al., 2004). In the more general case of systems over the ring $\mathcal{R}[\Delta]$, we therefore assume that the descriptor system (2.2) is ‘impulse controllable’, namely that it can be put in canonical form with $J = 0$, for instance, by a proportional feedback of the form

$$u(t) = +Fx(t)$$

if it satisfies the hypotheses of Proposition 2.2 or by a proportional plus DF of the form

$$u(t) = -Kx(t+1) + Fx(t)$$

if it satisfies the hypotheses of Proposition 2.3.

Let us assume that the descriptor system $\Sigma$ defined over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ by equations

$$\begin{cases} Ex(t + 1) = Ax(t) + Bu(t) + Dq(t), \\ y(t) = Cx(t) \end{cases}$$

(4.1)

with $s = \text{rank } E$ and the state $x(\cdot)$, the control input $u(\cdot)$, the output $y(\cdot)$ and the disturbance $q(\cdot)$ belong, respectively, to the free modules $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$, $Q = \mathbb{R}^q$, is algebraically solvable and unit index. Then there exist unimodular matrices $P$ and $Q$ such that

$$PEQ = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}, \quad PAQ = \begin{pmatrix} \tilde{A} & 0 \\ 0 & I \end{pmatrix}, \quad PB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad PD = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad CQ = (C_1 \quad C_2).$$

(4.2)

Denoting by $Q^{-1}x(t) = [z(t) \quad \tilde{z}(t)]'$, $z(t) \in \mathbb{R}^s$ and $\tilde{z}(t) \in \mathbb{R}^{n-s}$, we can write

$$\begin{cases} z(t + 1) = \tilde{A}z(t) + B_1u(t) + D_1q(t), \\ 0 = \tilde{z}(t) + B_2u(t) + D_2q(t), \\ y(t) = C_1z(t) + C_2\tilde{z}(t). \end{cases}$$
Moreover, since \( \tilde{z}(t) = -B_2u(t) - D_2q(t) \), the system (4.1) is equivalent to the retarded system \( \tilde{\Sigma} \)

\[
\tilde{\Sigma} = \begin{cases}
z(t + 1) = \tilde{A}z(t) + B_1u(t) + D_1q(t), \\
y(t) = C_1z(t) - C_2B_2u(t) - C_2D_2q(t).
\end{cases}
\tag{4.3}
\]

Existing geometric results on the DDPM for systems over rings concern systems without feed-through term, therefore we shall reformulate the problem as a DDPM for a suitably extended system of this form. Let us denote by \( \hat{\Sigma} \) the extended system defined over \( \mathbb{R} \) by the equations

\[
\hat{\Sigma} = \begin{cases}
w(t + 1) = \hat{A}w(t) + \hat{B}u(t) + \hat{D}q(t), \\
y(t) = \hat{C}w(t)
\end{cases}
\tag{4.4}
\]

with

\[
w(t) = \begin{bmatrix} z(t) \\ z_e(t) \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \tilde{A} & 0 \\ C_1 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ -C_2B_2 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D_1 \\ -C_2D_2 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & I \end{bmatrix}.
\]

**Proposition 4.1** With the above notations, denote by \( \hat{V}^* \) the maximum controlled invariant submodule of \( \hat{\Sigma} \) contained in \( \text{Ker} \hat{C} \). Then the DDPM for (4.3) is solvable if and only if

\[
\text{Im} \hat{D} \subseteq \hat{V}^* + \text{Im} \hat{B}.
\tag{4.5}
\]

We can express the solvability conditions in terms of the original data.

**Proposition 4.2** With the above notations, assume that \([V \ V']\) is a matrix whose columns span \( \hat{V}^* \). Then condition (4.5) holds if and only if there exist matrices \( L, M, G_1 \) and \( G_2 \) of suitable dimension such that

\[
\tilde{A}V = VL + B_1M, \quad C_1V + C_2B_2M = 0,
\]

\[
D_1 = VG_1 + B_1G_2, \quad C_2(D_2 - B_2G_2) = 0.
\tag{4.6}
\]

**Proof of Proposition 4.2**. First of all, \( \hat{V}^* \subseteq \text{Ker} \hat{C} \) implies \( W = 0 \) and (4.5) hold if and only if there exist matrices \( L, M, G_1 \) and \( G_2 \) of suitable dimension such that

\[
\hat{A} \begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} L + \hat{B}M, \quad \hat{D} = \begin{bmatrix} V \\ 0 \end{bmatrix} G_1 + \hat{B}G_2.
\]

