On the Jensen type inequality for generalized Sugeno integral

Marek Kałuszka, Andrzej Okolewski, Michal Boczek

Abstract

We prove necessary and sufficient conditions for the validity of Jensen type inequalities for generalized Sugeno integral. Our proofs make no appeal to the continuity of neither the fuzzy measure nor the operators. For several choices of operators, we characterize the classes of functions for which the corresponding inequalities are satisfied.

1 Introduction

The pioneering concept of the fuzzy integral was introduced and initially examined by Sugeno [20]. Further theoretical investigations of the integral and its generalizations have been pursued by many researchers. Among others, Ralescu and Adams [12] provided several equivalent definitions of the Sugeno integral and proved a monotone convergence theorem for the integral, Román-Flores et al. [13, 14] discussed level-continuity of fuzzy integrals and H-continuity of fuzzy measures, while Wang and Klir [22] presented an excellent general overview on fuzzy measurement and fuzzy integration theory. On the other hand, fuzzy integrals have also been successfully applied to various fields (see, e.g., [5], [8], [10], [21]).

The study of inequalities for Sugeno integral was initiated by Román-Flores and Chalco-Cano [16]. Since then, the fuzzy integral counterparts of several classical inequalities, including Chebyshev’s, Jensen’s, Minkowski’s and Hölder’s inequalities, are given by Flores-Franulic and Román-Flores [6], Agahi et al. [1, 2] Mesiar and Ouyang [9], Román-Flores et al. [18], and other researchers.
The purpose of this paper is to study not only the sufficient conditions for the validity of Jensen and reverse Jensen type inequalities for generalized Sugeno integral (cf. [18], [23], [2] and [4]), but also the necessary ones. In order to achieve this goal, a new method of proof of such inequalities is proposed. The results are obtained for quite universal integrals under no assumptions on continuity of the fuzzy measure and the operators. For some specific choices of operators, the corresponding characterizations of classes of functions for which Jensen’s inequality is satisfied are also presented.

The paper is organized as follows. In Sections 2 and 3 we set up notation and terminology and present our main results, some related results as well as several illustrative examples. Concluding remarks are given in Section 4.

2 Main results

Let \((X, \mathcal{F})\) be a measurable space and \(\mu: \mathcal{F} \to [0, \infty]\) be a monotone measure, i.e., \(\mu(\emptyset) = 0\), \(\mu(X) > 0\) and \(\mu(A) \leq \mu(B)\) whenever \(A \subset B\). The monotone measures are also referred to in the literature as fuzzy measures (see, e.g., [22]). Let \(Y \subset \mathbb{R}\) be an arbitrary nonempty set; usually \(Y = [0, 1]\), \(Y = [0, \infty]\), \(Y = [0, \infty)\) or \(Y = \mathbb{R}\). We denote the range of \(\mu\) by \(\mu(\mathcal{F})\). For a measurable function \(f: X \to Y\), we will define the Sugeno integral of \(f\) on a set \(A \in \mathcal{F}\) with respect to \(\mu\) and an operator \(\Delta: Y \times \mu(\mathcal{F}) \to Y\) as

\[
\int_A f \, \mu = \sup_{y \in Y} \left\{ y \, \mu(A \cap \{ f \geq y \}) \right\},
\]

(1)

where \(\{ f \geq y \}\) stands for \(\{ x \in X: f(x) \geq y \}\). When \(A = X\), we write \(\int_X f \, \mu\) instead of \(\int f \, \mu\).

The following theorem gives sufficient conditions under which Jensen type inequality for the integral (1) holds (cf. Theorem 1 of [18], Lemma 3.1 of [23],

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Throughout the paper $\circ, \star : Y \times \mu(\mathcal{F}) \to Y$ will denote arbitrary operators.

**Theorem 2.1.** Assume a function $H : Y \to Y$ is nondecreasing, left-continuous and satisfies the condition

(A1) for all $y \in Y$ and $b \in \mu(\mathcal{F})$,

$$H(y) \circ b \geq H(y \circ b).$$

(2)

If $\mu(\mathcal{F}) \ni b \mapsto y \circ b$ is nonincreasing for any fixed $y \in Y$ and $\int_A f \star \mu \in Y$ for $A \in \mathcal{F}$ and measurable $f : X \to Y$, then

$$\int_A H(f) \circ \mu \geq H\left(\int_A f \star \mu\right),$$

(3)

where $H(f)(x) = H(f(x))$.

