Abstract—The use of finite state Markov chains (FSMC) for the simulation of the Rayleigh channel has been generalized in the last years. Several parameters influence the construction of the chain. The chain can be defined in terms of the complex channel response, the envelope of the received signal or the Signal to Noise Ratio. Moreover the partition method has to be selected among various options. This paper presents an evaluation of the FSMC, including a novel partition method that makes a uniform division of the density function of the desired signal and a comparison with other methods of generation of the fading channel. First and second order statistics and other performance parameters are examined in order to shed more light on the advantages and limitations of the Markov modeling.

Index Terms—Finite State Markov Chain, flat fading channel, performance evaluation, ACF, second-order statistics

I. INTRODUCTION

The increasing demand for wireless communication has motivated further investigation of the channel dynamics and its modeling. An adequate representation of the communications channel is essential for studying the performance of a telecommunication system. In fact, the wireless channel has a complex mathematical characterization that turns out to be a handicap for the development of upper layer procedures (e.g. QoS provisioning) where it would be desirable to have information of the behavior of the underlying radio channel. For the incorporation of the channel effects in the design of high layer procedures, simpler models that capture the main properties of the wireless channel while avoiding complex mathematical formulations have been developed. Finite State Markov Chain (FSMC) [1] [2] is one of these models and has been incorporated for the representation of flat fading.

In a Finite State Markov Chain, the fading process can be represented with distinct signals. In the literature, the envelope of the channel response and the received Signal to Noise Ratio are preferred [3] [4]. Here, the complex channel response is also considered, as it could be useful for applications working with the in-phase and quadrature components of the channel response separately. With the selected signal for the fading process representation, the state-space is defined by dividing the range of the amplitude of the signal into several consecutive regions. Region \( i \) is mapped into state \( i \) of the chain and delimited by two thresholds. Thus, each state of the chain represents one fading region. The partition method determines the thresholds of the states. In this paper, we study the usual partition methods as well as a novel one that makes a uniform division of the density function of the desired signal.

Applications and analysis of the FSMC and its limitations can be found in the literature but always limited to particular cases. In this work, the evaluation of the FSMC is performed by means of statistic and other parameters often employed in the evaluation of Rayleigh channels generators and compared with other methods of simulation of the fading channel. We are particular concerned about the autocorrelation function (ACF) of the process, but also parameters such as the mean and the standard deviation, the computation time and the mean and maximum basis power margin are considered.

Tan and Beaulieu showed in [3] the limitations of a first order FSMC in the modeling of the fading process. In particular, they proposed a stochastic analysis of the problem concluding that it is only appropriate for applications that require analysis over a short period of time but not for those requiring a large number of consecutive samples. This conclusion comes from the fact that the autocorrelation function is exponentially decaying, in contrast to the theoretical Clarke’s model, which follows a Bessel function. The analysis was done for a specific partition method. It is interesting to extend the study to the rest of partition methods and signals, so that it is possible to choose the parameters of the chain that approach more precisely the desired ACF.

The remainder of the paper is organized as follows. Section II briefly describes the main properties of the wireless channel. Section III presents the Finite State Markov Chain and the definition of the main parameters of the chain. The evaluation, made in terms of one and second-order statistics and other performance parameters, is presented in Section IV. Finally, some concluding remarks are given in Section V.

II. WIRELESS CHANNEL

In wireless systems, signals travel through multiple paths between the transmitter and the receiver. Due to these multiple ways, the received signal is formed as the addition of different constructive and destructive components that the receiver perceives as variations of the amplitude, phase and angle of arrival of the signal. This phenomenon is known as multipath fading [5]. The received signal is therefore a set of attenuated, time-delayed, phase shifted replicas of the transmitted signal.

The fading is categorized into two groups: large-scale and small-scale fading. The large-scale fading refers to variations that occur over relatively large distances. The small-scale fading, characterizes the effects of small changes in the separation between a transmitter and a receiver. These changes can be
caused by the mobility of the transmitter, the receiver or the intermediate objects in the path of the signal. Variation due to small-scale fading occurs over very short distances, on the order of the signal wavelength. No deterministic model for the phenomenon exists and hence it is characterized statistically. When there is no predominant direct line of sight between the transmitter and the receiver (NLOS, Not Line Of Sight), the Rayleigh distribution approximates quite well the channel envelope and the fading is denoted Rayleigh fading.  

