Full Minimal Steiner Trees on Lattice Sets

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1. INTRODUCTION

Given a finite set of points $P$ in the Euclidean plane, the Steiner problem asks us to construct a shortest possible network interconnecting $P$. Such a network is known as a minimal Steiner tree. The Steiner problem is an intrinsically difficult one, having been shown to be $NP$-hard [7]; however, it often proves far more tractable if we restrict our attention to points in special geometric configurations. One such restriction which has generated considerable interest is that of finding minimal Steiner trees for nice sets of integer lattice points. The first significant result in this direction was that of Chung and Graham [4], which, in effect, precisely characterized the minimal Steiner trees for any horizontal $2 \times n$ array of integer lattice points. In 1989, Chung et al. [3] examined a related problem, which they described as the Checkerboard Problem. They asked how to find a minimal Steiner tree for an $n \times n$ square lattice, that is, a collection of $n \times n$ points arranged in a regular lattice of unit squares like the corners of the cells of

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a checkerboard. Although their paper gave a series of conjectured solutions to this problem, not all of which turn out to be correct, they were unable to suggest a method for proving their claims. The case $n = 2^k$ was recently solved in [1].

In this paper, we examine a more general situation, namely the nature of minimal Steiner trees for Steiner-closed lattice sets, which we define to be sets of integer lattice points satisfying two conditions, the first of which says that they have a spanning tree all of whose edges have length 1, and the second of which is a technical condition which we believe to be redundant. These conditions, which ensure that the points are not sparsely scattered, are given in Section 2. Our analysis converts the largely geometric problem of constructing these trees to a somewhat simpler combinatorial one, which we study in the sequel to this paper [2].

Let $T^*$ denote a minimal Steiner tree for a Steiner-closed lattice set. The key feature of the conjectured solutions of Chung et al. [3] for the cases where the Steiner-closed lattice set is an $n \times n$ square lattice is that they use as their principal building block for $T^*$ the minimal Steiner tree for the corners of a unit square (shown in Fig. 1), which we will denote by $X$. A Steiner tree, such as $X$, is full if each of its terminals have degree 1. The full components of $T^*$ can be thought of as being the smallest irreducible 'blocks' from which the $T^*$ is composed (by union at the terminals). When $n = 2^k$, all the full components of $T^*$ are $X$s. This is proved in [1] by showing that, per terminal, $X$ is in some sense the most efficient possible component forming part of $T^*$. If $n \neq 2^k$ then $T^*$ cannot be built up solely from $X$s, hence it becomes necessary to examine what other full trees can occur in checkerboards.

In Theorem 6.6 we completely classify all such full components. In particular, we show that all possible full components of $T^*$ belong to a small number of easily understood classes. This classification greatly simplifies the problem of constructing minimal Steiner trees for specific Steiner-closed lattice sets. Our next paper [2] will demonstrate how the classification allows us to find a minimal Steiner tree for any rectangular array of integer lattice points using the concept of excess established in [1].

The strategy for achieving the classification is as follows. We first establish some preliminary definitions and general techniques in Section 2. The results in this section are not specific to Steiner trees on lattice sets, and
can be thought of as comprising a basic ‘toolchest’ of techniques for constructing minimal Steiner trees in a wide range of different contexts. We then consider a full subtree, $T$, of $T^*$ and define $G(S_T)$ to be the graph on $S_T$, the collection of square cells and triangular half cells containing parts of $T$, with the obvious adjacency. In Section 3 it is shown that $G(S_T)$ is a tree and that there are precisely two Steiner points in each square of $S_T$ and one Steiner point in each triangle of $S_T$. Section 4 introduces the concept of quasi-leaves of $G(S_T)$ which allow us to further investigate the structure of $T$, in Section 5, as we move inwards from leaves of $G(S_T)$. This results in a structure theorem for $S_T$ which states that $G(S_T)$ has only two leaves and $S_T$ has a restricted internal structure. In this case $S_T$ is said to be a strip. Finally, in Section 6, we closely examine minimal steiner trees corresponding to strips to determine which ones can possibly occur as subtrees of $T^*$. This final classification is complete in the sense that it lists all full components that can occur in $T^*$, and every full component listed does occur for some choice of Steiner-closed lattice set.

2. PRELIMINARIES

A tree, $T$, in the Euclidean plane, consisting of vertices and straight-line edges connecting the points of $P$ is called a Steiner tree if the angle between any two edges meeting at a vertex is greater than or equal to $120^\circ$ and all vertices of $T$ not in $P$ have degree 3. Such vertices are called Steiner points, and it is clear that the edges meeting at a Steiner point make angles of precisely $120^\circ$ with each other. Any minimal length network interconnecting $P$ is a Steiner tree. A tree connecting the points of $P$ without the addition of any new vertices is called a spanning tree, and the shortest such tree a minimal spanning tree.

The points in $P$ are referred to as terminals. Throughout this paper we will denote the terminals by $a$, $b$, $c$, $d$, ... and indicate the terminals of a unit square by listing them counterclockwise from the top left-hand corner. Steiner points are usually denoted by $s$ with subscripts.

After Cockayne [5], $(ab)$ denotes the third vertex of the equilateral triangle $ab$ where the vertices are listed in counterclockwise order. To differentiate between open and closed line segments, we will denote the line segment between points $a$ and $b$ by $[ab]$ if it is closed (that is, includes $a$ and $b$) or simply by $ab$ if it is open.

Consider an infinite square unit lattice on the Euclidean plane. A finite subset, $P$, of vertices of this lattice will be said to form a Steiner-closed lattice set if it satisfies the following conditions:

(i) there exists a spanning tree for $P$ all of whose edges have length 1; and
(ii) given lattice points \( a \) and \( b \) such that \(|ab| = 1\), if a minimal Steiner tree for \( P \) intersects the interior of \( ab \) then \( a \) and \( b \) are elements of \( P \).

Note that if a set of lattice points \( P \) has the property that for any unit lattice edge meeting a lattice point not in \( P \) the interior of that edges lies entirely outside the convex hull of \( P \), then \( P \) is Steiner-closed. It follows, for example, that an \( n \times n \) square lattice forms a Steiner-closed lattice set. Indeed, we believe the following to be true:

**Conjecture.** Condition (ii) is redundant in the above definition of a Steiner-closed lattice set; that is, \( P \) is Steiner-closed if and only if there exists a spanning tree for \( P \) all of whose edges have length 1.

We will use the word *square* to refer exclusively to a unit square of a Steiner-closed lattice set, and the word *triangle* to refer exclusively to an isosceles right triangle whose vertices belong to a Steiner-closed lattice set and whose orthogonal edges have length 1. Hence, a triangle is half a square.

First we establish some simple facts.

**Lemma 2.1.** Suppose \( T \) is a minimal Steiner tree for a Steiner-closed lattice set.

(i) If \( p, q \) are two points (not necessarily vertices) in \( T \) and \( s \) and \( t \) are vertices of \( T \), adjacent to each other and lying in the path between \( p \) and \( q \), then \(|pq| \geq |st|\), and the inequality must be strict if \( s \) or \( t \) are Steiner points. Moreover, no edge of \( T \) has length greater than 1.

(ii) No edge of \( T \) intersects the interior of two orthogonal sides of a triangle or two opposite sides of a square.

(iii) No convex path in a full component of \( T \) intersects two parallel lines of distance two. Hence, the terminals at the ends of a convex path in a full component of \( T \) are the endpoints of either a side or a diagonal of a square.

(iv) If a path in \( T \) is convex with respect to a vertex of a right angle and crosses each of its two legs at exactly one point, then the path has only one Steiner point in the right angle and meets the legs at no more than 30°.

(v) No two parallel edges of \( T \) intersect the interior of a single side of a square.

**Proof.** Statement (i) follows immediately from the minimality of \( T \) and the fact that a Steiner-closed lattice set can be spanned by edges of length 1.

In order to see (ii) we argue by contradiction. Let \( abc \) be a triangle with right angle at \( a \), and assume a single edge of \( T \) intersects \( ab \) and \( ac \) at \( p \)
and \( q \) respectively. By definition, \( a \) is an element of the Steiner-closed lattice set, hence we can assume, without loss of generality, that there is a path in \( T \) from \( p \) to \( a \) not passing through \( q \) (otherwise swap the roles of \( p \) and \( q \)). But this implies \( T \) is not a minimal as we can replace the line segment \( pq \) by the shorter line segment \( qa \) to create a shorter tree. The second part of (ii) follows directly from (i).

The remaining statements have easy proofs. In particular, (iii) follows from (i); (iv) is a consequence of angle considerations and is independent of the minimality of \( T \); and (v) is a corollary of (ii).

We use four well-known techniques, outlined in the following propositions, to help eliminate non-optimal Steiner trees.

**Proposition 2.2 (The Simpson–Heinen Construction)** (See [9]). Let \( abc \) be a triangle, all of whose angles are less than \( 120^\circ \). Let \( S \) be the minimal Steiner tree on \( a, b, c \) with Steiner points \( s \) (as in Fig. 2). Then \( (ac) \) lies on the extended line \( bs \), and \( |S| = |b(ac)| \).

This proposition provides a convenient way of referring to the topology of a given Steiner tree. For example, the Steiner tree \( T \) on terminals \( d, u, p, q \) and \( r \) has topology \( (p(ud))(rq) \) only if it immediately follows, by repeated application of Proposition 2.2, that \( |T| = |(p(ud))(rq)| \). This topology is illustrated by the tree in solid lines in Fig. 3. The repeated use of Proposition 2.2 to calculate the length of a Steiner tree with a given topology forms the basis of Melzak's algorithm [10]. The repeated use of this proposition also gives a practical method for constructing the Simpson line of a Steiner tree from any point on the tree.

Suppose \( p_1p_2p_3p_4 \) is a convex quadrilateral. Let \( \phi(p_1p_2, p_3p_4) \) denote the angle at the intersection of the diagonals which faces \( p_1p_2 \).

![Fig. 2. The Simpson–Heinen Construction.](image-url)
Fig. 3. The tree in solid lines has topology \((p(ud))(rq)\) and is not minimal.

**Proposition 2.3 (Pollak’s Theorem) [11].** Suppose both full Steiner trees \((p_1p_2)(p_3p_4)\) and \((p_4p_1)(p_2p_3)\) exist, then \((p_2p_1)(p_4p_3)\) is minimal if \(\phi(p_1p_2,p_3p_4) \leq 90^\circ\).

Note that Proposition 2.3 can be applied to more than four points. For example, if \(v = (rq)\), \(|ud| \geq |up|\), \(|pe| > |dv|\) in Fig. 3, and if both trees \((p(ud))(rq)\) (in solid lines) and \((d(rq))(pu)\) (in broken lines) exist, then the former is longer than the latter, since \(\phi(up, vd) < 90^\circ\).

**Proposition 2.4 (The Variational Argument) [12].** Let \(T_1\) and \(T_2\) be two Steiner trees on the same set of terminals. We will consider \(|T_1|\) and \(|T_2|\) to be functions of \(x\) in the range \([x_1, x_2]\) measuring the lengths of the perturbed Steiner trees as we move the terminals from one position to another. Then \(|T_2(x_1)| \leq |T_1(x_1)|\) if

1. \(|T_2(x_2)| \leq |T_1(x_2)|\) and
2. \(\frac{d|T_2|}{dx} \geq \frac{d|T_1|}{dx} \geq 0\) or \(\frac{d|T_1|}{dx} \leq \frac{d|T_2|}{dx} \leq 0\).