**Proof.** Observe that $H(Y) \subset Y$. Therefore

$$\int_A H(f) \circ \mu \geq \sup_{a \in Y} \left[ a \circ \mu(A \cap \{H(f) \geq a\}) \right]$$

$$\geq \sup_{a \in H(Y)} \left[ a \circ \mu(A \cap \{H(f) \geq a\}) \right]$$

$$= \sup_{y \in Y} \left[ H(y) \circ \mu(A \cap \{H(f) \geq H(y)\}) \right].$$

As $H$ is nondecreasing, it follows that $\{f \geq y\} \subset \{H(f) \geq H(y)\}$. Combining the above with the monotonicity of $\mu$, the monotonicity of $b \mapsto y \circ b$ and (A1) yields

$$\int_A H(f) \circ \mu \geq \sup_{y \in Y} \left[ H(y) \circ \mu(A \cap \{f \geq y\}) \right]$$

$$\geq \sup_{y \in Y} \left[ H(y \circ \mu(A \cap \{f \geq y\})) \right].$$

(4)
Define \( g(y) = y \ast \mu(A \cap \{ f \geq y \}) \). By assumption, \( g(y) \in Y \) and \( \sup_{y \in Y} g(y) = \int_A f \ast \mu \in Y \). By the monotonicity of \( H \), \( H\left( \sup_{y \in Y} g(y) \right) \geq \sup_{y \in Y} H(g(y)) \). Clearly, for \( g(y_n) \uparrow \sup_{y \in Y} g(y) \)

\[
H(\sup_{y \in Y} g(y)) = H\left( \lim_{n \to \infty} g(y_n) \right) \\
= \lim_{n \to \infty} H(g(y_n)) \leq \sup_{y \in Y} H(g(y)).
\]

Hence

\[
H(\sup_{y \in Y} g(y)) = \sup_{y \in Y} H(g(y)). \tag{5}
\]

From (4) and (5) we conclude that

\[
\int_A H(f) \circ \mu \geq H\left( \sup_{y \in Y} [y \ast \mu(A \cap \{ f \geq y \})] \right) \\
= H\left( \int_A f \ast \mu \right),
\]

which is our assertion. \( \square \)

Obviously, if \( H \) is increasing, then \( \{ H(f) \geq H(y) \} = \{ f \geq y \} \), and the monotonicity restriction of \( b \mapsto y \circ b \) can be dropped from the statement. Moreover, if the supremum of the function \( g \) over \( Y \) is attainable, then the assumption of left-continuity can be removed. Note that the condition \( \int_A f \ast \mu \in Y \) is satisfied provided the limit of any nondecreasing sequence of elements from \( Y \) belongs to \( Y \). Among others, the sets \( Y = [0,1] \) and \( Y = [0,\infty] \) possess this property.

If we replace \( y \in Y \) by \( y \in H(Y) \) in the condition A1, the result of Theorem 2.1 is still true for the following integral

\[
\int_A f \bigtriangleup \mu = \sup_{y \in H(Y)} \left[ y \bigtriangleup \mu(A \cap \{ f \geq y \}) \right]. \tag{6}
\]

The integrals (1) and (6) are distinct, as is illustrated by the following example.
Example 2.1. Let $X = Y = (0, \infty)$, $H(Y) = (0, 1]$, $\mu$ be the Lebesgue measure, $f(x) = 2/x$ and $a \triangle b = a \wedge b$, where $a \wedge b = \min(a, b)$. Then

$$
\sup_{y \in Y} [y \wedge \mu(\{f \geq y\})] = \sup_{y > 0} [y \wedge (2/y)] = \sqrt{2},
$$
$$
\sup_{y \in H(Y)} [y \wedge \mu(\{f \geq y\})] = \sup_{y \in (0, 1]} [y \wedge (2/y)] = 1.
$$

Next theorem provides the necessary condition for the validity of Jensen’s inequality.

Theorem 2.2. Let $H : Y \to Y$ be a nondecreasing function, where $Y \subset [0, \infty]$ and $0 \in Y$. Assume that $a \star 0 = a \circ 0 = 0$ for any $a \in Y$, and the functions $y \mapsto y \circ b$ and $y \mapsto y \star b$ are nondecreasing for an arbitrary fixed $b \in \mu(F)$. If the inequality (3) is satisfied for arbitrary set $A \in F$ and function $f(x) = y1_A(x)$ with $y \in Y$, then the condition A1 holds true.