**Proof of Proposition 4.1**. By Proposition 3.1, (4.5) is a necessary and sufficient condition for the existence of a feedback

\[
\begin{cases}
x_a(t + 1) = Lx_a(t) + G_1q(t), \\
u(t) = -Mx_a(t) - G_2q(t)
\end{cases}
\tag{4.7}
\]

solving the DDPM for the system (4.4).
In fact, the transfer function from the disturbance to the output is, for the compensated system,

\[
\begin{bmatrix}
\hat{C} & 0
\end{bmatrix}
\begin{bmatrix}
zI - \hat{A} & \hat{B}M \\
0 & zI - L
\end{bmatrix}^{-1}
\begin{bmatrix}
-\hat{B}G_2 + \hat{D} \\
G_1
\end{bmatrix}
\]

that is identically zero, since, by (4.6)

\[
\begin{bmatrix}
-\hat{B}G_2 + \hat{D} \\
G_1
\end{bmatrix}
= \begin{bmatrix} \hat{V} \\ I \end{bmatrix} G_1
\]

and, denoting \( \hat{C}_e = [\hat{C} \ 0] \),

\[
\hat{V}_e = \begin{bmatrix} \hat{V} \\ I \end{bmatrix}
\]

is contained in \( \text{Ker} \hat{C}_e \) and is invariant with respect to the dynamic of the extended system. This also implies that the same feedback (4.7) solves the DDPM for the system (4.3). In fact, the equations of the compensated system

\[
\begin{align*}
z(t + 1) &= \hat{A}z(t) - B_1 M x_a(t) + (-B_1 G_2 + D_1)q(t), \\
x_a(t + 1) &= Lx_a(t) + G_1 q(t), \\
y(t) &= C_1 z(t) + C_2 B_2 M x_a(t) + C_2 (B_2 G_2 - D_2)q(t)
\end{align*}
\]

reduce to

\[
\begin{align*}
\begin{bmatrix}
z(t + 1) \\
x_a(t + 1)
\end{bmatrix} &= \begin{bmatrix}
\hat{A} & -B_1 M \\
0 & L
\end{bmatrix}
\begin{bmatrix}
z(t) \\
x_a(t)
\end{bmatrix}
+ \begin{bmatrix} \hat{V}G_1 \\ G_1 \end{bmatrix} q(t), \\
y(t) &= C_1 z(t) - C_1 V \begin{bmatrix} z(t) \\ x_a(t) \end{bmatrix}
\end{align*}
\]

since, by (4.6), \(-B_1 G_2 + D_1 = \hat{V}G_1 \) and \( C_2 (B_2 G_2 - D_2) = 0 \). Then the transfer function from the disturbance to the output is identically zero since

\[
\begin{bmatrix} \hat{V}G_1 \\ G_1 \end{bmatrix}
\]

is contained in the kernel of the output map and is invariant with respect to the dynamics, being

\[
\begin{bmatrix}
C_1 & -C_1 V
\end{bmatrix}
\begin{bmatrix} \hat{V} \\ I \end{bmatrix} G_1 = 0
\]

and

\[
\begin{bmatrix}
\hat{A} & -B_1 M \\
0 & L
\end{bmatrix}
\begin{bmatrix} \hat{V} \\ I \end{bmatrix} G_1 = \begin{bmatrix} \hat{A}V - B_1 M \\ L \end{bmatrix} G_1 = \begin{bmatrix} \hat{V} \\ I \end{bmatrix} LG_1.
\]

For lack of space, we have not discussed the stability of the compensated system. For a discussion on stability in the solution of DDPM for state space systems over a ring, the reader is referred to Conte & Perdon (2007).
EXAMPLE 4.1 Consider the generalized state system with delays $\Sigma_d$:
\[
\begin{align*}
\dot{x}_1(t) + 3\dot{x}_2(t - 4h) - \dot{x}_2(t - 3h) &= x_1(t - h) + x_2(t) - 2x_2(t - 3h) + 6x_2(t - 4h) \\
+ 3x_3(t - 2h) - x_3(t - h) - 3x_4(t - 3h) + x_4(t - 2h) \\
+ u_1(t) + 3u_2(t - 2h) - u_2(t - h) + 3q(t - 5h) - q(t - 4h) \\
\dot{x}_2(t - 2h) &= 2x_2(t - 2h) + x_3(t) - x_4(t - h) + u_2(t) + q(t - 3h) \\
- \dot{x}_3(t) &= -2x_2(t - h) + x_4(t) - q(t - 2h) \\
\dot{x}_2(t) - \dot{x}_2(t - 2h) &= 2x_2(t) - 2x_2(t - 2h) - x_3(t) + x_4(t - h) - u_2(t) + q(t - h) - q(t - 3h) \\
y(t) &= x_1(t) + x_3(t).
\end{align*}
\]

