Computability of Convex Sets
(Extended Abstract)

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Abstract. We investigate computability of convex sets restricted to rational inputs. Several quite different algorithmic characterizations are presented, like the existence of effective approximations by polygons or effective line intersection tests. We also consider approximate computations of the \(n\)-fold characteristic function for several natural classes of convex sets. This yields many different concrete examples of \((1, n)\)-computable sets.

1 Introduction

Convex sets play a prominent role in mathematical programming, computational geometry, convex analysis, and many other areas. There is a large number of papers dealing with polynomial time computable convex sets, but the basic question “Which convex sets are computable?” was scarcely studied.

In this paper we characterize computable convex sets and show that several quite different approaches lead to the same notion. We call a convex set \(A\) computable if there is an algorithm which computes the characteristic function of \(A\) for all rational inputs that do not belong to the boundary of \(A\). It turns out that all other apparently weaker notions which have been proposed in the literature are equivalent to this definition. For instance, the existence of an algorithm for the “weak membership problem” of Grötschel, Lovász, and Schrijver [GLS88], or the property of being “Turing located” introduced by Ge and Nerode [GN94]. On the other hand, the apparently stronger requirement that there is a computable double sequence of rational polygons converging to \(A\) from inside and from outside, is also equivalent. Finally, if it is decidable which rational lines intersect the interior of \(A\), then \(A\) is already computable.

In the second part we show that several natural classes of convex sets are “approximable”, in the sense that for some fixed \(n\) and any rational numbers \(x_1, \ldots, x_n\) we can effectively exclude one of the \(2^n\) possibilities for the \(n\)-fold characteristic function \((\chi_A(x_1), \ldots, \chi_A(x_n))\). Approximable sets have previously been studied in recursion theory and complexity theory where such sets have been constructed by diagonalization. In contrast, we provide many natural and intuitive examples. Our results are established by effectivizing the concept of the Vapnik-Chervonenkis-dimension.

In the following we will mostly deal with a special class of convex sets, namely, bounded convex sets which have an interior point. We will call these sets convex regions.
The following notations are used: we write \( xy \) for the line segment joining the points \( x \) and \( y \). The line through these points is denoted by \( l(x, y) \). As usual \( S(x, r) \) is the closed circle (sphere) with radius \( r \) and center \( x \). The interior, the closure, and the boundary of \( A \) are denoted by \( A^\circ, \overline{A}, \) and \( \partial A \), respectively. The complement of \( A \) is written as \( A^c \). We use \( d_H(A, B) \) to denote the Hausdorff distance between two subsets \( A \) and \( B \) of \( \mathbb{R}^2 \).

In section 2 we only present proof sketches, in section 3 we have to omit all proofs. For full details the reader is invited to consult [Sch94].

## 2 Computable Convex Sets

We are interested in establishing a suitable notion of computability for convex sets. For the sake of exposition we restrict ourselves to convex regions in \( \mathbb{R}^2 \), but with more technical and notational effort it is possible to generalize the definitions and results to higher dimensions.

As we have decided to use the rational model for our investigations the following definition might appear natural: a set \( A \) is called computable if we can effectively decide whether a rational point lies in \( A \) or not. Approaching convex sets in a geometric setting will show that this definition is not very helpful. Typically geometric operations depend on interior points and cannot distinguish two sets differing only on their boundary. Another problem is that the boundary of a set may turn out to be quite complicated (in the sense of computability) although the set itself and its complement are not: take for example the circle around the origin with radius \( 1 \). There are uncountably many sets that differ from this circle only in the rational points on the boundary, so most of them are not computable.

This motivates the following definition which will prove invariant to different approaches.

**Definition 1.** (Weak characteristic function) Let \( A \) be a subset of \( \mathbb{R}^2 \). The function
\[
\omega_A : \mathbb{Q}^2 \to \{0, 1\} : x \mapsto \begin{cases} 
1 & x \in A^\circ \\
0 & x \notin \overline{A} \\
\top & \text{else}
\end{cases}
\]
is called the *weak characteristic function* of \( A \).