Proof. Set $f = y1_A$, where $y \in Y$ and $A \in F$. By the monotonicity of $y \mapsto y \star b$ and the condition $a \star 0 = 0$, we obtain

$$
\int_A f \star \mu = \max\left( \sup_{a \leq y} [a \star \mu(A)], \sup_{a > y} [a \star \mu(\emptyset)] \right)
$$
$$
= \max [y \star \mu(A), 0] = y \star \mu(A), \quad (7)
$$
as $a \star b \in Y \subset [0, \infty]$. Proceeding similarly with $\circ$ we find that

$$
\int_A H(f) \circ \mu = \max\left( \sup_{a \leq H(y)} [a \circ \mu(A)], \sup_{a > H(y)} [a \circ \mu(\emptyset)] \right)
$$
$$
= \max [H(y) \circ \mu(A), 0] = H(y) \circ \mu(A). \quad (8)
$$
Combining (3), (7) and (8), we get

$$
H(y) \circ b \geq H(y \star b)
$$
for any $y \in Y$ and $b \in \mu(\mathcal{F})$. \qed

As a consequence of Theorems 2.1 and 2.2 we have the following result.

**Theorem 2.3.** Let $Y \subset [0, \infty]$, $0 \in Y$ and $H : Y \to Y$ be a nondecreasing and left-continuous function. Suppose $a \star 0 = a \circ 0 = 0$ for any $a \in Y$ and the functions $b \mapsto y \circ b$, $y \mapsto y \circ b$ and $y \mapsto y \star b$ are nondecreasing. For arbitrary set $A \in \mathcal{F}$ and measurable function $f : X \to Y$ such that \( \int_A f \star \mu \in Y \), the Jensen inequality

\[
\int_A H(f) \circ \mu \geq H\left( \int_A f \star \mu \right)
\]  

(9)

is satisfied iff $H(y) \circ b \geq H(y \star b)$ for any $y \in Y$ and $b \in \mu(\mathcal{F})$.

The Jensen type inequality for generalized Sugeno integral is a powerful tool that can be used to determine some other type inequalities (see, e.g., \([15], [17], [2]\) and \([23]\)).

We now examine for which functions $H : Y \to Y$ the inequality (9) is valid if either $Y = [0, 1]$ or $Y = [0, \infty)$, $\mu(\mathcal{F}) = Y$ and the operators $\circ, \star : Y \times Y \to Y$ are chosen from the following: $a \wedge b = \min(a, b)$, $a \cdot b$, $a + b$ for $a, b \in [0, \infty)$, $(a + b)/2$ for $a, b \in [0, 1]$, and $a \oplus b = (a + b - 1)_+$, where $(x)_+ = \max(x, 0)$. The last operator is known as the Łukasiewicz norm.

Denote by $\mathcal{H}(Y, \circ, \star)$ the class of all nondecreasing functions $H$ for which the Jensen inequality with operators $\circ, \star : Y \times Y \to Y$ holds true.

1. Let $\circ = \star = \wedge$ and $Y$ be a subset of $[0, \infty)$ containing zero. Then $A1$ takes the form

\[
H(y) \wedge H(b) \leq H(y) \wedge b, \quad y, b \in Y.
\]  

(10)
Obviously, (10) holds if \( H(y) \leq y \) for any \( y \in Y \). Moreover, if there exists \( \bar{y} \in Y \) such that \( H(\bar{y}) > \bar{y} \), then for \( b = y = \bar{y} \), by (10), we get \( H(\bar{y}) \leq \bar{y} \), a contradiction. This and Theorem 2.3 lead to the following characterization.

**Corollary 2.1.** Suppose \( Y \subset [0, \infty) \), \( 0 \in Y \) and \( H: Y \to Y \) is nondecreasing and left-continuous. For arbitrary \( A \in \mathcal{F} \) and measurable function \( f: X \to Y \) such that \( \int_A f \ast \mu \in Y \), the Jensen inequality for Sugeno integral

\[
(S) \int_A H(f) d\mu \geq H \left( (S) \int_A f d\mu \right)
\]

is fulfilled iff \( H(y) \leq y \) for \( y \in Y \), where \( (S) \int_A f d\mu = \sup_{y \in Y} \left[ y \wedge \mu(A \cap \{ f \geq y \}) \right] \).