Small-scale fading can be further divided into two types: frequency selective and frequency non-selective fading. The latter is also known as flat fading because all the frequency components of the transmitted signal are affected by the channel. In this work, we focus on flat Rayleigh channels.

### A. Representation of the fading process

In the case of flat fading, the effect of the channel over the transmitted signal can be represented through the low-pass equivalent of the channel response, that we denote \( h(t) = h_{\text{phase}}(t) + jh_{\text{quad}}(t) \). The phase and quadrature components are independent of each other and Gaussian distributed, with probability density function (pdf) \( f_C(h) \), mean zero and variance \( b_0 \).

The envelope of the fading channel, calculated from the phase and quadrature components as \( r(t) = \sqrt{h_{\text{phase}}^2(t) + h_{\text{quad}}^2(t)} \), is Rayleigh distributed, due to the underlying Gaussian components. We denote its pdf \( f_R(r) \), and it has mean \( \sqrt{b_0 \pi} \) and mean-square value \( 2 \cdot b_0 \).

Finally, the instantaneous Signal to Noise Ratio at the receiver, \( \gamma(t) \), is widely employed as a good indicator of the state of the channel. When the channel is "bad", the signal is severely degraded during its route from the transmitter to the receiver and the instantaneous SNR decreases. A high value of instantaneous SNR indicates that the channel is "good" and the signal is hardly affected by channel degradation.

The instantaneous SNR is proportional to the square of \( |h(t)| \):

\[
\gamma(t) = |h(t)|^2 \frac{E_s}{N_0} \tag{1}
\]

where \( E_s \) is the average energy per symbol and \( N_0 \) is the noise power spectral density. If the noise is Additive White Gaussian Noise (AWGN), \( \gamma(t) \) is exponentially distributed for Rayleigh channels, with pdf \( f_E(\gamma) = \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \), where \( \bar{\gamma} \) is the average Signal to Noise Ratio.

### B. Autocorrelation Function

An important characteristic of the multipath channel is its time-varying nature, since either the transmitter or the receiver is in motion and therefore the location of reflectors in the transmission path, which give rise to multipath, will change over time. The coherence time of the channel \( T_C \) is defined as the time over which the channel response can be considered invariant.

Fading can also be studied in the frequency domain. Whenever there is relative motion between the receiver and the transmitter, the received signal suffers a frequency shift which is just the manifestation of the fading phenomenon in the frequency domain. The maximum frequency shift is characterized by the Doppler frequency, \( f_D \), which is computed as \( v/\lambda \), where \( v \) is the relative velocity between the transmitter and receiver and \( \lambda \) is the wavelength of the transmitted signal. The Doppler shift and the coherence time are inversely proportional to one another, that is:

\[
T_C \approx \frac{1}{f_D} \tag{2}
\]

The variability of the channel over time is usually reflected through its autocorrelation function. This second-order statistic generally depends on the propagation geometry, the velocity of the mobile and the antenna characteristics. The isotropic scattering is a common assumption that means that the channel consists of many scatterers densely packed with respect to angle [6]. It is usually referred as Clarke’s model or the "classical" model. The impulse response of the channel is a wide-sense stationary random process (WSS) and the continuous-time ACF of the phase and quadrature components does not depend on the time \( t \) but just on the time difference \( \tau \):

\[
R_{h_{\text{phase}}}(\tau) = R_{h_{\text{quad}}}(\tau) = b_0 J_0(2\pi f_D \tau) \tag{3}
\]

where \( J_0(.) \) is the zeroth order Bessel function of the first kind and \( f_D \) is the maximum Doppler frequency in Hertz.

In the discrete-time domain, it is more convenient to express the Doppler frequency normalized by the sampling rate, i.e., multiplied by the symbol period \( T_S \) \( f_D \cdot T_S \). A low value of the product \( f_D \cdot T_S \) implies high correlation in the signal and, likewise, a high value of \( f_D \cdot T_S \) means low correlation. In the limit \( f_D \cdot T_S \rightarrow \infty \) there is no correlation and the samples are independent of each others.

If we are working with the Rayleigh envelope, the expression of the ACF, calculated from (3), is as follows:

\[
R_r(\tau) = \frac{\pi b_0}{2} {}_2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; (J_0(2\pi f_D \tau))^2 \right) \tag{4}
\]

where \( {}_2F_1(n, d, z) \) is the hypergeometric function.