The attribute “weak” is justified for convex regions as in this case the weak characteristic function is Turing reducible to the characteristic function (even uniformly). As we mentioned above, the converse is not true; one can even construct (by diagonalization) a convex compact set \( A \) such that \( \omega_A \) is partial recursive but \( \chi_A \) is not.

We will now list a couple of other natural algorithmic problems. First we consider two operations on sets which can be regarded as approximations (from a geometric viewpoint).
Definition 2. (Weak Membership Tests) For a set $A \subseteq \mathbb{R}^2$ every partial function $\alpha : \mathbb{Q}^2 \times \mathbb{Q}_+ \rightarrow \{0, 1\}$ which for all $x \in \mathbb{Q}^2$ and $r \in \mathbb{Q}_+$ satisfies the conditions

- $S(x, r) \subseteq A^c \Rightarrow \alpha(x, r) = 1$
- $S(x, r) \subseteq (A^c)^c \Rightarrow \alpha(x, r) = 0$

is called a Weak Membership Test (WMT) for $A$. We will write $\alpha_A$ for a WMT for $A$.

The name WMT is explained by the observation that $\alpha_A(x, r) = 1$ implies $d(x, A) \leq r$, and $\alpha_A(x, r) = 0$ implies $d(x, A^c) \leq r$. Note, however, that we do not require $\alpha$ to be everywhere defined. Also, if $S(x, r)$ intersects both $A$ and its complement then $\alpha_A(x, r)$ may give an arbitrary answer.

The following definition is due to Grötschel, Lovász, and Schrijver [GLS88].

Definition 3. (WMEM) For a set $A \subseteq \mathbb{R}^2$ we say that the Weak Membership Problem (WMEM) is solvable if there is a total recursive function $\beta_A$ from $\mathbb{Q}^2 \times \mathbb{Q}_+$ to $\{0, 1\}$ such that

- $\beta_A(x, r) = 1 \Rightarrow S(x, r) \cap \overline{A} \neq \emptyset$
- $\beta_A(x, r) = 0 \Rightarrow S(x, r) \cap (\overline{A})^c \neq \emptyset$

Analogically to WMTs we investigate another operation which is based on lines intersecting the set $A$.

Definition 4. (Line Intersection Tests) Given a set $A \subseteq \mathbb{R}^2$ every partial function $\gamma : \mathbb{Q}^2 \times \mathbb{Q}_+ \rightarrow \{0, 1\}$ which satisfies the conditions

- $l(x, y) \cap A^c \neq \emptyset \Rightarrow \gamma(x, y) = 1$
- $l(x, y) \cap A = \emptyset \Rightarrow \gamma(x, y) = 0$

is called a Line Intersection Test (LIT). We will write $\gamma_A$ for a LIT with respect to $A$.

With regard to LITs the following notion from convex geometry will prove helpful and can be used to establish another equivalent characterization:

Definition 5. (Extremal points) A point $x$ of a set $A \subseteq \mathbb{R}^2$ is said to be extremal (for $A$) if it does not lie on a line strictly between two other points of $A$. The set of extremal points of $A$ is denoted by $\text{ext}(A)$.

To define the next notion we need to recall a concept from recursive analysis [Ko91, PR89]. A real number $x \in \mathbb{R}^k$ is represented by a function $\phi : N \rightarrow \mathbb{Q}^k$ if $|\phi(n) - x| < 2^{-n}$ for all $n$. A real function $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ is computable if there is an oracle Turing machine $M$ such that for every $x$ and every representation $\phi$ of $x$, $\lambda n. M^{\phi}(n)$ is a representation of $f(x)$, i.e., $M$ computes a representation of $f(x)$ if it is given a representation of $x$ as an oracle.