Note that if \( Y = [0, 1] \), then (11) is satisfied for any convex function \( H \) for which \( H(0) = 0 \).

2. Let \( \circ = \ast = \cdot \). We will consider first the case of \( Y = [0, 1] \). The condition A1 holds iff

\[
H(by) \leq bH(y), \quad 0 \leq b, y \leq 1.
\]

Taking \( b = 0 \) gives \( H(0) = 0 \). For \( b, y > 0 \), the condition

\[
\frac{H(by)}{by} \leq \frac{H(y)}{y}, \quad 0 < b, y \leq 1,
\]

is equivalent to (12). In consequence, we can formulate the following characterization for the \( N \)-integral, called also the Shilkret integral (see, [19]),

\[
(N) \int_A f d\mu = \sup_{y \in [0,1]} \left[ y \mu \{ f \geq y \} \right],
\]

in which \( \mu \) is a fuzzy measure with values in the interval \([0, 1]\).
Corollary 2.2. Assume $H : [0, 1] \to [0, 1]$ is nondecreasing and left-continuous. For any measurable $A \subset [0, 1]$ and any measurable function $f : X \to [0, 1]$, the Jensen inequality
\[
(N) \int_A H(f) \, d\mu \geq H \left( (N) \int_A f \, d\mu \right)
\]
holds true iff $H(0) = 0$ and $y \mapsto H(y)/y$ is nondecreasing.

Denote by $C_0$ the family of convex functions $H : [0, 1] \to [0, 1]$ such that $H(0) = 0$. Then
\[
H(by) = H(by + (1 - b)0) \leq bH(y) + (1 - b)H(0) = bH(y),
\]
so $C_0 \subset \mathcal{H}([0, 1], \cdot, \cdot)$. The set $\mathcal{H}([0, 1], \cdot, \cdot)$ also contains nonconvex and even discontinuous functions, e.g., $H(y) = y1_{[a,1]}(y)$, $0 < a < 1$.

We now turn to the case $Y = [0, \infty)$. It is easy to check that validity of (13) for $0 < b, y < \infty$ implies that $H(0) = 0$ and $H(y)/y$ is nonincreasing as well as nondecreasing. Hence $H(y) = cy$ for $c > 0$ and $\mathcal{H}([0, \infty), \cdot, \cdot) = \{H : H(y) = cy, y \geq 0, c \geq 0\}$.

3. Set $\circ = \star = +$ and $Y = [0, \infty)$. The condition $a \circ 0 = a \star 0 = 0$ is not fulfilled, so we can use Theorem 2.1, only. Since $A1$ takes the form $H(y + b) \leq H(y) + b$ for $y, b \geq 0$, Jensen’s inequality holds for all nondecreasing functions $H$ satisfying the Lipschitz condition with the constant $L = 1$, e.g., $H(y) = c(y + 1)/2$, where $c \leq 1$, is an element of $\mathcal{H}([0, \infty), +, +)$.

For $Y = [0, 1]$ and $\circ = \star = \oplus$, in which $\oplus = (a + b)/2$, $H$ satisfies $A1$ if
\[
H((y + b)/2) \leq H(y + b)/2, \quad 0 \leq y, b \leq 1.
\]
Putting $y = b = 0$ in (14) we have $H(0) = 0$. If $H : [0, 1] \to [0, 1]$ is a convex function, then $H$ fulfills (14) as $H(y) \leq y$ for $0 \leq y \leq 1$. 

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4. Assume that $a \circ b = a \star b = (a + b - 1)_+$ and $Y = [0, 1]$. By Theorem 2.3, the inequality (9) is valid for any $f$ iff $H$ meets the condition

$$H((y + b - 1)_+) \leq (H(y) + b - 1)_+, \quad 0 \leq b, y \leq 1.$$  \hspace{1cm} (15)

An example of such a function is $H(y) = (y - c)_+$, $0 \leq c \leq 1$. Substituting $b = 0$ in (15) gives $H(0) = 0$. We claim that if $H$ has the left-hand side derivative $H'$ at a point $y$ such that $H(y) > 0$, then $H'(y) \geq 1$. Indeed, since $H(y) > 0$, it is seen that for $b$'s sufficiently close to 1

$$\frac{H(y) - H(y + b - 1)}{1 - b} \geq 1,$$

hence taking the limit of the left-hand side as $b \to 1$ yields $H'(y) \geq 1$.