Finally, the ACF of the Signal to Noise Ratio is:

\[
R_\gamma(\tau) = \tau^2_0 + (\gamma_0 J_0(2\pi f_D \tau))^2 \tag{5}
\]

### III. Finite State Markov Chain

In a Finite State Markov Chain, the range of the amplitude of the signal is divided into several consecutive regions. Region \( i \) is mapped into state \( i \) of the chain and is delimited by two thresholds, \( \alpha_i \) and \( \alpha_{i+1} \). Thus, each state of the chain represents one fading region. The steady state probability for state \( i \) is just the probability that the received signal is between the thresholds of the region:

\[
\pi_i = \int_{\alpha_i}^{\alpha_{i+1}} f(\alpha)\,d\alpha \tag{6}
\]

where the density function \( f(x) \) depends on the signal of interest: Gaussian (in-phase and quadrature components), Rayleigh (envelope) or exponential (SNR).
A. Partitioning

If the selected signal for the partition is the in-phase and quadrature components of the channel response, then the distribution is Gaussian and the thresholds are denoted as \( \mu_i = i \cdot n + 1 \) for a chain of \( n \) states. The symmetry of the pdf has to be taken into account and thus the first and last thresholds are \( \mu_1 = -\infty \) and \( \mu_{n+1} = \infty \) and the remaining have to be chosen preserving the symmetry, so that the ones in the positive part of the pdf match up the ones in the negative axis with opposite sign.

The envelope of the received signal follows a Rayleigh distribution. In that case, the thresholds are denoted as \( \delta_i = 1..n + 1 \) and the first and last threshold are predetermined by the range of variation of a Rayleigh pdf, i.e., \( \delta_1 = 0 \) and \( \delta_{n+1} = \infty \). The same applies when the desired parameter is the exponentially distributed received SNR and thresholds are obtained from them.

Four different methods of partition have been studied: uniform partition, equal probabilities, linearly increasing probabilities and exponentially increasing probabilities.

1) Uniform partition: This partition is not found in the literature, even though similar approaches for systems with adaptive modulation have been employed [7]. The methodology consists of defining first the desired thresholds and then deriving the state probabilities from them. In particular, the continuous pdf is divided with thresholds that are uniformly spaced. The distribution is truncated by means of a selected maximum finite value, so that the last threshold in this partition is not infinite but that finite number. Thresholds defined this way give rise to state probabilities that follow the shape of the pdf, doing a sampling of the density function.

In the rest of methods the approach is opposite: the state probabilities are fixed and the corresponding thresholds are obtained from them.

2) Equal probabilities: This is the very common approach for the partitioning of a FSMC. The thresholds are selected in such a way that the steady state probabilities of being in any state are equal:

\[
\pi_1 = \pi_2 = \ldots = \pi_n = \frac{1}{n} \tag{7}
\]

3) Linearly increasing probabilities: In the linearly increasing probabilities partition, the goal is to increase linearly the probability of the states with higher amplitude. Thus, state probabilities are:

\[
\pi_i = i \pi_1 \quad \text{where} \quad \pi_1 = \frac{2}{n^2 + n} \tag{8}
\]

4) Exponentially increasing probabilities: This partition is similar to the partition above but with exponential increments in the probability:

\[
\pi_{i+1} = 2 \pi_i \quad \text{where} \quad \pi_1 = \frac{1}{2^n - 1} \tag{9}
\]

In the case of \( f_G(x) \), slight changes have to be done in (8) and (9) due to the symmetry of the pdf.

B. Output vector and transition probabilities

Once the partition is defined, it is necessary a discrete value of the amplitude of the signal within the interval that represents the state of the chain. This value is the statistical mean between the two thresholds of the state. The set of those values is called output of the Markov chain and denoted \( \mathbf{x} = \{x_1, x_2, \ldots, x_n\} \).

In the case of the gaussian distribution:

\[
\begin{align*}
\pi_i & = \frac{\int_{\mu_i}^{\mu_{i+1}} h \cdot f_G(h)dh}{\pi_i} \\
& = \frac{1}{\pi_i} \left( \frac{\sqrt{b_0}}{2\pi} \left( e^{-\frac{\mu_i}{2b_0}} - e^{-\frac{\mu_{i+1}}{2b_0}} \right) \right) \\
& = \frac{1}{\pi_i} \left( \frac{\sqrt{b_0}}{2\pi} \left( e^{-\frac{\mu_i}{2b_0}} - e^{-\frac{\mu_{i+1}}{2b_0}} \right) \right) \tag{10}
\end{align*}
\]