Definition 6. (Turing located) A set $A \subseteq \mathbb{R}^2$ is called Turing located if the real function $x \mapsto d(x, A)$ is computable.
This definition was introduced by Ge and Nerode [GN94] who were looking for an effective version of the Krein-Milman theorem. They proved that a convex compact set \( A \) is Turing located iff the extreme points of \( A \) can be approximated in the following sense: There is a uniformly computable sequence of real numbers \( \{ y_i \}_{i \in \mathbb{N}} \) such that their closure is the closure of the set of extreme points of \( A \), and there is a computable function \( g \) such that \( d_H(\text{conv}(y_1, \ldots, y_{g(n)}), A) \leq 2^{-n} \) for all \( n \).

We can now state our main theorem which says that for convex regions all the above definitions are equivalent.

**Theorem 7.** Let \( A \) be a convex region. Then the following are equivalent:

(i) \( \omega_A \) is partial recursive.
(ii) There are computable sequences \( (P_i)_{i \in \mathbb{N}} \) and \( (Q_i)_{i \in \mathbb{N}} \) of rational polygons such that \( P_i \subseteq A \subseteq Q_i \) and \( d_H(P_i, Q_i) < 2^{-i} \) for all \( i \).
(iii) There is a partial recursive WMT \( \alpha_A \).
(iv) There is a partial recursive LIT \( \gamma_A \).
(v) The Weak Membership Problem for \( A \) is solvable.
(vi) \( A \) is Turing located.

**Sketch.** For the proof we introduce the following variants of (iii) and (iv):

(iii') The WMT \( \alpha_A(x, r) := \begin{cases} 1 & S(x, r) \subseteq A^0 \\ 0 & S(x, r) \subseteq (A^c)^0 \end{cases} \) is partial recursive.

(iv') The LIT \( \gamma_A(x, y) := \begin{cases} 1 & l(x, y) \cap A^c \neq \emptyset \\ 0 & l(x, y) \cap \overline{A} = \emptyset \end{cases} \) is partial recursive.

The proof has several parts:

(ii) \( \Rightarrow \) (iii') and (ii) \( \Rightarrow \) (iv') This is a direct construction: We search for an \( i \) such that \( S(x, r) \subseteq (P_i)^c \) or \( S(x, r) \cap Q_i = \emptyset \). The search terminates iff \( (x, r) \in \text{dom}(\alpha_A) \). The argument for LITs is similar.

(ii') \( \Rightarrow \) (iii) and (iv') \( \Rightarrow \) (iv) Trivial.

(iii) \( \Rightarrow \) (i) To prove this we need the following technical notion:

**Definition 8.** (Wedgeset) Call a set \( A \subseteq \mathbb{R}^2 \) a **wedgeset** if there is a triangle \( \Delta \) such that for every \( x \in A \) there is a congruent copy of \( \Delta \) which is a subset of \( A \) and has \( x \) as a vertex.

Note that without loss of generality we may assume that the triangle is isosceles with vertex \( x \).

**Lemma 9.** Every convex region and its complement are wedgesets.

The simple proof is omitted. Now, all we need is the following lemma.
Lemma 10. Let $A \subseteq \mathbb{R}^2$ be a set for which $A^c$ is a wedgeset and a WMT $\alpha_A$ is partial recursive. Then the rational points of $A^c$ are recursively enumerable.

The last two lemmas together directly yield the desired implication. We will only sketch the geometric idea behind the proof of the last lemma.

*Sketch.* The rather intuitive idea of the proof is that a point which is closely surrounded by a ring of discs which do not lie in the complement of $A$ has to be an interior point. Naturally there have to be some conditions concerning the location of the discs to render this argument correct.

As $A^c$ is a wedgeset there is a triangle $\triangle$ with legs of (rational) length $\varepsilon$ and $\delta$ the (rational) length of the base. For a point $x$ we search for a set of discs which fulfill the following conditions

1. The centers of all discs have distance $r \in \mathbb{Q}_+$ from $x$ with $r < \frac{\varepsilon}{2}$.
2. The discs have radius $\frac{r + \varepsilon}{2}$.
3. The discs cover the circle with radius $r$ and center $x$.
4. $\alpha_A$ is 1 for all discs in the set.

\[\text{Fig. 1. Construction in Lemma 10}\]
First of all, it is obvious that for each interior point $x$ of $A$ there is such a set of discs (choosing $r$ small enough) and we are able to find it effectively by dovetailing the computations. The interesting part is to prove that whenever we find such a set of discs for a point $x$, this point is, in fact, an interior point. Granted we have found a set of discs satisfying (1) to (4), suppose for contradiction that $x \notin A^e$, that is $x \in A^c$ or $x \in \partial A$.