The basic principle, from [12], for computing the relative size of \(d|T_1|/dx\) and \(d|T_2|/dx\) is as follows. If each of the terminals, \(a_i\), being moved is perturbed at a particular instant in the direction of a unit vector \(v_i\), then the contribution of an edge incident with \(a_i\) to the derivative is minus the cosine of the angle between \(v_i\) and the edge. The derivative is the sum of all such contributions.

The following lemma, which will prove useful in Sections 4 and 5, represents a typical example of an application of Proposition 2.4.

**Lemma 2.5.** The tree \(T_1\) in solid lines in Figure 4(a) cannot be part of a Steiner minimal tree for a Steiner-closed lattice set.

**Proof.** We will show that the tree \(T_2\), drawn in broken lines in Fig. 4a, is shorter than \(T_1\). Let \(p\) move to \(a\) along \(da\), and \(q\) to \(b\) along \(cb\), and perturb the two trees appropriately. Clearly \(-\cos(\angle s_2pa) > -\cos(\angle s_1p_1) > 0\)
and $- \cos(\angle s_1 qb) > - \cos(\angle s_3 qb) > 0$. Hence, $d |T_2|/dx > d |T_1|/dx > 0$, and in the end, the Steiner point of $T_1$ adjacent to $d$ degenerates into $d$ and $|T_1| = |T_2|$ (Fig. 4b). Thus, the lemma holds by Proposition 2.4.

In applying the proposition in this section, it is useful to have the concept of a left-turn path. Let $s$ be a terminal or Steiner point of the Steiner tree $T$ and let $s_1$ be an adjacent Steiner point. Consider a walk starting at $s$ in the direction towards $s_1$, turning left at each Steiner point, and finishing at the first terminal reached, say $t$. We refer to the path traced by this walk as the left-turn path $ss_1 \cdots$ (terminating at $t$). A right-turn path is defined similarly.

**Proposition 2.6 (Non-minimal Paths)** [13]. Let $p \cdots rq$ be a path in a Steiner tree $T$ such that $p \cdots rq$ is a simple polygon and $\angle prq \leq 60^\circ$. Let $m$ be the point on the line through $rq$ such that $\angle mpr = \angle prq$ (Fig. 5). Suppose that every terminal that exists inside or on the boundary of the polygon $p \cdots rm$ is connected to $q$ via $r$ (and in particular that $q$ is not a terminal if $q$ lies on $[rm]$). Then $T$ is not a minimal Steiner tree.

The following useful lemma is a corollary of this result.
**Lemma 2.7.** Suppose \( p \ldots uvrq \) is a path in a minimal tree \( T \) so that \( p \ldots uvrq \) is a simple polygon, \( v, r \) and \( q \) are Steiner points, and every terminal that exists inside the polygon \( p \ldots uvrq \) is connected to \( q \) via \( v \). If \( |pu| \geq |uv| \) and \( |pu| \geq |vr| \), then \( \angle puv > 120^\circ \) (Fig. 6).

**Proof.** Assume, on the contrary, that \( \angle puv \leq 120^\circ \), and that, consequently, \( \angle prv \leq 120^\circ \). If \( \angle prv \leq 60^\circ \), then \( T \) is not minimal by applying Proposition 2.6 to \( p \ldots vr \). Hence, \( \angle uvr < 60^\circ \). Since \( |pu| \geq |uv| \), it follows by the geometry of the triangle \( puv \) that \( \angle puv \geq 60^\circ \). Furthermore, by the geometry of the quadrilateral \( puvr \) it is easy to see that \( \angle prv \geq 60^\circ \) since \( |pu| \geq |vr| \), \( \angle uvr = 120^\circ \) and \( \angle puv \leq 120^\circ \). Hence, \( \angle prq \leq 60^\circ \) which again contradicts the minimality of \( T \) by applying Proposition 2.6 to \( p \ldots rq \).

A weakness of Proposition 2.6 is that the condition that every terminal inside the polygon is connected to \( q \) via \( r \) is often difficult to check. The following theorem is, in a sense, a stronger version of the proposition which provides a method of overcoming this difficulty in many situations.

**Theorem 2.8.** Suppose \( s_1 \) is a Steiner point in a Steiner tree \( T \) and \( s_0, s_2 \) are two vertices adjacent to \( s_1 \). Let \( p \) be a point in \( T \) such that \( p \) lies on the same side of the line through \( s_0s_1 \) as \( s_2 \). \( s_0 \) lies on the path connecting \( p \) and \( s_1 \), and \( \angle ps_0s_1 \leq 60^\circ \). Let \( c \) be the point on \( s_0s_1 \) or its extension such that \( pc \parallel s_1s_2 \), and let \( c' \) be the point on \( s_1s_2 \) or its extension such that \( pc' \parallel s_0s_1 \). Define the trap region of \( p \ldots s_0s_1 \), \( R \), as follows:

![Diagram](image_url)

**Fig. 6.** The path \( p \ldots uvrq \) makes \( T \) non-minimal if \( \beta \leq 120^\circ \).
(i) \( R = p(cp) \cap pc \), if \( \angle s_0 s_1 p \leq 120^\circ \) (Fig. 7);

(ii) \( R = p(c'p) \cap pc' \), if \( \angle s_0 s_1 p > 120^\circ \) and \( \angle ps_2 s_1 \leq 120^\circ \);

(iii) \( R = \triangle s_1(pc)(c'p) \cup p(cp) \cap pc' \cap p(c'p) \) otherwise (Fig. 8).

If there are no terminals in the interior of \( R \), then \( T \) is not minimal.

**Proof.** Assume \( T \) is minimal. We consider three cases corresponding to the three possibilities for \( R \).

(i) If \( (pc) \) lies on \( s_0 s_1 \), then \( |p(pc)| \leq |s_0 (pc)\) which contradicts the minimality of \( T \). Hence we assume \( s_1 \) lies on \( c(pc) \). Let \( s_0 s_1 s_2 \cdots s_{k+1} \) be a path in \( T \) such that: \( s_i s_{i+1} \parallel s_0 s_2 \) if \( i \) is odd; \( s_is_{i+1} \parallel p(pc) \) if \( i \) is even and \( s_i \) lies in \( \triangle pc(p(c) \cap p(cp) \) if \( i \) is even and \( s_i \) lies in \( \triangle cp(cp) \); and \( s_is_{i+1} \) intersects \( p(cp) \) or \( p(pc) \) at a point \( u \) (see Fig. 7). Let \( j \) be the largest integer less than \( k \) such that \( s_j s_{j+1} \) intersects \( [cp] \), say at the point \( r \). It immediately follows that \( |pu| \leq |rs_{j+1}| \leq |s_j s_{j+1}| \), contradicting Lemma 2.1(i).

(ii) Clearly, this case is symmetric to Case (i).

(iii) Consider the path \( s_0 s_1 s_2 \cdots \) where \( s_i s_{i+1} \parallel s_0 s_1 \) if \( i \) is even and \( s_is_{i+1} \parallel s_1 s_2 \) if \( i \) is odd (as in Fig. 8). Clearly this path intersects \([cp]\) or \([c'p]\), say at \( u \). Let \( s_j s_{j+1} \) be the edge of \( T \) such that either \( u = s_i \) or \( u \) lies in the interior of \( s_j s_{j+1} \). Then \( \angle ps_j s_{j+1} \leq 60^\circ \) and \( \angle p_{j+1} s_j < 120^\circ \), hence we can apply the argument in Case (i), since the region \( p(up) u(pu) \) lies in \( R \). This completes the proof.

\[ \text{Fig. 7. Theorem 2.8, Case (i).} \]
Given a line segment $pq$, the polygon $p((qp)p)(qp)q((pq))p(pq))$, shown in Fig. 9, is referred to as the butterfly of $pq$ and $q$ is referred to as its head.

**Corollary 2.9.** Suppose $p \cdots rs_0$ is a path in a Steiner tree $T$ and $s_0$ is a Steiner point. Let $q$ be a point on $rs_0$ such that $\angle pqs_0 \leq 60^\circ$. If there is no terminal in the interior of the butterfly of $pq$ with head $q$, then $T$, is not minimal.

**Proof.** Clearly, the trap region defined in Theorem 2.8 is completely covered by the butterfly of $pq$. 

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![Fig. 8. Theorem 2.8, Case (iii).](image)

![Fig. 9. The butterfly of $pq$.](image)
LEMMA 2.10. Let $abcd$ be a square, and let $T$ be a minimal Steiner tree for a Steiner-closed lattice set. Suppose there is an edge $s_1s_2$ of $T$ intersecting $ad$ at $p$ such that $s_2$ lies in $abcd$ and $\angle s_1pd \leq 60^\circ$. Then $s_1$ lies on the path from $s_2$ to $d$ (see Fig. 12).

Proof. The butterfly of $pd$ contains no terminals, hence the result follows by Corollary 2.9.

Finally, in Lemma 2.12, we give some very general local conditions which allow us to move a terminal of a Steiner tree $T$ along a circle whose center lies on the Simpson line at that terminal without increasing the length of $T$. This will be used in the next section to show that there are strong restrictions on the way an edge of a minimal Steiner tree can cross a lattice edge. Lemma 2.12 is preceded by a necessary technical lemma.

LEMMA 2.11. Let $T$ be a Steiner tree containing a Steiner point $s$ adjacent to a terminal $p$. If $T$ is perturbed by moving $p$ and fixing the other terminals of $T$, such that the topology of $T$ remains unchanged, then the trajectory of $s$ makes an angle of at least $60^\circ$ to the edge of $T$ incident with $p$.

Proof. Suppose $p$ moves a very small distance to a point $p'$, such that $s$ moves to a different point $s'$. Let $q$ denote the other end of the Simpson line originating at $p$, and let $r$ denote the other end of the Simpson line originating at $s$ and on the same side of $pq$ as $p'$. Let the point of intersection of $s'q$ and $sr$ be $O$. Then angles $qsO$ and $rs'O$ are both $60^\circ$, so the triangles $qsO$ and $rs'O$ are similar. Furthermore, $|sq| > |sr|$, since $|sq|$ is the total length of two subtrees of $T$ at $s$ whilst $|sr|$ is the length of one of those subtrees. Thus $|sO| > |s'O|$. For $|pp'|$ arbitrarily small, angle $sOs'$ is arbitrarily close to $60^\circ$, and the lemma follows.

LEMMA 2.12. Let $p$ be a terminal of a full Steiner tree $T$ with terminal set $A \cup \{p\}$ and let $q$ be the other end of the Simpson line from $p$. Choose points $r$, $s$ and $O$ such that $s$ and $O$ are on $pq$, and such that $|Op| = |Or|$, $\angle psr \leq 60^\circ$, and no Steiner point lies in the interior of $ps$. Suppose further that if $p'$ is any point on the circle through $p$ and $r$ centred at $O$, and lying on the smaller of the two arcs, $C$, between $p$ and $r$, then for any full Steiner tree with terminals $p'$ and a subset of $A$, the other end of the Simpson line from $p'$ does not lie inside the triangle $Opr$. Then there exists a tree with terminals $A \cup \{r\}$ which is no longer than $T$.