The Level Crossing Rate (LCR) of a process is the average number of times per unit interval that a fading signal crosses a given signal level \( \alpha_i \). The transition probabilities from state \( i \) to state \( i+1, p_{i,i+1} \), can be approximated by the ratio of the level crossing rate at threshold \( \alpha_{i+1} \) and the average number of packets per second staying in state \( i \). Similarly, the transition probability \( p_{i,i-1} \) is approximately the ratio of the LCR at threshold \( \alpha_i \) and the average number of packets per second staying in state \( i \). These approximations will be valid as long as \( \text{LCR} \ll \frac{\pi_i}{T_S} \), i.e., the fading process is sufficiently slow and the transitions of the chain occur only between adjacent states \( (p_{k,i} = 0, \quad i \neq j \text{ or } |k - i| > 1) \).

The expression of the transition probabilities for the in-phase and quadrature components is then:

\[
\begin{align*}
p_{i,i+1} & \approx \frac{N_G(\mu_{i+1}) \cdot T_S}{\pi_i} = \frac{1}{\pi_i} T_S \sqrt{2} f_D e^{-\mu_{i+1}/(2b_0)} \tag{11} \\
p_{i,i-1} & \approx \frac{N_G(\mu_i) \cdot T_S}{\pi_i} = \frac{1}{\pi_i} T_S \sqrt{2} f_D e^{-\mu_i/(2b_0)} \tag{12}
\end{align*}
\]

When the envelope is considered, the resulting output vector and transition probabilities are:

\[
\begin{align*}
r_i &= \int_{\delta_i}^{\delta_{i+1}} r \cdot f_R(r)dr \\
& = \frac{1}{\pi_i} \left( \delta_i e^{-\left(\frac{\delta_i^2}{b_0}\right)} - \delta_{i+1} e^{-\left(\frac{\delta_{i+1}^2}{b_0}\right)} \right) \\
& - \frac{1}{\pi_i} \sqrt{\frac{b_0}{2}} \left( erf \left( \frac{\delta_i}{\sqrt{2b_0}} \right) + erf \left( \frac{\delta_{i+1}}{\sqrt{2b_0}} \right) \right) \tag{13}
\end{align*}
\]

\[
\begin{align*}
p_{i,i+1} & \approx \frac{1}{\pi_i} T_S \int_{\delta_i}^{\delta_{i+1}} f_D e^{-\delta_i^2/(2b_0)} \tag{14} \\
p_{i,i-1} & \approx \frac{1}{\pi_i} T_S \int_{\delta_i}^{\delta_{i+1}} f_D e^{-\delta_i^2/(2b_0)} \tag{15}
\end{align*}
\]

Finally, the expressions of the output vector and transition probabilities of the received SNR are as follows:

\[
\gamma_i = \frac{1}{\pi_i} \left( (\Gamma_i + \gamma_0) e^{-\frac{\gamma_i}{\gamma_0}} - (\Gamma_{i+1} + \gamma_0) e^{-\frac{\gamma_{i+1}}{\gamma_0}} \right) \tag{16}
\]

\[
\begin{align*}
p_{i,i+1} & \approx \frac{1}{\pi_i} T_S \frac{2\pi \Gamma_{i+1}}{\gamma_0} f_D e^{-\Gamma_{i+1}/\gamma_0} \tag{17} \\
p_{i,i-1} & \approx \frac{1}{\pi_i} T_S \frac{2\pi \Gamma_i}{\gamma_0} f_D e^{-\Gamma_{i}/\gamma_0} \tag{18}
\end{align*}
\]
C. Autocorrelation function

A parameter of interest in this paper is the autocorrelation function of the FSMC. It is one of the limitations of the Markov modeling, as it decays exponentially in contrast to the theoretical Bessel function in (3), (4) and (5). The analytical expression of the ACF is:

\[ R[m] = \mathbb{E}[x(x_0)x(x_m)] = \sum_{i=1}^{n} x(i)\mathbb{P}(x_m = j, x_0 = i) \]

\[ = \sum_{j=1}^{n} x(j)\pi_j \sum_{i=1}^{n} x(i)p_{ij} \]

where \( x(\cdot) \) is the output of the chain in the corresponding state at instant \( m \). In matrix form we can write:

\[ R[m] = x^T\Phi P^m x \]  

with \( \Phi \) a diagonal matrix whose elements are the state probabilities and \( P \) the transition matrix.