If $x \in A^c$ we know that there is a triangle with vertex $x$ which is a subset of $A^c$. We claim that one disc out of the set of discs wholly lies inside the triangle. Otherwise there is a point on the circle with radius $r$ and center $x$, which is not covered by one disc out of the set of discs. Actually the point where the bisector of $\Delta$ in $x$ intersects the circle has distance $> \frac{r}{3}$ from the nearest point on the boundary of $\Delta$. As the diameter of the discs is smaller than $\frac{r}{2}$ this point is not covered. Thus there is a disc which lies inside $\Delta$ and thereby in the interior of $A^e$ which contradicts the fact that it was valued 1 by $a_A$. For $x \in \partial A$ we use the same argument for a point $x' \in A^e$ which is sufficiently near to $x$.

(i) $\iff$ (ii) The direction (ii) $\Rightarrow$ (i) is easy.

The existence of rational polygons as in (ii) is guaranteed by the well-known Lemma of Hadwiger (see [Mar77]). For the direction (i) $\Rightarrow$ (ii) we need an effective proof of this Lemma. The idea is to compute $\omega_A$ on a sufficiently fine rational grid which covers $A$. We dovetail the computation of $\omega_A$ until for every vertex $p$, either $\omega_A(p)$ is defined or $\omega_A(q)$ is defined where $q$ is one of the neighbors of $p$ (if all of them lie on the boundary then $p$ is an interior point, thus one of the computations must be defined). Let $T$ be the set of all vertices $p$ for which we discovered that $\omega_A(p) = 1$. The convex hull of $T$ gives us the approximation from inside. For the approximation from outside we take a convex polygon that covers all point of distance $\delta$ from $T$. Suitable values of the grid size and $\delta$ can be computed from any triangle $\Delta$ witnessing that $A$ is a wedge set.

(iv) $\Rightarrow$ (i) The proof consists of two lemmas which prove the recursive enumerability of the rational points of $(A^c)^e$ and $A^e$ respectively. The easy part is the following

**Lemma 11.** Let $A$ be a convex region which has a computable LIT $\gamma_A$. Then the rational points of $(A^c)^e$ and $A^e$ respectively. The easy part is the following

**Sketch.** Since $A^e \neq \emptyset$ there is a rational point $p \in A^e$. Then for all $x \in \mathbb{Q}^2$ (in fact for all $x \in \mathbb{R}^2$):

$$x \in (A^c)^e \Rightarrow (\exists y, z \in \mathbb{Q}^2) [x \notin l(y, z) \wedge \gamma_A(y, z) = 0 \wedge xp \cap l(y, z) \neq \emptyset] ,$$

i.e. there is a rational line $l(y, z)$ which separates $x$ and $p$ without intersecting the interior of $A$. For the implication from left to right the boundedness of $A$ is important as the example of a half plane with noncomputable slope shows.
We now show that the rational points in the interior of $A$ are recursively enumerable. This is more complicated and requires the notion of extremal point as defined above.

The main theorem on extremal points is by Minkowski (later generalized as the famous Krein-Milman theorem of functional analysis):

**Proposition 12 Minkowski [Min11, Jac71, Brø83].** Let $A$ be a convex and compact subset of $\mathbb{R}^2$ then $A$ is the convex hull of its extremal points:

$$A = \mathrm{conv}(\mathrm{ext}(A))$$

The following lemma proves an interesting fact about extremal points which makes it possible to approximate them:

**Lemma 13.** Let $A$ be a convex region, $x \in \mathrm{ext}(\overline{A})$. Then for every $\delta > 0$ there are two rational points with distance less than $\delta$ from $x$ which do not lie in $\overline{A}$ but the line segment joining the two points contains an interior point of $A$.