Proof. Perturb $T$ by fixing all terminals in $A$ and moving $p$ along $C$ towards $r$ either to the first point where the topology of $T$ is about to change or, if no such point exist, all the way to $r$. Denote the new position of $p$ by $p'$, denote the new tree by $T'$, and denote the intersections of $p'q$ with $Or$ and $sr$ by $O'$ and $s'$ respectively. Then $q$ is at the other end of the
Fig. 10. Under the conditions of Lemma 2.12, $p$ can travel along the circular arc about $O$ to $r$ without increasing the length of $T$.

Simpson line of $T'$ at $p'$, and using Lemma 2.11 the assumptions of the lemma are satisfied with $t, p, O$ and $s$ replaced by $t', p', O'$ and $s'$ (see Fig. 10). There are two possible reasons that the tree topology must change. One is that a Steiner vertex is about to collide with a terminal. In this case we can replace $T'$ by the full component containing $p'$ of the tree which results at the collision. The other possible reason is that two Steiner vertices collide. In this case the tree can be cut into two Steiner trees with two crossing edges, and $p'$ is a terminal in one of them. In both cases the conditions of the lemma are still satisfied (since the terminals of the new tree contain $p'$ and a subset of $A$) and so the process can be continued. However, if in the continuation, an edge of the tree crosses a terminal of one of the trees which have been cut off, that edge needs to be cut short at that terminal in order to ensure that the present tree, together with all the trees cut off, form a connected network. After a finite number of stages, we must have $p' = r$. At all stages of the process, the movement of $p$ is about a circle centred on a point lying on the Simpson line from $p$, and so the length of the tree is not increased, as required.

3. $G(S_T)$ IS A TREE

Throughout the remainder of this paper, let $T$ be a full subtree of a minimal Steiner tree $T^*$ for a Steiner-closed lattice set $P$. Let $S_T$ be the set of all squares in the lattice whose interiors contain parts of $T$. Define $S_T$ from
$S_T$ as follows: for each square $abcd$ of $S_T$, if there is a triangle in the lattice such that the part of $T$ contained in the interior of $abcd$ is completely contained in that triangle then replace $abcd$ by that triangle. So, for example, if $T$ is a unit lattice edge then $S_T$ is empty, whereas if $T$ is the Steiner tree in Fig. 11 then $S_T$ contains two squares and two triangles, as shown in the figure. A square or triangle is said to be adjacent to another square or triangle if they share a side. Let $G(S_T)$ be the graph on $S_T$ with the adjacency as defined above. Note that $G(S_T)$ is a connected graph (since $T$ is full), and that all vertices of the squares and triangles of $S_T$ are elements of $P$.

The word component will be used to refer to a connected component of the intersection of $T$ with the interior of a given square or triangle.

**Lemma 3.1.** There is only one component in a square or a triangle of $S_T$.

**Proof.** We prove the lemma only for squares since the proof is similar and easier for triangles. Suppose, on the contrary, there are two components in the square $abcd$. By Lemma 2.1(ii) each component has at least one Steiner point. Let $P_1$ and $P_2$ be convex paths in separate components, each reaching from one edge of $abcd$ to another, such that no part of $T$ lies between them. It is clear from Lemma 2.1 that $P_1$ and $P_2$ cannot both join the interiors of opposite sides of the square without forming a loop. Hence we can assume $P_1$ is part of the left-turn path $s_1s_2s_3\ldots$ where $s_1$ meets $[ad]$, $s_2$ is in the interior of the square $abcd$ and $s_2s_3$ meets $[ab]$. It follows from Lemmas 2.1(iv) and 2.10, that $s_1$, $s_2$ and $s_3$ lie on the path in $T^*$ from $d$ to $b$ (see Fig. 12). Similarly, assume $P_2$ is part of the left-turn path $s'_1s'_2s'_3\ldots$ where $s'_1$ lies in the interior of $abcd$. By symmetry we can assume that $s_1$ does not lie on the path in $T^*$ joining $s_2$ and $s_3$. This immediately tells us that $P_2$ cannot meet $[ad]$. Furthermore, it is clear that $P_2$ does not join $ab$ to $bc$ by angle considerations, and does not join $bc$ to $cd$ by Lemma 2.10 (since otherwise $s_1$ lies on the path joining $s_2$ and $s'_2$). Hence $P_2$ must join $ab$ to $cd$ with $s_3=s'_3$ and $s_3s'_3$ meeting $ab$, as shown in Fig. 12.

Now let $iadh$ and $dcef$ be squares adjacent to $abcd$. By Lemma 2.1(iii) the left-turn Steiner path $s'_1s'_2s'_3\ldots$ cannot reach the line through $ef$; it must terminate at $h$ or $i$. Applying Lemma 2.7 to $d\ldots s'_1s'_2s'_3s'_4$ we conclude that $T$ is not minimal.

A useful consequence of this lemma is the following corollary.

![Fig. 11. Here, $S_T$ contains two squares and two triangles.](image-url)
COROLLARY 3.2. The interior of any edge of a square intersects at most one edge of $T$ and no Steiner points of $T$.

We now wish to show that there are at most two Steiner points in each square of $S_T$. The key to this lies in the following lemma.

**Lemma 3.3.** Let $abcd$ be a square of the Steiner-closed lattice set $P$. Suppose that $s_1s_2$ is an edge of $T^*$ between Steiner vertices $s_1$ and $s_2$ crossing $bc$ at $q$ such that $s_1$ lies above $bc$ and the extensions of other edges of $T^*$ at $s_1$ do not intersect the interior of the interval $qc$. Then the interval $[c(ad)]$ does not intersect the extension of $qs_1$.

**Proof.** Arguing by contradiction, assume the extension of $qs_1$ intersects the interval $[c(ad)]$. First, consider the subtree $T_0$ of $T^*$ containing $s_1$ obtained by cutting $T^*$ at $q$. Suppose $c$ is in $T_0$, or in other words $s_1$ lies on the path in $T^*$ from $c$ to $s_2$, and apply Theorem 2.8(i). Since $\angle s_1s_2c < 120^\circ$ and the extension of $qs_1$ intersects $[c(ad)]$, it easily follows (by considering extreme cases) that the trap region of $c \cdots s_1s_2$ contains no terminals, contradicting the minimality of $T^*$. Hence $c$ is not a terminal of $T_0$. Let $O$ denote the point of intersection of the Simpson line of $T_0$ at $q$ and $(ad)c$. Note that $\angle (ad)cq = 75^\circ$. We now consider two separate cases.

Assume, in the first case, that angle $s_1qc < 75^\circ$. Observe that $|Oq| > |Oc|$, and also that $|Ox_1| < |Oc|$, since $\angle cs_1O \geq 120^\circ$. Hence there exists a point $q_0$ on the interval $s_1q$ such that $|Oq_0| = |Oc|$. Since $T_0$ does not contain $c$, the hypotheses of Lemma 2.12 are satisfied with $s = s_1$, $r = c$, $p = q_0$ and $T = T_0 - qq_0$. Thus $T_0$ can be replaced by a shorter tree connecting the terminals of $T^*$ in $T_0$ to $c$, contradicting the minimality of $T^*$.
So assume, on the other hand, that $\angle s_1qc \geq 75^\circ$. Applying Lemma 2.12, as in the previous case, we conclude that there exists a tree, containing $c$ and all the terminals of $T^*$ in $T_0$, whose length is at most $|T_0| + |qq_0|$. We complete the contradiction by finding a tree containing all terminals of $T^*$ not in $T_0$ whose length is less than $|T^* - T_0| - |qq_0|$. If $q_0$ lies on $qs_2$ then the contradiction immediately follows, so we may assume $|qq_0| \geq |qs_2|$.

First, note that the maximum value of $|qq_0|$ occurs when $O = (ad)$ and $|(ad)q_0| = |(ad)c|$. Thus $|qs_2| \leq 2 + \sqrt{3} - (1 + \sqrt{3}/2) < 0.0658$. Now let $s_3$ and $s_4$ denote the two Steiner vertices adjacent to and below $s_2$, and let $w$ denote the distance between the parallel edges of $T^*$ below them (see Fig. 13). Consider the hexagon $s_2s_3r_3r_1r_2s_4$, where all angles of the hexagon are $120^\circ$ and $|s_3r_1| = |s_4r_1| = w$. Since the convex path in $T^*$ containing $s_3$, $s_2$ and $s_4$ reaches two distinct terminals, it follows, by the minimality of $T^*$, that there must be a terminal on or inside this hexagon. In particular, some terminal must be within distance $w + 2w \tan 30^\circ$ below $s_2$. Hence $0.0658 + w + 2w/\sqrt{3} > 1$, so $w > 0.433$.

Now consider cutting $T^*$ apart at $s_2$, and let $T_3$ and $T_4$ be the two subtrees containing $s_3$ and $s_4$ respectively. Note that $|T_3| > 1$ and $|T_4| > 1$, since each subtree contains at least two terminals of $T^*$. Consider joining $T_3$ and $T_4$ directly to $f$ instead of through $s_2$, where $f$ is the point of intersection of the extensions of the third edges at $s_3$ and $s_4$ as shown in Fig. 13. For each $i \in \{3, 4\}$ let $O_i$ be the point on the Simpson line for $T_i$ at $s_2$ such that $|O_is_2| = 1$. It is clear that we can apply Lemma 2.12 to each subtree ensuring that $r$ is the point $f$, and $O$ is the point $O_i$ in each case. Hence it follows from Lemma 2.12 that the total decrease in length of $T_3$ and $T_4$ when they are joined directly at $f$ is at least $(|O_3s_2| - |O_3f|) + (|O_4s_2| - |O_4f|) = 2 - (|O_3f| + |O_4f|)$. Observe that $f$ lies on a line segment from $O_3s_2$ to $O_4s_2$ parallel to $O_3O_4$ and of length $2w$. In particular,
this line segment depends only on \( w \) and is otherwise independent of the positions of \( s_3 \) and \( s_4 \). Hence the minimum decrease in the sum of lengths of \( T_3 \) and \( T_4 \) occurs when \( f \) is, say, on the line \( O_4s_2 \), in which case
\[
|s_2f| = w/\sin 60^\circ, \quad |Of|^2 = (\sqrt{w \tan 30^\circ} + 1)^2 + w^2
\]
and the decrease is at least \(|s_2f| + 1 - |Of| > 0.177 > 0.0658\), as required.

**Lemma 3.4.** There are at least two Steiner points in a square of \( S_T \).

**Proof.** Suppose, on the contrary, there is only one Steiner point \( s_1 \) in the square of \( S_T \), \( abcd \). Consider three rays radiating from \( s_1 \) in the opposite directions to the edges themselves. Since there are three 120° angles formed by these rays, at least two corners of \( abcd \), say \( b \) and \( c \), lie in the same 120° angle. Let the edge of \( T, s_1s_2 \), lying in this angle intersect the boundary of \( abcd \) at \( p \). Without loss of generality, we may assume that either \( p \) lies on \( ab \) or \( p = b \) or \( p \) lies on \( bc \).