Let us now consider several cases of distinct operators $\circ$ and $\star$.

5. The set $\mathcal{H}([0, 1], \wedge, \cdot)$ includes the nondecreasing and left-continuous functions $H$ such that $H(y) \leq y$. It follows from the fact that $H(yb) \leq H(y)$ and $H(yb) \leq H(b) \leq b$ for $0 \leq b, y \leq 1$. The function $H(y) = \sqrt{y}$ is not an element of $\mathcal{H}([0, 1], \wedge, \cdot)$.

The only element of the set $\mathcal{H}([0, 1], \cdot, \wedge)$ is $H(y) = 0, y \geq 0$. In fact, if $H(y \wedge b) \leq H(y)b, 0 < y, b \leq 1$, and $H(\bar{y}) > 0$ for $\bar{y} < 1$, then $H(\bar{y}) \leq H(\bar{y})\bar{y}$, so $\bar{y} = 1$, a contradiction.

The family $\mathcal{H}([0, \infty), \cdot, \wedge)$ is constituted of all functions given as $H(y) = h(y)1_{(1, \infty)}(y)$, in which $h(\cdot)$ is an arbitrary nondecreasing and left-continuous function with values in $[1, \infty)$.

6. A nondecreasing function $H$ belongs to the class $\mathcal{H}([0, 1], \cdot, \oplus)$ if

$$H((y + b - 1)_+) \leq H(y)b, \quad 0 \leq y, b \leq 1.$$ \hspace{1cm} (16)

Taking $b = 0$ we see that $H(0) = 0$. Any linear function $H(y) = cy, 0 \leq c \leq 1$, is an element of $\mathcal{H}([0, 1], \cdot, \oplus)$ as it meets (16). Substituting $x = y + b - 1 > 0$ into
we conclude that
\[ \frac{H(x + (1 - b)) - H(x)}{(1 - b)H(x)} \geq \frac{1}{b} \]
for all \( x \in (0, b] \) such that \( H(x) > 0 \). Letting \( b \to 1 \) we see that \( H'(x)/H(x) \geq 1 \) provided \( H(x) > 0 \).

7. The functions \( H \) from \( H([0, 1], \wedge, \oplus) \) fulfill the inequality
\[ H((y + b - 1)_+ \leq H(y) \wedge b, \quad 0 \leq y, b \leq 1. \]  
Writing \( y = 1 \) we deduce that \( H([0, 1], \wedge, \oplus) \) is the set of all nondecreasing functions which are bounded from above by the function \( H_0(y) = y \).

8. Let \( \circ = \cdot, \ast = + \) and \( Y = [0, \infty) \). Then the condition A1 can be rewritten as
\[ H(y + b) \leq H(y)b, \quad y, b \in [0, \infty). \]
Putting \( b = 0 \) one can assert that the only element of the set \( H([0, \infty) \circ, \ast) \) is \( H(y) = 0, y \geq 0 \). The same one-element set can be obtained for \( Y = [0, 1] \) and \( a \ast b = (a + b)/2 \).

If \( a \circ b = (a + b)/2 \) and \( a \ast b = a \cdot b \) for \( a, b \in [0, 1] \), then \( H([0, 1], \circ, \ast) \) consists of the functions \( H \) for which
\[ 2H(yb) \leq H(y) + b, \quad y, b \in [0, 1]. \]  
For example, any function which is nondecreasing and bounded from above by the function \( H_0(y) = y \) fulfills (18) because
\[ H(yb) - H(y) \leq 0 \leq b - H(b) \leq b - H(yb). \]
If \( a \circ b = a + b \) and \( a \ast b = ab \) for \( a, b \geq 0 \), then the elements of \( H([\mathbb{R}_+, \circ, \ast]) \) satisfy the inequality
\[ H(yb) \leq H(y) + b, \quad y, b \geq 0. \]
An example of such a function is $H(y) = (\ln y)_+$. Substituting $x = yb$ into (19) we get the following condition

$$H(x) - H(y) \leq \frac{x}{y}, \quad 0 < y < x,$$

which is equivalent to (19), by the monotonicity of $H$.