We will omit the proof (it is by contradiction).

**Lemma 14.** Let $A$ be a convex region which has a partial recursive LIT $\gamma_A$. Then the rational points of $A^e$ are recursively enumerable.

**Sketch.** Let $p$ be a rational interior point of $A$. Search for triplets $(y, z, w)$ of rational points which satisfy

1. $\gamma_A(y, w) = 0$ ($\Rightarrow l(y, w) \cap A^e = \emptyset$)
2. $\gamma_A(z, w) = 0$ ($\Rightarrow l(z, w) \cap A^e = \emptyset$)
3. $\gamma_A(y, z) = 1$ ($\Rightarrow l(y, z) \cap \overline{A} \neq \emptyset$)
4. $pw$ intersects $yz$.

The diagram shows the relative position of the points.

*Fig. 2. Construction in Lemma 14*
In this situation we know that the triangle $\triangle(w, y, z)$ contains a point of $\overline{A}$. On the other hand the preceding lemma guarantees that every extremal point of $A$ can be approximated in this manner.

As every interior point of $A$ lies in the interior of the convex hull of at most four extremal points of $\overline{A}$ and conversely, we have an algorithm at hand to detect whether $x \in \mathbb{Q}^2$ is an interior point: search for approximations of (three and four) extremal points as described above and check whether $x$ is in the intersection of all convex hulls generated by vertices in the approximating triangles. If $x$ is indeed an interior point there will be a sufficiently close approximation of extremal points (because of the preceding lemma) such that $x$ is contained in all of these convex hulls. On the other hand only interior points will satisfy these conditions. The search can be done effectively by dovetailing.

$(iii) \iff (v)$ Let the WMEM problem be solvable by $\beta_A$. We claim that $\beta_A$ is a WMT:

- $S(x, r) \subseteq A^c \Rightarrow S(x, r) \cap (\overline{A})^c = \emptyset \Rightarrow \beta_A(x, r) \neq 0 \Rightarrow \beta_A(x, r) = 1$
- $S(x, r) \subseteq (A^c)^c \Rightarrow S(x, r) \cap \overline{A} = \emptyset \Rightarrow \beta_A(x, r) \neq 1 \Rightarrow \beta_A(x, r) = 0$

On the other hand the existence of a partial recursive WMT implies $(ii)$. So we can effectively approximate $A$ by $P_i$ and $Q_i$ from inside and outside. To solve the Weak Membership Problem we wait until $S(x, r)$ intersects either $P_i$ or $Q_i$ (this computation terminates, because $d_H(P_i, Q_i) \to 0$). In the first case we let $\beta_A$ be 1 and 0 else. It is easy to see that the resulting function solves WMEM.

$(ii) \Rightarrow (vi)$ This follows using the fact that if $x$ is a computable real, then the distance from $x$ to any given rational polygon is also (uniformly) computable.

$(vi) \Rightarrow (iii)$ To prove this implication it must be shown that a WMT is computable if $A$ is Turing located. For a pair $(x, r) \in \mathbb{Q}^2 \times \mathbb{Q}_+$ compute $d \in \mathbb{Q}_+$ such that $|d(x, A) - d| < \frac{r}{2}$ (this is effectively possible since $A$ is Turing located). Define $\alpha_A(x, r)$ to be 1 if $d < \frac{r}{2}$ and 0 else. We have to check that this is indeed a WMT:

- $S(x, r) \subseteq A^c \Rightarrow d(x, A) = 0 \Rightarrow d < \frac{r}{2} \Rightarrow \alpha_A(x, r) = 1$
- $S(x, r) \subseteq (A^c)^c \Rightarrow d(x, A) > r \Rightarrow d > \frac{r}{2} \Rightarrow \alpha_A(x, r) = 0$

This finishes the proof sketch.

Remarks. (1) Item $(ii)$ of the theorem can be substituted by a seemingly weaker variant:

$(ii')$ There is a computable sequence $(P_i)_{i \in \mathbb{N}}$ of rational polygons such that $d_H(P_i, A) < 2^{-i}$ for all $i$.