Firstly, suppose that \( p \) lies on \( ab \) or \( p = b \). Let the other two edges incident with \( s_1 \) intersect the boundary of \( abcd \) at \( q \) and \( r \) (reading counterclockwise around the square from \( p \)). Then, by geometry and the fact that \( abcd \) is a square of \( S_T \), it follows that \( q \) lies on \( cd \), \( r \) lies on \([ad]\) and \( r \neq d \). The extension of \( ps_1 \) must meet \( cd \); otherwise we are done by applying Lemma 3.3 to \( r \). Hence the point \((ba)\) must lie above the extension of \( qs_1 \); otherwise we are done by Lemma 3.3 applied to \( q \). But now the variation which moves \( r \) towards \( a \) and \( p \) towards \( b \), with \( q \) fixed, rotates the edge \( qs_1 \) downwards, and so \((ba)\) must remain above the extension of \( qs_1 \). This contradicts the fact that it lies on the extension of \( qs_1 \) when \( p = b \) and \( r = a \).

So finally suppose that \( p \) lies on \( bc \). We can assume by symmetry that \((ad)\) lies on or to the left of the extension of \( ps_1 \). Then we are done by Lemma 3.3.

Suppose there are \( m_1 \) squares and \( m_2 \) triangles in \( S_T \). By a simple induction argument the number of terminals of \( T \) is less than or equal to \( 2 + 2m_2 + m_1 \), with equality occurring only if \( G(S_T) \) is a tree. Since \( T \) is full, the number of Steiner points of \( T \) is less than or equal to \( 2m_2 + m_1 \). But clearly each triangle must contain at least one Steiner point, and by Lemma 3.4 each square contains at least two Steiner points. Hence, the above inequalities are forced to be equalities. This immediately implies the following two lemmas:

**Lemma 3.5.** There are precisely two Steiner points in each square of \( S_T \) and one Steiner point in each triangle of \( S_T \).

**Lemma 3.6.** \( G(S_T) \) is a tree.
4. LEAVES AND QUASI-LEAVES OF $G(S_T)$

Before introducing the concept of a quasi-leaf we require three lemmas. The first is an elementary observation, the second is a technical result which will prove useful in this and the following section, and the third gives us valuable information about the structure of the part of $T$ inside triangles of $S_T$.

**Lemma 4.1.** Let $abcd$ be a square in the Steiner-closed lattice set. Let $s_1s_2$ be an edge of $T$ such that $s_1s_2$ intersects $ab$, say at $p$, and $s_2$ lies in the interior of $abcd$. If $60^\circ < \angle app_1 < 120^\circ$ then $abcd$ is a square of $S_T$.

**Lemma 4.2.** Let $abcd$ and $dcef$ be adjacent squares in the Steiner-closed lattice set. Let $u, t, q$ be points on the line segments $[bc], [ad]$ and $[ef]$ respectively, let $p$ lie on $ce$, and let $r$ be a point on either $[df]$ or $aq$ (Fig. 14). Let $T_1$ be a full Steiner tree on $d, u, t, p, q$ and $r$ with topology $(ut)p(rq)$ if $t = d$, or topology $(a(td))p(rq)$ if $t \neq d$ (as in the figure). Let $s'$ be the Steiner point in $abcd$ adjacent to $u$, and suppose $\angle s'ac \leq 30^\circ$. Then

(i) $T_1$ exists only if $r$ lies on $[df]$, and

(ii) $T_1$ is not a subtree of $T$.

**Proof.** Let $T_1$ be a subtree of $T$. If $t \neq d$, let $t'$ be the point where the line through the two Steiner points in $abcd$ intersects $ad$; otherwise let $t' = d$. Let $T_1'$ be the full Steiner tree on $t', u, p, q$ and $r$ with topology $(ut')p(rq)$. We first show that $T_1'$ exists only if $r$ lies on $[df]$, from which (i) immediately follows.

Suppose, on the contrary, that $r$ lies on the interior of $aq$ and $T_1'$ exists. First note, by Corollary 3.2, that $q = e$. By Proposition 2.2, $\angle (ut')p(rq) < 120^\circ$. We will show this cannot occur. Observe that $\angle (bd)c(e) = 120^\circ$. It is clear that $\angle (ut')c(b) \leq \angle (bd)c(b)$ and $\angle (rq)c(e) \leq \angle (fe)c(e)$; hence $\angle (ut')c(qr) > 120^\circ$. If we let $p$ move along the line segment $ce$ from $c$ to $e$, then $\angle (ut')p(qr)$ increases at first, since $\angle (ut')c(b) > \angle (rq)c(e)$, then decreases again as $p$ approaches $e$. However $\angle (ut')e(qr) > 120^\circ$, hence it follows that $\angle (ut')p(qr) > 120^\circ$ for any point $p$ on $ce$, giving the desired contradiction.

Thus $r$ lies on $[df]$ and either $q$ lies on $ef$ or $q = e$ (Fig. 14). Define $s_1$ to be the Steiner point in $dcef$ adjacent to $r$ and $q$, and $s_2$ to be the Steiner point adjacent to $s_1$ and $p$. Note, by Lemma 2.10, that $q$ lies on the path from $s_1$ to $e$. Let $bb'c'e$ and $cc'e'e$ be the squares below $abcd$ and $dcef$ respectively, and let the next vertices in the left-turn path $s_1s_2\cdots$ be $s_3, s_4$ and $s_5$. Clearly $s_1$ lies in the interior of $cc'e'e$. If the right-turn Steiner path $u\cdots s_3s_4\cdots$ intersects the interior of $cc'$ then, by Lemma 4.1, $bb'c'e$ is a
square of $S_T$, contradicting Lemma 3.6. Hence the right-turn path $u \cdots s_2s_3 \cdots$ terminates at $c$ or $c'$. Assume, in the first case, that the path ends at $c$ (Fig. 14a). By angle considerations, there must be a single Steiner point, say $s_3$, between $s_2$ and $c$. Hence $s_3s_4$ intersects $ee'$ since it is parallel to $cs_a$ and $\angle csc_e < 45^\circ$. Let $L_1$ be the line through $e$ parallel to $s_1s_2$. Let $\theta$ be the acute angle between $L_1$ and $ce$. Note that $15^\circ < \theta \leq 30^\circ$. We first show that $s_3$ lies above $L_1$. Suppose, on the contrary, that $[s_2, s_3]$ intersects $L_1$. Then $[cs_a]$ intersects $L_1$, say at $x$, and $|cs_a| \geq |cx| = 2 \sin(\theta)/\sqrt{3}$. Let $y$ be the point where $T$ intersects $cd$ and let $y'$ be the point where the line through $(bd)$ parallel to $cs_a$ intersects $cd$. By Proposition 2.2, $|cy| = |cy'|$, and a simple calculation shows that $|cy'| = 1 - \sqrt{2} \sin(45 - \theta)/\sin(30 + \theta)$. It can now be checked that, over the domain $15^\circ < \theta \leq 30^\circ$,

$$1 - \sqrt{2} \sin(45 - \theta) \leq \frac{2 \sin(\theta)}{\sin(30 + \theta)}$$

with equality when $\theta = 30^\circ$. It follows that $T$ is not minimal, since we can replace $cs_a$ by $cy$ to form a shorter tree. Hence $s_3$ lies above $L_1$ and $\angle cs_3s_4 < 60^\circ$. Let $z$ be the point on $s_3s_4$ such that $ez \parallel s_2s_3$. Clearly there are no terminals in the region $e(ze)z(ze)$, so $T$ is not minimal by Theorem 2.8.

If, on the other hand, the right-turn path $u \cdots s_2s_3 \cdots$ ends at $c'$ (Fig. 14b) then $s_4$ lies in $cc'e'e$ and it is clear, by another easy angle argument, that $s_3s_4$ intersects $ee'$. Note that $\angle es_3s_4 < 60^\circ$. Let $z$ be the point on $s_4s_5$ such that $ez \parallel s_2s_3$. Again, since there are no terminals in $e(ze)z(ze)$, $T$ is not minimal by Theorem 2.8.

**Lemma 4.3.** The Steiner point in a triangle of $S_T$ is adjacent to the terminal at the right angle.

**Proof.** Let $bcd$ be a triangle of $S_T$ with right angle at $c$. By Lemma 3.5, this triangle contains a unique Steiner point, $s_1$. By Corollary 3.2, $s_1$ is adjacent to at least one vertex of $\triangle bcd$. Suppose, contrary to the lemma, that $s_1$ is not adjacent to $c$, but is adjacent to $d$. Without loss of generality, let the second edge incident with $s_1$ meet $[bc]$ at $u$, and the third edge incident with $s_1$ intersect the interior of $dc$. If $\triangle bcd$ shares dc with another triangle $\triangle dce$ then there are two possible cases: either both right angles occur at $c$ (Fig. 15a) or one occurs at $c$ and the other at $d$ (Fig. 15b). In the first case an edge must cross $dc$, since $\triangle dce$ contains exactly one Steiner point; in the second case the part of $T$ in the two triangles is clearly non-minimal by Pollak’s Theorem (Proposition 2.3). Thus, in each case there is a contradiction. Consequently, we may assume that $bcd$ is adjacent to a square of $S_T$, $dcfe$. Let the right-turn Steiner path starting with $as_1$ be $as_1s_2s_3$. We consider two cases.
Fig. 14. The Steiner tree $T_1$ and two possibilities for a neighbouring square.

Fig. 15. Trees in two adjacent triangles.
(i) Assume $s_3s_1$ meets $ce$ at $p$. If $p \neq c$ then $T$ is not optimal, by Lemma 4.2. So it follows that $p = c$ (Fig. 16). Let $v = (rq)$ and note that the tree $(cq)(dv)$, shown in broken lines in the figure, does exist. Consequently, by Proposition 2.3 and the remark following it, we have a contradiction to the minimality of $T$.

(ii) Assume $s_1$ is also in $deef$ (Fig. 17a). By Corollary 3.2 it follows that $s_1$ is adjacent to $c$ and its third edge intersects $ce$, say at $p$. Another edge incident with $s_2$ meets $df$ or $ef$, say at $q$. Let $T_1$ be this Steiner tree on $u,c,p,q$ and $d$; let $T_2$ be $uc$ plus the tree $(dq)(pc)$ (shown in broken lines in Fig. 17a). We now argue by variation (Proposition 2.4). Let $p$ move along $ce$ to $c$ and let $q$ move along $qe$ to $e$. The resulting trees are shown in Fig. 17b. This process decreases the length of $T_1$ at a greater rate than the length of $T_2$, but the length of the tree resulting from $T_2$ is clearly less than or equal to the length of the tree resulting from $T_1$. So $T_1$ is not minimal, and consequently neither is $T$.