9. It is easy to check that the set $H(\mathbb{R}_+, +, \wedge)$ is constituted of all nondecreasing functions $H : \mathbb{R}_+ \to \mathbb{R}_+$. In the case of $a \circ b = (a + b)/2$ and $a \star b = a \wedge b$, the set $H([0, 1], \circ, \star)$ includes functions satisfying the inequality

$$H(y \wedge b) \leq (H(y) + b)/2.$$

In particular, any nondecreasing function $H(y) \leq y, y \in [0, 1]$, meets this condition. What is more, the validity of (21) implies that $H(y) \leq y$ for $y \in [0, 1]$. Indeed, if we suppose that $H(\bar{y}) > \bar{y}$ for some $\bar{y}$, then writing $y = \bar{y} = b$ into (21), we get $H(\bar{y}) \leq \bar{y}$, a contradiction.

It is worth to point out that equality holds in (3) for all $f$ and $A$ if $H$ is increasing, $H(Y) = Y$ and $H(y) \circ b = H(y \star b)$ for all $y \in Y$ and $b \in \mu(\mathcal{F})$. Finding functions satisfying these conditions amounts to solving the corresponding functional equations. For example, $\circ = \star = \wedge$ leads to $H(x) = x$, only, while $\circ = \star = \cdot$ and $Y = [0, \infty)$ implies $H(x) = cx, c > 0$.

3 Related results

Since the assumption $A1$ is too restrictive for some applications (see, [13], Theorem 1), it becomes desirable to develop other Jensen type inequalities for the Sugeno integral. The following theorem deals with this issue.

**Theorem 3.1.** Let $p = \int_A f \star \mu \in Y$. Suppose that $H : Y \to Y$ is nondecreasing, left-continuous and satisfies the condition
(C1) for any \( y \leq p \) and \( b \in \mu(F) \), \( H(y) \circ b \geq H(y \ast b) \).

Suppose also

(\( C2 \)) \( \sup_{y \leq p, y \in Y} [y \ast \mu(A \cap \{ f \geq y \})] = p \).

If \( b \mapsto y \circ b \) is nondecreasing for each \( y \in Y \), then

\[
\int_A H(f) \circ \mu \geq H \left( \int_A f \ast \mu \right).
\]

**Proof.** Arguing analogously as in the proof of Theorem 2.1, we obtain

\[
\int_A H(f) \circ \mu \geq \sup_{y \leq p, y \in Y} \left[ H(y) \circ \mu(A \cap \{ f \geq y \}) \right] \\
\geq \sup_{y \leq p, y \in Y} \left[ H(y \ast \mu(A \cap \{ f \geq y \})) \right],
\]

by \( C1 \). From \( C1 \) and \( C2 \), we conclude that \( p = \sup_{y \leq p, y \in Y} (y \ast \mu(A \cap \{ f \geq y \})) \in Y \), hence

\[
\int_A H(f) \circ \mu \geq H \left( \sup_{y \leq p, y \in Y} \left[ y \ast \mu(A \cap \{ f \geq y \}) \right] \right) = H(p).
\]

The condition \( C2 \) is valid for \( \ast = \wedge \). Conversely, suppose that there exists \( \bar{y} > p \) such that

\[
\sup_{y \leq p} \left[ y \wedge \mu(A \cap \{ f \geq y \}) \right] < \bar{y} \wedge \mu(A \cap \{ f \geq \bar{y} \}) \leq p,
\]

which means that

\[
p \wedge \mu(A \cap \{ f \geq p \}) < \bar{y} \wedge \mu(A \cap \{ f \geq \bar{y} \}) \leq p.
\]
Combining this with \( \bar{y} > p \), we get \( \mu(A \cap \{ f \geq p \}) < \mu(A \cap \{ f \geq \bar{y} \}) \), a contradiction.

Theorem 2.3 can be applied to establish the necessary and sufficient conditions for the validity of Liapunov type inequality for Sugeno integral (cf., e.g., [3] and [7]).