IV. EVALUATION

With the fundamentals of a FSMC presented above, the performance evaluation of the model is done next. Due to lack of space, we include only the results for the envelope of the channel response. It has to be marked, though, that for the other two signals, channel response and SNR, we have obtained similar behaviors and conclusions. A program coded in MATLAB generates the desired number of realizations of the FSMC with the parameters specified by the user: number of samples, Doppler frequency, symbol period, signal for the partition, partition method and number of states.

The estimated mean \( \mathbb{E} \) and standard deviation \( \sigma \) of the fading envelope are shown in Table I, and compared to the theoretical, with a realization of \( 10^6 \) samples, \( b_0 = 1, 10 \) states and \( f_D \cdot T_S = 10^{-2} \). All the partition methods estimate correctly the mean. For the standard deviation, the uniform partition is the one that obtains the best approximation for the specified length of the simulation.

As it was previously mentioned, the ACF is expected to be exponentially decreasing. We are interested in checking if it is possible to improve the grade of approximation to the theoretical bessel function by selecting properly the parameters of the chain (partition method and number of states).

The ACF function of the different partition methods is compared in Fig. 1 (a), with the rest of parameters of the chain fixed (\( f_D \cdot T_S = 10^{-2} \) and 10 states). The theoretical ACF from Clarke’s model is plotted with solid line. In the figure, one can observe that the uniform and equal probabilities and linearly increasing probabilities obtain similar results, being the first two partitions slightly better, as they are closer to Bessel’s function in the first samples of observation (the only ones that the FSMC can approximate properly). A zoom of the figure is shown in Fig. 1 (b) to corroborate it.

On the other hand, the influence of the number of states in the ACF function is shown in Figure 2. In principle, defining more states would correspond to better approximations to Clarke’s model. However, we may keep in mind the assumption of transitions only happening to adjacent states. If the number of states is increased too much, it will not be true anymore. The dependence with the correlation of the channel is closely related with the number of states, e.g. given a number of states, if the channel is too fast the assumption is not accomplished.

To finish with the study of the ACF, we evaluate the power margin quality measures as another parameter of quality performance. In particular, the mean basis power margin and the maximum basis power margin, defined as:

\[ \mathcal{G}_{\text{mean}} = \frac{1}{\sigma_2} \text{trace}(C_g C_g^{-1} C_g^T) \]

\[ \mathcal{G}_{\text{max}} = \frac{1}{\sigma_2} \max(\text{diag}(C_g C_g^{-1} C_g^T)) \]

where \( \sigma_2 = b_0 \) is the variance of the theoretical distribution, \( L \) is the number of points used in the calculation of the metrics. \( C_g \) is the \( L \times L \) covariance matrix of \( L \) neighboring samples of the theoretical fading process, and \( C_g \) is the \( L \times L \) covariance matrix of \( L \) neighboring samples of the fading model. The results are shown in Table II, for the partitions that approximate better Clarke’s model: uniform and equiprobable. Perfect performance corresponds to 0 dB for both measures. An autocorrelation sequence length of 300 was considered with the same value of \( f_D \cdot T_S \) fixed above. The measures of the FSMC are compared with three representative methods for Rayleigh channel generation: the simulator provided by MATLAB, the Sum-of-Sinusoids method proposed in [8] and simulated with 15 sinusoids and the IDFT simulator in [9]. The results show the poor approximation of the FSMC to the theoretical ACF compared with the rest of models.

The main advantage of a Markov model for Rayleigh channels is the simplicity of the model, which leads to short times for generating long realizations. We show in Table III the results of the time comparison for the same four methods. One can observe the big differences in terms of computation time for generating 5e6 samples of the complex channel response, being the FSMC 100 times better than the IDFT method and 1000 times better than the one provided by MATLAB.

V. CONCLUSIONS

In this paper, the Finite State Markov Chain (FSMC) for the modeling of flat fading is evaluated. The chain can be constructed based on different signals that represent the fading process: complex channel response, envelope of the channel response and Signal to Noise Ratio. Different partition methods have been considered, including a novel proposal, uniform
partition, that is based on a uniform partition of the density function. Results show the advantages and limitations of the chain. As advantages, we mark the computational efficiency and the good approximation of first order statistics (mean and standard deviation). In the limitations, the autocorrelation function always follows an exponential decay instead of the theoretical bessel function. Thus, the use of a FSMC is limited to applications that require only the first coefficients of the ACF function. Among the different partition methods, the proposed uniform partition together with the one that fixes equal state probabilities have been demonstrated to be the best.

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