We define the direction of the edges of $T$ to range from $-15^\circ$ to $165^\circ$ from the horizontal. Of the three directions of edges in $T$, one must be either in the range

\[-15^\circ, 0^\circ\], \text{ and called negative horizontal, or}

\[0^\circ, 15^\circ\], \text{ and called positive horizontal, or}

\[75^\circ, 90^\circ\], \text{ and called positive vertical, or}

\[90^\circ, 105^\circ\], \text{ and called negative vertical.}

Note that these directions, referred to as main directions, are exclusive. That is, $T$ cannot have two directions which are both main directions.

If a leaf of $G(S_T)$ is a triangle then, by Lemma 4.3, precisely one edge of $T$ in triangle is incident with one of the acute angles of the triangle, and clearly lies in the main direction (Fig. 18a). If a leaf is a square, the it is as shown in Figure 18b. (The other possible Steiner tree, is not minimal by Proposition 2.3, as shown in [1].) The edge joining two Steiner points $s_1$ and $s_2$ is in the main direction.

![Fig. 16. The tree in solid lines is not minimal.](image)
By the variational argument, the tree in solid lines in (a) is not minimal. We will call the sides of a square whose interiors intersect $T$ shared sides. Suppose $V_m$ is a vertex of degree two in $G(S_T)$. $V_m$ is called a quasi-leaf if there is a sequence of adjacent vertices $V_1, V_2, \ldots, V_m$ in $G(S_T)$ such that

1. $V$ is a leaf,
2. $V_i, 1 \leq i < m$, are quasi-leaves,
3. $V_m$ is either a triangle or square of degree 2 in $G(S_T)$, and in the latter case its shared sides are opposite.

If a quasi-leaf is a triangle, one of the edges intersecting its sides is in a main direction (Fig. 19a). It is quasi-leaf $abcd$ is a square with two Steiner points $s_1$ and $s_2$, then $s_1s_2$ is in a main direction. The square is referred to as normal if $s_1, s_2$ are adjacent to the endpoints of an unshared side (Fig. 19b). Otherwise $s_1, s_2$ are adjacent to the endpoints of a diagonal $ac$ or $bd$, and the square is referred to as abnormal (Fig. 19c). Note that whether a square is normal or abnormal depends on the topology of $T$. In all cases we will classify a leaf or quasi-leaf by its main direction, for example as positive horizontal if that is its main direction. Similarly, we can classify $T$ by its main direction.

We conclude this section with a few useful lemmas on quasi-leaves, beginning with a simple observation. This is a stronger version of Lemma 4.1.
**Fig. 19.** The three possible topologies in quasi-leaves of $G(S_T)$.

**Lemma 4.4.** (i) If an edge of $T$ intersects the interior of a side of a triangle of $S_T$, then the angle between them is in $(0, 30^\circ)$.

(ii) If an edge of $T$ intersects a shared side of a square leaf or square quasi-leaf, then the angle between them is in $(15^\circ, 45^\circ)$.

Note that this result tells us that if an edge of $T$ intersects a shared side of a leaf or quasi-leaf then the angle between them is less than $45^\circ$.

**Lemma 4.5.** In any sequence of adjacent square quasi-leaves in $G(S_T)$ at most one square is abnormal.

**Proof.** Let $V_1 V_1+1 \cdots V_1+m$ be a sequence of adjacent square quasi-leaves in $G(S_T)$. Suppose, contrary to the lemma, that $V_i$ and $V_i+m$ are both abnormal quasi-leaves with no abnormal quasi-leaves lying between them. Note, by angle considerations, that $m$ must be odd. Let $V_i = abcd$ and $V_i+m = a'b'c'd'$ and let $T$ intersect $ab$ at $p$, and $c'd'$ at $q$. Let $T_1$ be the subtree of $T$ in these $m+1$ squares (Fig. 20). Let $T_2$ be the Steiner tree on $p, q$ and the terminals of $T_1$ whose topology in each square is the same as that of a normal quasi-leaf (Fig. 19b), as shown in broken lines in the figure. As $p$ moves along $ab$ towards $a$ and $q$ moves along $c'd'$ towards $c'$ it is clear that $d' |T_2|/dx > d |T_1|/dx > 0$, and eventually $|T_1| = |T_2|$ when $p$ coincides with $a$ and $q$ with $c'$. Hence, by Proposition 2.4, we have a contradiction to the minimality of $T$. 

**Lemma 4.6.** Let $abcd$ be a square of $S_T$ and let $V_1$ be an adjacent vertex of $G(S_T)$ lying to the right of $abcd$. Suppose $V_1$ is either a leaf or quasi-leaf.

(i) If $abcd$ has degree two in $G(S_T)$ then $T$ is horizontal.

(ii) If $abcd$ is adjacent to another vertex of $G(S_T)$ which is a leaf or quasi-leaf, then $T$ is horizontal.

**Proof.** Suppose $V_1$ lies to the right of $abcd$, but is not horizontal. By symmetry, we may assume without loss of generality that $V_1$ is negative
vertical. $V_1$ is not a square, by our previous descriptions of leaves and quasi-leaves; hence $V_1$ is a triangle, with Steiner point $s_1$. Let $s_2$ be the Steiner point in $abcd$ adjacent to $s_1$, and let $s_3$ be the other Steiner point in $abcd$. Clearly, $s_3$ must lie on the left-turn path $cs_1s_2s_3$ (as in Fig. 21), otherwise $abcd$ would not be a square of $S_T$. Since $V_1$ is negative vertical, the extension of $s_2s_3$ intersects $ad$. Hence, one edge incident with $s_3$ intersects $ad$. By Corollary 3.2, $s_3$ has to be adjacent to $d$. Since $\angle s_2dc > 45^\circ$, the third edge incident with $s_3$ cannot end at $b$, but rather intersects the interior of $ab$. This implies that $abcd$ has degree three in $G(S_T)$, proving (i).

The vertex of $G(S_T)$ above $abcd$ cannot be a leaf or quasi-leaf by Lemma 4.4 since $ad$ meets an edge incident with $s_3$ at more than $60^\circ$. Furthermore, the vertex to the left of $abcd$ is not a leaf or quasi-leaf since $ab$ meets an edge incident with $s_3$ at more than $45^\circ$. Hence, by contradiction, (ii) is also true.

**LEMMA 4.7.** Let $V_1V_2\ldots V_m$ be a sequence of adjacent vertices of $G(S_T)$ so that $V_1$ is a leaf and the others are quasi-leaves. Let $V_{m+1} = abcd$ be a square of $G(S_T)$ adjacent to $V_m$ along the side $cd$. Suppose the component of $T$ in $abcd$ has Steiner points $s_1$ and $s_2$ so that $s_1$ is adjacent to $b$, $s_2$ is adjacent to $d$ and one of the edges incident with $s_1$ intersects $ad$ at $p$ (Fig. 22). Then $T$ is positive horizontal.

![Fig. 21. The part of $T$ in $abcd$, for $T$ vertical.](image)
Proof. By Lemma 4.6, \( T \) is horizontal. Assume, contrary to the lemma, that the main direction is negative horizontal. If \( V_m \) is a triangle, then the part of \( T \) in \( V_m \) and \( V_{m+1} \) fails to be minimal by Lemma 2.5. Hence \( V_m \) must be a square and is clearly normal since the main direction is negative horizontal. Let \( j \) be the largest element of \( \{1, \ldots, m-1\} \) such that \( V_j \) is either

(i) a triangle (Fig. 22a), or

(ii) a square leaf or abnormal quasi-leaf (Fig. 22b).

If \( V_j \) is an abnormal quasi-leaf, let \( q \) be the point where \( T \) intersects the edge shared by \( V_j \) and \( V_{j-1} \) and let \( d' \) be the terminal on \( V_j \) adjacent to a Steiner point in \( V_{j-1} \) (as in the figure). Note that in Case (i) \( m-j \) is necessarily even, while in Case (ii) \( m-j \) is odd. Let \( T_1 \) be the part of \( T \) in \( V_j V_{j+1} \cdots V_{m+1} \).

The lemma now follows from a variational argument similar to that used in Lemma 2.5. Observe that the orientation of a leaf, triangle or abnormal quasi-leaf determines whether the main direction is positively or negatively nearly horizontal. Hence, as we move \( p \) to \( a \) and (in the case of \( V_j \) being an abnormal quasi-leaf) \( q \) to \( d' \), the main direction of \( T_1 \) cannot change from negative horizontal to positive horizontal. This forces \( T_1 \) to degenerate into an alternating series of \( X \)s and edges when \( p \) coincides with \( a \) (and \( q \) with \( d' \)). In each case, let \( T_2 \) be the Steiner tree shown in broken lines in Fig. 22. Clearly, as \( p \) moves to \( a \) (and \( q \) to \( d' \)), \( T_2 \) is perturbed to
the same alternating series of $X$s and edges, except that the $X$ in $V_{m+1}$ is differently oriented; but $T_2$ increases in length faster than $T_1$. Hence, by Proposition 2.4, $T_1$ is not minimal.

5. THE STRUCTURE OF $G(S_T)$

The aim of this section is to establish a structure theorem for $S_T$. In essence, we show that all vertices of $G(S_T)$ are leaves or quasi-leaves, and consequently that there can be no branching in $G(S_T)$. This theorem follows from Corollary 5.2, Lemma 5.4, and Lemma 5.5, which systematically demonstrate that certain vertices which are neither leaves nor quasi-leaves do not occur in $G(S_T)$. Moreover, using simple angle arguments we are able to further restrict the structure of $S_T$ to a form we describe as a strip.

The first lemma follows directly from Lemma 4.6 and the fact that the main direction is exclusive.

**Lemma 5.1.** Suppose a square of $S_T$, $abcd$, is adjacent to two vertices of $G(S_T)$, $V_1$ and $V_2$, each of which is either a leaf or quasi-leaf.

(i) If $V_1$ lies to the right or left of $abcd$, then $V_1$ is horizontal; if $V_1$ lies above or below $abcd$, it is vertical.

(ii) $V_1$ and $V_2$ lie on opposite sides of $abcd$.

An immediate consequence of this lemma is the following result.

**Corollary 5.2.** $G(S_T)$ has no vertex of degree four adjacent to three vertices, each of which is a leaf or quasi-leaf.

Before proving our next main result, we need a small technical lemma.

**Lemma 5.3.** None of the trees $T_1$, drawn in solid lines in Fig. 23, can be subtrees of $T$.

**Proof.** We will show that in each case the tree $T_2$, drawn in broken lines in Fig. 23, is shorter than $T_1$, using Proposition 2.4. In Fig. 23a and 23b, let $p, q, u, v$ move to the corners $i, b, c, h$ respectively. Clearly, we always have $d |T_2| \, dx > d |T_1| \, dx > 0$. In the end, $|T_1| = \sqrt{11} + 6 \sqrt{3}$, while $|T_2| = 2 + \sqrt{5} + 2 \sqrt{3}$ in Fig. 23a and $|T_2| = 3 + \sqrt{3}$ in Fig. 23b. In both cases, $|T_1| \geq |T_2|$. Hence, by Proposition 2.4, $T_1$ is not minimal. In Fig. 23c, let $q, u, v$ move to $b, d, i$ respectively. It then follows that $d |T_2| \, dx > d |T_1| \, dx > 0$ for $q, v$, and $d |T_1| \, dx \leq d |T_2| \, dx < 0$ for $u$. In the end, $|T_1| = |T_2|$. So, again in this case $T_1$ is not minimal.
Fig. 23. The Steiner trees in solid lines are not minimal.