The associated system $\Sigma = (E, A, B, C, D)$ over the ring $R = \mathbb{R}[\Delta]$ is given by
\[
E = \begin{bmatrix} 1 & 3\Delta^4 - \Delta^5 & 0 & 0 \\ 0 & \Delta^2 & 0 & 0 \\ 0 & -\Delta & 0 & 0 \\ 0 & 1 - \Delta^2 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \Delta & 6\Delta^4 - 2\Delta^3 + 1 & 3\Delta^2 - \Delta & -3\Delta^3 + \Delta^2 \\ 0 & 2\Delta^2 & 1 & -\Delta \\ 0 & -2\Delta^2 & 0 & 1 \\ 0 & -2\Delta^2 + 2 & -1 & \Delta \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3\Delta^2 - \Delta \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 3\Delta^5 - \Delta^4 \\ \Delta^3 \\ -\Delta^2 \\ -\Delta^3 + \Delta \end{bmatrix}, \quad C = [1 \ 0 \ 1 \ 0].
\]

Left multiplication by the unimodular matrix
\[
P = \begin{bmatrix} 1 & -3\Delta^2 + \Delta & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & \Delta & 0 \\ 0 & \Delta & 1 & \Delta \end{bmatrix},
\]

which corresponds to row operations on the system’s equations, leads to the equivalent system in canonical form:
\[
E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \Delta & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \Delta \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1 \ 0 \ 1 \ 0]
\]

that can be written as
\[
\begin{align*}
z(t + 1) &= \begin{bmatrix} \Delta & 1 \\ 0 & 2 \end{bmatrix} z(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \Delta \end{bmatrix} q(t), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} q(t),
\end{align*}
\]
\[
\begin{align*}
  z(t+1) &= \begin{bmatrix} \Delta & 1 \\ 0 & 2 \end{bmatrix} z(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \Delta \end{bmatrix} q(t), \\
  y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 & -1 \end{bmatrix} u(t).
\end{align*}
\]

The extended system \( \hat{\Sigma} \) becomes
\[
\begin{align*}
  w(t+1) &= \begin{bmatrix} \Delta & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \Delta \\ 0 \end{bmatrix} q(t), \\
  y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} w(t).
\end{align*}
\]

The maximum controlled invariant submodule in \( \text{Ker} \hat{C} \) is \( \hat{V}^* = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). Since
\[
\text{Im} \hat{D} = \text{span} \begin{bmatrix} 0 \\ \Delta \\ 0 \end{bmatrix} \subseteq \hat{V}^* + \text{Im} \hat{B} = \text{span} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{span} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},
\]

(4.5) holds and we can find the matrices that satisfy (4.6). This computation can be performed by using the package control.cpkg (see CoCoATeam, n.d.; Perdon et al., 2006) or, for instance, Maple®.

Several solutions are possible, we propose
\[
L = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} \Delta & 1 \\ -1 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -1 \\ \Delta \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The dynamic feedback solving the DDPM for \( \hat{\Sigma} \) is
\[
\begin{align*}
  x_a(t+1) &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} x_a(t) + \begin{bmatrix} -1 \\ \Delta \end{bmatrix} q(t), \\
  u(t) &= -\begin{bmatrix} \Delta & 1 \\ -1 & 0 \end{bmatrix} x_a(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t).
\end{align*}
\]

By formally replacing the algebraic indeterminate \( \Delta \) with the delay operator \( \delta \), we obtain the dynamic feedback solving the DDPM for the descriptor system with delays defined by (4.10)
\[
\begin{align*}
  \dot{x}_{a1}(t) &= -q(t), \\
  \dot{x}_{a2}(t) &= 2x_{a2}(t) + q(t - h), \\
  u_1(t) &= -x_{a1}(t - h) - x_{a2}(t) - q(t), \\
  u_2(t) &= x_{a1}(t).
\end{align*}
\]
5. Concluding remarks
In this paper, we have investigated the DDP via measurable disturbances for delay differential descriptor systems with delays, using as models systems over a ring of polynomials.

Conditions for the existence of a, possibly dynamic, state feedback that achieves decoupling are given in geometric terms. The solvability conditions can be checked and the feedback solving the problem, if it does exist, can be found using the routines contained in the package control.cpkg, written in the freely available software CoCoA (CoCoATeam, n.d.) and several new ones developed for singular systems over a ring or by other software such as Maple®.

Acknowledgment
The authors thank the anonymous reviewers for their constructive comments and valuable suggestions that were helpful in improving the article.

REFERENCES