Now $(ii)$ obviously implies $(ii')$. The other implication follows from $(ii') \Rightarrow (vi)$. The implication $(vi) \Rightarrow (ii')$ was proved by Ge and Nerode in their paper [GN94].

(2) We have the following, by no means obvious, result:

A convex region is Turing located iff its complement is Turing located.
If the complement of a set is Turing located we can show as in the proof of (vi) ⇒ (iii) that a WMT is computable. On the other hand we have (vi) ⇒ (ii) and (ii) implies that the complement of a convex region is Turing located as in (vi) ⇒ (vi).

(3) Grötschel, Lovász and Schrijver in their book [GLS88] take WMEM as a starting point to prove further equivalent variants (even preserving polynomial time computability) like optimization of linear functions, separation by hyperplanes and others.

At this point the question of recursive enumerability of convex sets suggests itself. Adhering to our concept (excluding the boundary) we call a set \( A \subseteq \mathbb{R}^2 \) recursively enumerable if the rational interior points of \( A \) are recursively enumerable. A list of results rather similar to those of the theorem can be obtained. For example (using the same method as above) one easily proves that \( A \) is r.e. iff the rational circles which intersect \( A \) are recursively enumerable. The same result (although with a new proof) holds for intersecting lines. Furthermore \( A \) is r.e. iff it can be approximated from within by rational polygons. The convergence in this case will in general not be effective, because by the theorem this would automatically imply that \( A \) is recursive. Analogically a weaker variant of the concept Turing located can be used to give another equivalent characterization.

### 3 Approximable Convex Sets

In this section we do not require our geometric sets to be computable. Nevertheless, in many cases we can still compute certain types of approximations which are currently intensively studied in recursion theory [BGGO93, Ga91] and complexity theory [BKS94]. Natural examples of noncomputable sets which are approximable in this sense were rare and so it came as a surprise that the well-known convex figures dealt with in elementary geometry like circles and polygons should prove quite resourceful in this respect. The results are established in terms of an effective variant of the Vapnik-Cervonenkis-dimension [Ass83, VC71] used in probability theory ([Dud84]), learning theory ([BE89]) and computational geometry ([HW87]).

**Definition 15.** (Approximable sets) A set \( A \subseteq \mathbb{Q}^d \) is \((1, n)\)-computable if there is a recursive function \( f \) from \((\mathbb{Q}^d)^n\) into \(\{0, 1\}^n\) such that for all rational points \( x_1, \ldots, x_n: (\chi_A(x_1), \ldots, \chi_A(x_n)) \neq f(x_1, \ldots, x_n) \). A set is called approximable if there is an \( n \) such that the set is \((1, n)\)-computable. In this case the greatest \( n \) such that the set is not \((1, n)\)-computable is called the effective Vapnik-Cervonenkis-dimension of \( A \) and is denoted by VC\(_{eff}\)(\( A \)). The effective Vapnik-Cervonenkis-dimension of a concept class \( \mathcal{C} \) is defined to be the supremum of the effective Vapnik-Cervonenkis-dimension of its members (the concepts).

In other words, the effective Vapnik-Cervonenkis-dimension of a concept class is the smallest \( n \) such that for every concept from the class there is an algorithm to exclude one of the \( 2^n + 1 \) possibilities of the \( (n + 1)\)-fold characteristic function.
This is related to the original VC-dimension as follows: The VC-dimension of a concept class is the greatest $n$ such that there is a set of $n$ points which is \textit{shattered} by the concept class (i.e. every subset of the $n$ points can be written as the intersection of the set of $n$ points with a concept). In other words, for any $n + 1$ points $x_1, \ldots, x_{n+1}$ there is a vector $v \in \{0,1\}^{n+1}$ such that $v \neq (\chi_A(x_1), \ldots, \chi_A(x_{n+1}))$ for all $A \in \mathcal{C}$. For all natural concept classes $\mathcal{C}$, there is a \textit{computable} function $f$ such that $f(x_1, \ldots, x_{n+1}) = v$. So, every concept from the class is $(1, n + 1)$-computable via the same $f$; in particular, $\text{VC}(\mathcal{C}) \geq \text{VC}_{\text{eff}}(\mathcal{C})$. However, note that in the definition of $\text{VC}_{\text{eff}}$, $f$ can depend on $A$, and therefore the inequality may be strict.