**Lemma 5.4.** $G(S_T)$ has no vertex of degree three which is adjacent to two vertices, $V_1$ and $V_2$, each of which is either a leaf or quasi-leaf.

**Proof.** Assume $abcd$ is such a vertex of $G(S_T)$, and that $V_1$ lies on its left side and $V_2$ on its right side. Also assume $V_1$ and $V_2$ are positive horizontal and the third vertex is above $abcd$. By Lemma 5.1 all these assumptions can be made without any loss of generality. Let $s_2$ and $s_3$ be the Steiner points in $abcd$ such that $s_2$ is adjacent to a Steiner point, $s_1$, in $V_1$. Since $V_1$ is a leaf or quasi-leaf, the edge incident with $s_2$ in the main direction cannot meet $[ab]$ or $c$ by angle considerations, or intersect $cd$ by Lemma 4.4. Also, it cannot meet $ad$, since in that case $s_1$, being adjacent to $a$ would force one of the edges adjacent to $s_3$ to intersect $ab$ (Fig. 24) whereas $s_1$ being adjacent to $b$ would clearly force $abd$ to be a triangle of $S_T$. Hence, the edge in the main direction is $s_2s_3$. Moreover, since $abcd$ is a vertex of degree three in $G(S_T)$, exactly one of the corners of $abcd$ is adjacent to $s_2$ or $s_3$. Since $abcd$ is positive horizontal and the third vertex in $G(S_T)$ is above $abcd$, this corner cannot be $c$. This leaves three cases to be eliminated. In each case, the nearby vertices of the checkerboard are labelled as indicated in the figures.

(i) Assume $s_2$ is adjacent to $b$ (Fig. 25).

Both $V_1$ and $V_2$ are squares by Lemma 4.4. Let $s_4$ and $s_5$ be the next vertices on the left-turn path $s_2s_3s_4s_5$. Since $G(S_T)$ is a tree, $jkat$ is not a square of $S_T$, which means $T$ enters this square only if $jui$ is a triangle of $S_T$. So, by Lemma 4.4, the path $s_2s_3s_4s_5 \cdots$ cannot intersect $ai$. Hence, $s_6,$
the next Steiner point on the right-turn path \( s_3s_2s_4 \) lies in \( iadhi \) by Lemma 3.5. Since \( hdgj \) is not a square of \( S_T \), \( s_6 \) must be adjacent to \( d \). Now if the third edge of \( s_6 \) meets \([ih]\), Proposition 2.3 is contradicted, and if it meets \( dh \), \( T \) is not minimal by Lemma 2.5.

(ii) Assume \( s_2 \) is adjacent to \( a \) (Fig. 26).
Let \( s_4, s_5 \) and \( s_6 \) be vertices of \( T \) as defined above. If \( V_1 \) is a square or \( s_4s_5 \) meets \([ih]\) then the minimality of \( T \) is again contradicted by the argument in (i). Hence, \( V_1 \) is a triangle and \( s_4s_5 \) intersects \( ai \). As before, \( s_6 \) must be adjacent to \( d \) since \( V_2 \) is clearly a square. If the third edge incident with \( s_6 \) meets \( dh \) (Fig. 26a), \( T \) is not minimal by Lemma 5.3, Fig. 23a. If the third edge incident with \( s_6 \) meets \([ih]\) (Fig. 26b), \( T \) is not minimal by Lemma 5.3, Fig. 23b.

(iii) Assume \( s_3 \) is adjacent to \( d \) (Fig. 27).
Again \( V_2 \) is a square by Lemma 4.4. Let \( s_4 \) and \( s_5 \) be the next Steiner points on the right-turn path \( s_3s_2s_4 \) and let the third edge incident with
Fig. 26. The case where $s_2$ is adjacent to $a$.

$s_4$ end at the vertex $s_6$. We will assume, in the first case, that $s_6$ does not lie in the interior of the square $iadh$ (Fig. 27a). This implies that $s_3$ lies in $iadh$. If $s_7$ is the next vertex on the right-turn path $s_3, s_2, s_4, s_5, s_7$ then $s_5s_7$ must meet [ih] since $hdfg$ is not a square of $S'_{T}$ (by Lemma 3.6). It is now easily seen that the left-turn path $s_3s_5 \cdots$ cannot intersect the interior of $ai$. It immediately follows from Lemma 4.2(i) that $s_5$ is adjacent to $h$ (replace $abcd$ in the statement of that lemma with $dabc$ here). If $s_6$ lies in the interior of $jkai$ then, by Lemma 4.2(ii), $T$ is not minimal. If, on the other hand, $s_6 = a$ then $T$ is not minimal by Lemma 5.3, Fig. 23c.

Thus $s_6$ must lie in $iadh$ (Fig. 27b). In this case, $s_6$ lies on the path connecting $i$ and $s_4$ by Lemma 2.10. Note that $s_5s_4$ intersects $ad$ at less than
60°. Hence, the line through $s_3s_4$ intersects $dc$ or $ai$, from which it follows that either $\angle s_3s_4c > 60°$ or $\angle s_3s_4i > 60°$. If the first of these possibilities holds then the left-turn path $c \cdots s_3s_4$ shows that $T$ is not minimal by Proposition 2.6. If the second holds then the left-turn path $i \cdots s_3s_4$ shows that $T$ is not minimal, again by Proposition 2.6. This completes the proof of the lemma.

**Lemma 5.5.** Every vertex of degree two in $G(S_T)$ adjacent to a leaf or quasi-leaf is itself a quasi-leaf.

**Proof.** Much of this proof parallels that of the previous lemma. Assume $abcd$ is a vertex of $G(S_T)$ which is not a quasi-leaf, but is adjacent to a leaf
or quasi-leaf, $V$. By Lemma 4.6, if $V$ lies to the right of $abcd$ then $T$ is not vertical. By symmetry, we can now assume, without loss of generality, that $V$ lies to the right of $abcd$. $T$ is horizontal and the second vertex in $G(S_T)$ adjacent to $abcd$ lies above $abcd$. Let $s_2$ and $s_3$ be the Steiner points in $abcd$, such that $s_1$ is adjacent to a Steiner point in $V$. It is clear that $s_2s_3$ is in the main direction. Since $abcd$ is a vertex of degree two, two corners of $abcd$ are adjacent to $s_2$ or $s_3$. This results in three cases to eliminate.

(i) Assume $s_2$ is adjacent to both $a$ and $b$.

In this case $s_2s_3$ must be positive horizontal. The possibilities for $T$ correspond to those in Figure 26, where $s_1$ now coincides with $b$. It follows that $T$ is not minimal by the argument used in the proof of Lemma 5.4, Case (ii).

(ii) Assume $s_2$ is adjacent to $b$ and $s_3$ is adjacent to $d$.

There are two subcases. If $s_2s_3$ is negative horizontal (as in Fig. 22), then $T$ is not minimal by Lemma 4.7. If $s_2s_3$ is positive horizontal, then here the possibilities for $T$ correspond to those in Figure 27, where $s_1$ now coincides with $b$. Again it follows that $T$ is not minimal by the argument used in the proof of Lemma 5.4, Case (iii).

(iii) Assume $s_2$ is adjacent to $b$ and $s_3$ is adjacent to $c$ (Fig. 28).

Here $s_2s_3$ is negative horizontal. Let $s_4$ and $s_5$ be the next Steiner points on the right-turn path $d \cdots s_3s_4s_5$ which intersects $cd$ at $q$, and intersects $ad$ at $p$. Let $s_6$ be the next Steiner point on the left-turn path $s_2s_4s_6$. Since

![Figure 28](image-url)
the right-turn path \( s_3 s_4 s_5 \cdots \) cannot intersect \( d b \), \( s_6 \) lies in \( i a d b \). By Lemma 2.10, \( s_6 \) is on the path connecting \( i \) and \( s_4 \). Let \( \alpha \) be the absolute value of the slope of the main direction. If, for a fixed \( p \), we wish to maximise \( \alpha \), we should choose \( d c e f \) to be an abnormal quasi-leaf. In this case, \( f e f' \) cannot be a triangle, by Lemma 4.1, nor can \( f e' f' \) be an abnormal quasi-leaf, by Lemma 4.5. It follows that \( \alpha \) is maximised if \( f e f' \) is a square leaf. Now construct the full Steiner tree, \( T' \) on \( i, a, b, c, d, e, f, e', f' \) shown in broken lines in Figure 28, where \( f e f' \) is a square leaf, \( d c e f \) is an abnormal quasi-leaf, and the part of \( T' \) inside \( a b c d \) is similar in topology to the part of \( T \) inside \( a b c d \), as shown. Suppose \( T' \) intersects \( a d \) at \( p' \). It is easy to calculate that \( |a p'| = 1/3 \) and the main direction of \( T' \) is \(-11.565^\circ\). If \( p \) lies on \( [a p] \), then it immediately follows from the construction that \( \alpha \leq 11.565^\circ \), and hence that \( \angle b s_4 s_5 \leq 60^\circ \). Applying Theorem 2.8 to \( i \cdots s_4 s_5 \), we conclude that \( T \) is not minimal. Similarly, if \( p \) lies on \( p'd \) and the line through \( s_2 s_4 \) intersects \( [i a] \) then again \( T \) is not minimal. But if, on the other hand, \( p \) lies on \( p'd \) and the line through \( s_2 s_4 \) intersects \( i b \), then a simple calculation (using the fact that \( |s_4 s_5| < 1 \)) shows that the subtree \( (b s_4)(q c) \) is not minimal by Proposition 2.3, again contradicting the minimality of \( T \).

Before stating the main theorem of this section we require some definitions. Given an infinite unit square lattice in the Euclidean plane, we define a \textit{ladder} to be a finite sequence of adjacent squares all lying in the one row or column. A ladder is said to be \textit{horizontal} if the squares all lie in the same row, and \textit{vertical} if they all lie in the same column. We define a \textit{staircase} to be a finite sequence of adjacent triangles in the square lattice with the property that they are adjacent along unit edges and all the hypotenuses of the triangles are parallel. A staircase is said to be \textit{ascending} if the hypotenuses lie at an angle of \( 45^\circ \) from the horizontal and \textit{descending} if they lie at an angle of \( 135^\circ \) from the horizontal.

Let \( S \) be a finite alternate sequence of adjacent ladders and staircases, with the adjacencies occurring at the ends of the ladders and staircases. A staircase in \( S \) is said to be \textit{internal} if it is adjacent to two ladders, and \textit{external} if it is adjacent to precisely one ladder. We say that \( S \) is a \textit{strip} if it satisfies the following conditions:

(i) Either all ladders in \( S \) are horizontal, or all ladders in \( S \) are vertical. Likewise, all staircases in \( S \) are ascending, or all are descending.

(ii) If \( S \) contains no ladders, then \( S \) contains exactly one or an even number of triangles. If \( S \) contains one or more ladders, then all internal staircases of \( S \) contain an even number of triangles, and all external staircases of \( S \) contain an odd number of triangles.