**Theorem 3.2.** Let \( Y \subset [0, \infty] \) be an interval, \( 0 \in Y \) and \( \mu(\mathcal{F}) = Y \). Let \( \circ, \star : Y \times Y \to Y \) be operators such that \( a \ast 0 = a \circ 0 = 0 \) and let \( b \mapsto y \circ b, y \mapsto y \circ b \) and \( y \mapsto y \ast b \) be nondecreasing. Assume that \( U, V : Y \to Y \) are increasing functions with ranges \( U(Y), V(Y) \) equal to \( Y \). For arbitrary \( A \in \mathcal{F} \) and measurable \( g : X \to Y \), the Liapunov type inequality

\[
U^{-1} \left( \int_A U(g) \circ \mu \right) \geq V^{-1} \left( \int_A V(g) \ast \mu \right)
\]  

holds iff for any \( a, b \in Y \)

\[
U^{-1}(U(a) \circ b) \geq V^{-1}(V(a) \ast b).
\]  

**Proof.** The necessary and sufficient condition of Theorem 2.3 for \( H(y) = U \left( V^{-1}(y) \right) \), \( y \in Y \), and \( f(x) = V \left( g(x) \right) \) takes the form

\[
U \left( V^{-1}(y) \right) \circ b \geq U \left( V^{-1}(y \ast b) \right), \quad y \in Y, \ b \in \mu(\mathcal{F}).
\]

Putting \( a = V^{-1}(y) \) gives (23). \( \square \)

In the case of \( \circ = \ast = \land \), the condition (23) is satisfied if \( V(y) \geq U(y) \), \( y \in Y \). Moreover, if \( \circ = \ast = \cdot \), then (23) holds for \( U(x) = x^s \) and \( V(x) = x^r \), where \( 0 < s < r, \ x \in [0, 1] \).
The assumption of Theorem 3.2 that $U(Y) = V(Y) = Y$ cannot be dropped.

To see this observe first that, for $Y = [0, \infty]$, $A = X$, and $B \in \mathcal{F}$,

$$\int U(a1_B) \circ \mu = \max \left( \sup_{0 \leq y \leq U(0)} [y \circ \mu(X)], \sup_{U(0) < y \leq U(a)} [y \circ \mu(B)] \right)$$

$$= \max \left( U(0) \circ \mu(X), U(a) \circ \mu(B) \right).$$

Hence, if $U(0) > 0$, then for a properly chosen set $B$,

$$\int U(a1_A) \circ \mu = U(0) \circ \mu(X),$$

so the left-hand side of (22) is meaningless for $U(0) \circ \mu(X) < U(0)$. For example, if $\mu(X) = 1$, $U(0) = 2$ and $\circ = \wedge$, then (S) $\int U(a1_B)d\mu = 1$ and $1 \notin U(Y)$.

Now we give the sufficient and necessary conditions for the reverse inequality to (3) to hold. Note that these conditions are different than those of Theorems 2.1 and 2.2. Since the proofs are analogous to the proofs of Theorems 2.1 and 2.2, they will be omitted.

**Theorem 3.3.** Let $H: Y \to \mathbb{R}$ be an increasing function such that $Y \subset H(Y)$, let $\circ: H(Y) \times \mu(\mathcal{F}) \to \mathbb{R}$ and

$$(B1) \text{ for any } y \in Y \text{ and } b \in \mu(\mathcal{F}), H(y) \circ b \leq H(y \star b).$$

If $f: X \to Y$ is measurable and $\int f \star \mu \in Y$, then $\int H(f) \circ \mu \leq H\left( \int f \star \mu \right)$.

**Theorem 3.4.** Let $Y \subset [0, \infty]$, $0 \in Y$ and $H: Y \to Y$ be an increasing function such that $H(Y) = Y$. Suppose $a \ast 0 = a \circ 0 = 0$ for any $a \in Y$ and the functions $y \mapsto y \circ b$ and $y \mapsto y \star b$ are nondecreasing. For arbitrary $A \in \mathcal{F}$ and $f: X \to Y$ such that $\int f \star \mu \in Y$, the Jensen inequality $\int H(f) \circ \mu \leq H\left( \int f \star \mu \right)$ holds iff $H(y) \circ b \leq H(y \star b)$ for $y \in Y$ and $b \in \mu(\mathcal{F})$. 

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In the literature some studies focus on the properties of $S$-type integral defined as
\[
\int_A f \triangle \mu = \inf_{y \in Y} \left[ y \triangle \mu(A \cap \{f > y\}) \right],
\]
in which $f: X \to Y$, $Y \subset \mathbb{R}$ and $\triangle: \mathbb{R} \times \mu(F) \to \mathbb{R}$ (see, e.g., [11]). We provide the sufficient conditions for Jensen type inequality for $S$-type integral.