A lower bound for $\text{VC}(\mathcal{C})$ is shown by exhibiting a configuration of points which cannot be shattered by $\mathcal{C}$. In contrast, a lower bound for $\text{VC}_{\text{eff}}(\mathcal{C})$ requires a more complicated construction. To show that $\text{VC}_{\text{eff}}(\mathcal{C}) \geq n$, we construct a particular concept $A \in \mathcal{C}$ such that $A$ is not $(1, n)$-computable. $A$ is constructed as the limit of concepts $A_0, A_1, \ldots$ such that $A_i$ is not $(1, n)$-computable via the first $i$ recursive functions. To this end we need to consider “local shatterings”, i.e., we are given a concept $B \in \mathcal{C}$ and any $\epsilon > 0$ and need a configuration of $n$ points that is shattered by $\{C \in \mathcal{C} : d_H(C, B) < \epsilon\}$. An example where this becomes quite involved is the class of all triangles.

The following table summarizes our results:

<table>
<thead>
<tr>
<th>concept class $\mathcal{C}$</th>
<th>$\text{VC}(\mathcal{C})$</th>
<th>$\text{VC}_{\text{eff}}(\mathcal{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>half spaces in $\mathbb{R}^d$</td>
<td>$d + 1$</td>
<td>$d$</td>
</tr>
<tr>
<td>bounded intervals in $\mathbb{R}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>bounded intervals with rational length</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>triangles</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>convex $n$-gons</td>
<td>$2n + 1$</td>
<td>probably $2n + 1$</td>
</tr>
<tr>
<td>spheres in $\mathbb{R}^d$</td>
<td>$d + 1$</td>
<td>$d + 1$</td>
</tr>
<tr>
<td>discs with rational radius</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>spheres in $\mathbb{R}^d$ with rational center</td>
<td>$d + 1$</td>
<td>1</td>
</tr>
</tbody>
</table>

The results concerning $\text{VC}$ are well-known (see [WD81]). The results on $\text{VC}_{\text{eff}}$ are new. Here is an example of how the $\text{VC}_{\text{eff}}$ column should be read: an effective Vapnik-Cervonenkis-dimension of 7 for triangles means that every triangle is $(1, 8)$-computable but there is a triangle which is not $(1, 7)$-computable.

The upper bounds of the above table can be stated in a stronger form: Beigel, Kummer and Stephan in [BKS94] introduced the concept of polynomially approximable sets, that is approximable sets which are approximable by polynomial
time functions. All of the examples in the table are even polynomially approximable, thus we also obtain nontrivial examples in this context. Agrawal and Arvind [AgAr94] have recently shown that even infinite nonuniform families of half spaces share many properties of approximable sets in the resource-bounded setting.

The results of the table can be generalized.

**Definition 16.** A subset \( A \) of \( \mathbb{Q}^d \) is **algebraic** if it is a Boolean combination (i.e. union, intersection, and complement) of sets of the form \( \{ x \in \mathbb{Q}^d : f(x) > 0 \} \) where \( f \) is a real polynomial.

Algebraic sets comprise most of the sets usually dealt with in elementary geometry. This holds especially for convex sets like spheres, half spaces, conics.

**Theorem 17.** Every algebraic set is approximable.

This theorem is a direct consequence of some effective versions of results proved in [Dud84]. The effective versions appear in [Sch94]. The theorem yields another interesting conclusion: it is known that approximable sets (i.e. sets with \( VC_{\text{eff}}(A) < \infty \)) are not \( m \)-complete (since the halting problem is not approximable, see e.g. [Ga91, Theorem 23]), so we have:

**Corollary 18.** No algebraic set is \( m \)-complete.

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**References**