**Theorem 5.6.** \( S_T \) is a strip.
Proof. Let \( S^* \) be the subset of all elements of \( S_T \) which are not leaves or quasi-leaves of \( G(S_T) \). If \( S^* \) is non-empty then \( G(S^*) \) is clearly a tree. Let \( L \) be a leaf of \( G(S^*) \). Clearly \( L \) has degree 4, 3 or 2 in \( G(S_T) \). However, these possibilities contradict, respectively, Corollary 5.2, Lemma 5.4 and Lemma 5.5. It follows that two vertices in \( G(S_T) \) are leaves and all other vertices are quasi-leaves. Hence, \( S_T \) consists of a sequence of adjacent ladders and staircases. Moreover, condition (i) follows easily from the fact that the main direction of \( T \) is exclusive. For example, if \( S_T \) has both kinds of ladders, horizontal and vertical, then there are two directions, resulting in a contradiction. The fact that condition (ii) is satisfied follows from condition (i).

Recall that a tree is called a caterpillar if the subtree obtained by removing all leaves forms a path (i.e., a caterpillar is a tree which is a path in Autumn). From the description of \( T \) in leaves and quasi-leaves of \( G(S_T) \) we have the following result.

**Corollary 5.7.** \( T \) is a caterpillar.

6. CLASSIFYING THE FULL COMPONENTS OF \( T^* \)

In the previous section we showed that \( S_T \) is a strip. The aim of this final section is to completely classify those strips whose vertices can be spanned by a full minimal Steiner tree. This will provide us with a list of all possible full components \( T \) for any \( T^* \), and their lengths. The key to this classification is the following geometric construction for computing lengths and main directions of such trees, which is based on a more general result for caterpillars to appear in [14]. Throughout this section, let \( T \) be a positive horizontal full minimal Steiner tree for a Steiner-closed lattice set.

Assume \( S_T \) contains more than one square or triangle. Let \( V_1, V_2, \ldots, V_k \) be the sequence of adjacent squares and triangles in \( S_T \) ordered from left to right. Assume the set \( \{V_j, V_{j+1}, \ldots, V_k\} \) contains no abnormal squares. Later in this section we will show that abnormal squares in fact never occur in \( S_T \). Let \( s_1, s_2 \) be adjacent Steiner points in \( T \) such that \( s_2 \) is to the right of \( s_1 \), and \( s_1 \) lies in \( V_j \) (see Fig. 29). Since \( T \) is a caterpillar, all the Steiner points to the right of \( s_1 \) lie on a path \( s_1s_2 \ldots s_m \). Let \( x \) be the terminal in \( T \) adjacent to \( s_2 \). Let \( p_2 = x \); let \( q \) be the terminal of \( T \) such that \( q \) is adjacent to \( s_1 \) and \( |p_2q| = 1 \); and let \( p_1 \) be the terminal of \( T \) such that \( p_1p_2 \) is a unit edge of \( V_{j-1} \) and \( \angle p_1p_2q = 90^\circ \) (in particular, if \( V_{j-1} \) is not an abnormal square then \( p_1 \) is adjacent to \( s_1 \)). Let \( s_{m-1} \) be the terminal of \( T \) adjacent to \( s_m \) such that \( s_{m-1}s_ms_{m+1} \) is a left-turn path if \( V_k \) is a square or a right-turn path if \( V_k \) is a triangle. We now construct a path
$p_1p_2 \cdots p_{m+1}$, denoted $M_x$, and defined as follows: $|p_ip_{i+1}| = 1$ for all $1 \leq i \leq m$; $\angle p_ip_{i+1}p_{i+2} = 150^\circ$ for all $1 \leq i \leq m - 1$; and if we walk along the path from $p_1$ to $p_{m+1}$ we turn left at $p_i$ if $s_{i-1}s_is_{i+1}$ is part of a left-turn path and right at $p_i$ if $s_{i-1}s_is_{i+1}$ is part of a right-turn path. Note that $T$ is divided into three subtrees by $s_1$. Let $T_2$ be the subtree containing $s_2$. A simple inductive argument, using the methods of Melzak, shows that $p_{m+1}$ lies on the line through $s_1s_2$ and $|s_1p_{m+1}| = |T_2|$. (This is illustrated in Fig. 29. By the inductive hypothesis, the end of the constructed path beginning $p_2q$ coincides with the end of the Simpson line from $s_1$ shown in the figure. By the Simpson–Heinen construction, if we swing this path around $p_2$ by $60^\circ$ then $\angle p_1p_2p_3 = 150^\circ$ and $p_{m+1}$ coincides with the end of the Simpson line from $s_2$.)

If $S_T$ contains no abnormal squares, we can extend the construction of $M_x$ to a construction for all of $T$ as follows. Let $s_2$ be the left-most Steiner point of $T$, let $s_1$ be the Steiner point of $T$ adjacent to $s_2$, let $s_i$ be the terminal of $T$ adjacent to $s_2$ such that $s_1s_2s_3 \cdots$ is a right-turn path if $V_1$ is a square or a left-turn path if $V_1$ is a triangle, and let $x$ be the other terminal adjacent to $s_2$. Then we define the path $M_x = p_1p_2 \cdots p_{m+1}$ as in the previous paragraph, and we define $M_T$ to be $M_x$ orientated by rotation so that $p_1$ is the leftmost point of $M_T$ and $p_ip_{i+1}$ is horizontal whenever $s_is_{i+1}$ is in the main direction. It follows, again by the methods of Melzak, that the line through $p_1$ and $p_{m+1}$ is in the main direction of $T$ and $|T| = |p_1p_{m+1}|$.

These results are summarized in the following lemma.
Lemma 6.1. Let $T, s_1, s_2, T_2, M_x$ and $M_T$ be defined as above.

(i) If $M_x = p_1 p_2 \cdots p_{m+1}$ then the line through $s_1, s_2$ passes through $p_{m+1}$ and $|s_1 p_{m+1}| = |T_2|$. 

(ii) If $M_T = p_1 p_2 \cdots p_{m+1}$ then the line through $p_1$ and $p_{m+1}$ is in the main direction of $T$ and $|p_1 p_{m+1}| = |T|$.

The above definition for $M_T$ can be extended to negative horizontal Steiner trees. Let $\bar{T}$ be a horizontal full minimal Steiner tree for a strip containing no abnormal squares. If $\bar{T}$ is negative horizontal, let $T$ be the reflection of $\bar{T}$ about a vertical line. In this case we define the path $M_T = p_1 p_2 \cdots p_{m+1}$ to be the reflection of $M_T$ about a vertical line.

We can also define the following useful quantities on $T$. Define $D_H(M_T)$ to be the horizontal distance between $p_1$ and $p_{m+1}$, that is, the distance between the vertical lines through $p_1$ and $p_{m+1}$. Similarly, define $D_V(M_T)$ to be the vertical distance between $p_1$ and $p_{m+1}$. Note that $D_H(M_T)^2 + D_V(M_T)^2 = |T|^2$.

The next lemma shows that the condition that there are no abnormal squares in $S_T$ holds for all $T$.

Lemma 6.2. $S_T$ contains no abnormal squares.

Proof. Let $V_1, \ldots, V_k$ be a sequence of adjacent squares forming a ladder of $S_T$, and assume, contrary to the lemma, that the square $abcd = V_j$ is abnormal for some $1 < j < k$. By Lemma 4.5 there are no other abnormal squares in this ladder, and it immediately follows by an easy angle argument that $k$ is odd and $j$ is even. Let $L_{ad}$ be the line through $ad$ and let $L_{bc}$ be the line through $bc$. Let $T'$ be the part of $T$ lying between $L_{ad}$ and $L_{bc}$. Furthermore, let $p$ be the rightmost point of $T'$ lying on $L_{ad}$, and let $e$ be the rightmost terminal of $T'$ lying on $L_{bc}$. Similarly, let $q$ be the leftmost point of $T'$ lying on $L_{ad}$, and let $f$ be the leftmost terminal of $T'$ lying on $L_{bc}$ (as in Fig. 30). Let $s_1$ and $s_2$ be the two Steiner points of $T'$ lying in $abcd$. Applying Melzak's construction to $T'$ we obtain the Simpson line $p^*q^*$ for $T'$ passing through $s_1, s_2$. By the proof of Lemma 6.1, and noting that $j$ is even, it follows that $p^*$ lies on $L_{ad}$ and $q^*$ lies on $L_{bc}$. We can construct an alternative Steiner tree, $T''$, on $p, q$ and the terminals of $T'$ by placing an $X$ in each $V_i$ for $i$ odd, and connecting the tree with unit edges and

![Fig. 30. The subtree $T'$.](image-url)
the edges \( ep \) and \( fq \). Again using the proof of Lemma 6.1, we conclude 
\[ |T^*| = |q^*b| + |ap^*| < |q^*p^*| = |T'|. \] Hence \( T' \) is not minimal, giving the desired contraction.

Note that, since every square of \( G(S_T) \) is a leaf or normal quasi-leaf, it follows that the topology of \( T \) is completely determined up to reflection or rotation by \( S_T \).

We say that a full Steiner tree \( \hat{T} \) on the vertices of a strip is \emph{locally minimal} if its topology (up to rotation or reflection) in leaves of \( G(S_T) \) is as in Fig. 18 and in quasi-leaves of \( G(S_T) \) is as in Figs. 19a and 19b. In view of Lemma 6.2 it follows that every full minimal tree for a strip is locally minimal. Let \( A_{2k} \) be the locally minimal positive horizontal full Steiner tree for a \( 2k \)-ladder, that is, for a ladder containing \( 2k \) adjacent squares (An example is illustrated in Fig. 31). Let \( B_{2k+1} \) be the locally minimal positive horizontal full Steiner tree for a \( 2k \)-ladder with a triangle attached to one end, and let \( C_{2k+2} \) be the locally minimal positive horizontal full Steiner tree for a \( 2k \)-ladder with a triangle attached to each end such that the two hypotenuses are parallel (as in Fig. 31). A simple argument shows that \( A_{2k} \), \( B_{2k+1} \) and \( C_{2k+2} \) exist as full Steiner trees for all \( k \). Define \( Q(T) \) to be the main direction of \( T \). It follows from the proof of Lemma 6.1 that
\[ Q(A_{2k}) > Q(B_{2k+1}) > Q(C_{2k+2}) > Q(A_{2k+2}). \]

These definitions and inequalities are used in the proof of the following lemma.

**Lemma 6.3.** Let \( Z \) be a strip which is not a square, and which contains at least one ladder. Suppose there exists a full minimal Steiner tree on the lattice points of \( Z \). Then the following statements hold:

(i) every ladder in \( Z \) contains an even number of squares;
(ii) each external staircase of \( Z \) contains precisely one triangle and each internal staircase of \( Z \) contains precisely two triangles;

![Fig. 31. The Steiner trees \( A_2 \), \( B_3 \), and \( C_4 \).](image-url)
(iii) all ladders in $Z$ contain the same number of squares; and

(iv) if $Z$ contains more than one ladder then $Z$ contains either zero or two external staircases.

**Proof.** We can assume the full minimal Steiner tree on the lattice points of $Z$ is positive horizontal. Let $Z = ST$. We prove each of the four statements in turn. Statement (i) follows directly from Lemma 6.2.