**Theorem 3.5.** Let $H: Y \to \mathbb{R}$ be an increasing function such that $Y \subset H(Y)$. Suppose that $\circ: \mathbb{R} \times \mu(F) \to \mathbb{R}$ and the assumption $A1$ of Theorem 2.1 holds. If $\int_A f \star \mu \in Y$, then
\[
\int_A H(f) \circ \mu \geq H \left( \int_A f \star \mu \right).
\]

**Proof.** Since $Y \subset H(Y)$ and $\{f > y\} = \{H(f) > H(y)\}$, we have
\[
\int_A H(f) \circ \mu \geq \inf_{y \in Y} \left[ H(y) \circ \mu(A \cap \{H(f) > H(y)\}) \right]
\]
\[
= \inf_{y \in Y} \left[ H(y) \circ \mu(A \cap \{f > y\}) \right]
\]
\[
\geq \inf_{y \in Y} \left[ H(y \star \mu(A \cap \{f > y\})) \right] \geq H \left( \int_A f \star \mu \right).
\]

**Example 3.1.** Let $\circ = \star = \lor$, where $a \lor b = \max(a, b)$. The condition $A1$ is valid if $H(y) \lor b \geq H(y) \lor H(b)$ for $y \in Y$ and $b \in \mu(F)$. For example, this is the case if $H$ is increasing and $H(y) \leq y$ for $y \in Y$.

**Example 3.2.** Let $\circ = \lor$, $\star = \cdot$ and $Y = [0, 1]$. The condition $A1$ is satisfied for any increasing function as $H(yb) \leq H(y)$, so $H(y) \lor b \geq H(yb)$ for $y, b \in Y$.
Next theorem gives a necessary condition for Jensen type inequality for $S$-type integral.

**Theorem 3.6.** Assume that $\mu(F) = Y = H(Y) \subset [0, \infty]$ and $0 \in Y$. Assume also that $a \circ 0 = 0 \circ a = a$ for $a \in Y$ and the function $y \mapsto y \circ b$ is nondecreasing for $b \in Y$, where $\circ = \circ$ and $\circ = \star$. If $\int_A H(f) \circ \mu \geq H \left( \int_A f \star \mu \right)$ for arbitrary $A \in \mathcal{F}$, measurable $f$ and increasing $H$, then $H(b) \leq b$ for $b \in Y$.

**Proof.** Observe that for $f = a \mathbb{1}_X$, we have
\[
\int_A H(f) \circ \mu = \min \left[ \inf_{y < H(a)} (y \circ \mu(A)), \inf_{y > H(a)} (y \circ 0) \right]
\]
\[
= \min \left[ 0 \circ \mu(A), H(a) \circ 0 \right] = H(a) \wedge \mu(A).
\]
Similarly, $\int_A f \star \mu = a \wedge \mu(A)$. By Jensen’s inequality, $H(a) \wedge \mu(A) \geq H(a) \wedge H(\mu(A))$ for $a \in Y$ and $A \in \mathcal{F}$, so $H(a) \wedge b \geq H(a) \wedge H(b)$ for $a, b \in Y$. Writing $a = \sup_{y \in Y}$, we get our assertion.

\[\Box\]

From Example 3.2 it follows that the condition $H(b) \leq b$ is necessary and sufficient for the case of $\circ = \star = \vee$. Unfortunately, we are not able to establish analogous conditions for other pairs of operators.

### 4 Conclusions

We have presented the necessary and sufficient conditions for Jensen type, reverse Jensen type and Liapunov type inequalities for generalized Sugeno integral. Our method of proof provides results with no restriction on continuity of the fuzzy
measure and the operators. For specific choices of operators, we have characterized classes of functions for which the corresponding inequalities are satisfied.

References


MAREK KALUSZKA
INSTITUTE OF MATHEMATICS
LODZ UNIVERSITY OF TECHNOLOGY
UL. WÓLCZAŃSKA 215
90-924 LODZ
POLAND
E-MAIL: MAREK.KALUSZKA@P.LODZ.PL

ANDRZEJ OKOLEWSKI
INSTITUTE OF MATHEMATICS
LODZ UNIVERSITY OF TECHNOLOGY
UL. WÓLCZAŃSKA 215
90-924 LODZ
POLAND
E-MAIL: OKO@P.LODZ.PL

MICHAL BOCZEK