Now consider the full Steiner tree for the strip consisting of two triangles sharing a vertical edge (Fig. 32). The main direction of this strip is $\arctan(1/(4 + \sqrt{3})) > 9.896\%$. Hence, if $Z$ contains two triangles sharing a vertical edge, then $Q(T) > 9.896\%$. But the main direction of the full Steiner tree for the strip consisting of two squares sharing a vertical edge is $\arctan(1/(4 + 3\sqrt{3})) < 6.206\%$. So, if $Z$ contains a ladder then $Q(T) < 6.206\%$, and consequently $Z$ does not contain two triangles sharing a vertical edge. This immediately implies Statement (ii), noting that external staircases have an odd number of triangles.

To see Statements (iii) and (iv), we divide $T$ into component subtrees by cutting $T$ at each of the points where an edge of $T$ in an internal staircase intersects the interior of a horizontal unit edge of the lattice. Note that the parts of $T$ contained in each ladder of $Z$ lie in separate components. Suppose, contrary to Statement (iii), that the ladders corresponding to two component subtrees $T_1$ and $T_2$ contain $2k_1$ and $2k_2$ squares respectively, where $k_1 < k_2$. Then,

$$Q(T_1) > Q(C_{2k_1 + 2}) > Q(A_{2k_1 + 2}) > Q(A_{2k_2}) > Q(T_2),$$

contradicting the uniqueness of the main direction of $T$ (where $Q(T_1)$ and $Q(T_2)$ are defined in the obvious way).

Finally, to prove Statement (iv), assume $Z$ contains exactly one external staircase. Let each of the ladders of $Z$ contain $2k$ squares. Then it is clear, by the construction of $MT$ in Lemma 6.1, that the main direction of $T$ is equal to $Q(B_{2k + 1})$ and consequently that each of the component subtrees is a full subtree on a subset of the vertices of $Z$. If $Z$ contains more than one ladder this contradicts the fact that $T$ is full. 

**Lemma 6.4.** If $Z$ is a staircase, or if $Z$ is a strip containing at least one ladder and satisfying Statements (i), (ii), (iii), and (iv) in Lemma 6.3, then all minimal Steiner trees on the lattice points of $Z$ are full.

**Proof.** We can assume that $Z$ is orientated so that its ladders are horizontal and its staircases are ascending. We will first show that if $Z$ satisfies the hypotheses of the lemma then there exists a full locally minimal...
Steiner tree $\bar{T}$ such that $S_\bar{T}=Z$. To complete the proof, we then prove that $\bar{T}$ is strictly shorter than any Steiner tree for $Z$ containing more than one full component.

Consider a horizontal $2k$-ladder. Let $L_1$ be the horizontal line passing through the top vertices of this ladder and $L_0$ the horizontal line passing through its bottom vertices. Let $e$ be the rightmost vertex of the ladder lying on $L_1$ and let $f$ be the leftmost vertex of the ladder lying on $L_0$. Let $p$ be a point on $L_1$ lying on or to the right of $e$ and let $q$ be a point on $L_0$ lying on or to the left of $f$. By the construction for Lemma 6.1 it follows that there exists a full Steiner tree on $p$, $q$ and the vertices of the ladder which is locally minimal in the squares of the ladder and whose main direction is

$$\arctan\left(\frac{\frac{1}{2}}{k(2 + \sqrt{3}) + \frac{\sqrt{3}}{2} + |eq| + |ep|}\right).$$

Now let $Z$ be a strip satisfying the hypotheses of the lemma and containing $l$ $2k$-ladders labelled (from left to right) $Z_1$, $Z_2$, ..., $Z_l$. If $l=1$ then the construction above clearly gives a suitable full locally minimal Steiner tree $\bar{T}$, where $p$ and $q$ are respectively the top rightmost and bottom leftmost vertices of $Z$. So suppose $l>1$ and, moreover, $Z$ has no external staircases. For $1 \leq i \leq l-1$ let $L_i$ be the horizontal line passing through the top vertices of $Z_i$, let $e_i$ be the rightmost vertex of $Z_i$ lying on $L_i$, and let $p_i$ be the point on $L_i$ lying to the right of $e_i$ such that $|e_ip_i|=(l-i)/l$. Finally, let $p_0$ be the bottom leftmost vertex of $Z$ and let $p_l$ be the top rightmost vertex of $Z$. As above, for each $i$ we can construct a full Steiner tree $T_i$ on $p_{i-1}$, $p_i$ and the vertices of $Z_i$ whose main direction is

$$\arctan\left(\frac{\frac{1}{2}}{k(2 + \sqrt{3}) + \frac{\sqrt{3}}{2} + \frac{l-1}{l} + \frac{l-i}{l}}\right)$$

$$= \arctan\left(\frac{\frac{1}{2}}{k(2 + \sqrt{3}) + \frac{\sqrt{3}}{2} + \frac{l-1}{l}}\right).$$

Since each of the $T_i$s has the same main direction, their union forms a full Steiner tree, $\bar{T}$, for all $Z$. It immediately follows from the construction that

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$T$ is locally minimal. If, on the other hand, $Z$ contains two external staircases then we can use exactly the same argument to construct a suitable $T$ by choosing the points $p_i$ such that $|e_i p_i| = i/l$. Similarly, we can use this argument to construct a suitable $T$ for the case where $Z$ is a 2$k$-staircase by viewing the staircase as a collection of $k$ 0-ladders separated by internal 2-staircases.

Now suppose, contrary to the lemma, there exists a minimal Steiner tree $T'$ on the vertices of $Z$ such that $T'$ is not full. Let $\{T'_i\}$ be the set of full components of $T'$. It is clear, by an easy angle argument, that $T'$ contains no vertical unit edges. Hence, we can assume that all full components of $T'$ are horizontal. Furthermore, there is no square $abcd$ in the strip such that $ad$ and $bc$ are both unit edges of $T'$ (as two such unit edges can always be replaced by a suitably oriented minimal Steiner tree for a triangle to form a shorter tree). It immediately follows, by a simple induction argument for example, that $\sum_i D_H(M_{T'_i}) \geq D_H(M_T)$.

We next show that $\sum_i D_V(M_{T'_i}) \geq D_V(M_T)$. This follows from the fact that, for almost any horizontal strip $S$ with full minimal Steiner tree $T$, $D_V(M_T) = k/2$, where $k$ is the number of ladders in $S$ and where, as previously, a 2$k$-staircase is considered to contain $k$ 0-ladders. The sole exception to this is the case where $S$ is a single square, $T = X$ and $D_V(M_T) = 0$. However, it is clear that for any ladder of $Z$ there is no minimal Steiner tree on the vertices of that ladder consisting only of $X$s and unit edges (since the ladder contains an even number of squares). The inequality above now easily follows.

Using a standard inequality, we deduce that

$$|T'| = \sum_i |T'_i|$$

$$= \sum_i (D_H(M_{T'_i})^2 + D_V(M_{T'_i})^2)^{1/2}$$

$$\geq \left( \left( \sum_i D_H(M_{T'_i}) \right)^2 + \left( \sum_i D_V(M_{T'_i}) \right)^2 \right)^{1/2}$$

$$\geq (D_H(M_T)^2 + D_V(M_T)^2)^{1/2}$$

$$= |T|.$$

This contradict the minimality of $T'$.  

Note that the tree $\bar{T}$ constructed in the above proof is indeed a minimal Steiner tree for $Z$ since, up to rotation and reflection, any full locally minimal Steiner tree on $Z$ is unique.

The next lemma tells us that any minimal Steiner tree on the vertices of a strip occurs as a subtree of a minimal Steiner tree of some Steiner-closed lattice set, since the vertices of the strip are themselves Steiner-closed.

**Lemma 6.5.** Let $Z$ be a strip. The set of lattice points corresponding to the vertices of $Z$ forms a Steiner-closed lattice set.

**Proof.** Suppose, contrary to the proposition, there exists a minimal Steiner tree, $T'$, on the vertices of $Z$, such that $T'$ contains an edge $s_1 s_2$ crossing a lattice edge not contained in $Z$. Clearly $s_1$ and $s_2$ are both Steiner points at least one of which lies outside $Z$. If both $s_1$ and $s_2$ lie outside $Z$, then an easy exercise shows that the four distinct terminals of the two convex paths through $s_1 s_2$ cannot all be vertices of $Z$, giving a contradiction. If, on the other hand, $s_1$ lies in $Z$ and $s_2$ lies outside $Z$, then the four distinct terminals of the two convex paths through $s_1 s_2$ can only lie in $Z$ if there exists another edge of $T'$ between two Steiner points both of which lie outside $Z$. So again, by the previous argument, we obtain a contradiction to the existence of $T'$.

In order to complete our classification of full components of $T^*$ we need to introduce some new notation for some special kinds of strips. A $[2k, l]$-strip is defined to be a strip consisting of $\ell 2k$-ladders separated by $l - 1$ internal 2-staircases. Similarly, a $\langle 2k, l \rangle$-strip is a $[2k, l]$-strip with an external 1-staircase (that is, a single triangle) on one end, while a $\langle 2k, l \rangle$-strip is a $[2k, l]$-strip with external 1-staircases on both ends. Since the topology of $T$ is completely determined up to reflection or rotation by $S_T$, we can also use this notation to describe $T$. Finally, let $Y$ denote the full Steiner tree for a triangle.

The following classification now follows from Lemma 6.3, 6.4 and 6.5.

**Theorem 6.6.** Let $T$ be a full component of $T^*$, containing at least one Steiner point. Up to reflection or rotation, $S_T$ is either

(i) a triangle;
(ii) a square;
(iii) a $2k$-staircase;
(iv) a $\langle 2k, 1 \rangle$-strip;
(v) a $[2k, l]$-strip; or
(vi) a $\langle 2k, l \rangle$-strip.

In each case the main direction and length of $T$ are as shown in Table 1.
TABLE 1

Complete Classification of All Possible Full Components $T$ of $T^*$

| $T$        | Main direction of $T$ | $|T|$                      |
|------------|-----------------------|---------------------------|
| Unit edge  | $0^\circ$             | $1$                       |
| $Y$        | $15^\circ$            | $\sqrt{2} + \sqrt{3}$    |
| $X$        | $0^\circ$             | $1 + \sqrt{3}$           |
| $2k$-staircase | $\arctan \left( \frac{1}{2} \left( \frac{1}{k+1} \right) \right)$ | $k \left( \frac{1}{4} \left( \frac{3}{2} + \frac{k+1}{k} \right) \right)$ |
| $\langle 2k, 1 \rangle$-strip | $\arctan \left( \frac{1}{2} \left( \frac{1}{k(2k+\sqrt{3})+\frac{3}{2}+1} \right) \right)$ | $\frac{1}{2} \left( k(2+\sqrt{3})+\frac{3}{2}+1 \right)$ |
| $[2k, l]$-strip | $\arctan \left( \frac{1}{2} \left( \frac{1}{k(2+\sqrt{3})+\frac{3}{2}+1} \right) \right)$ | $\frac{1}{4} \left( k(2+\sqrt{3})+\frac{3}{2}+1 \right)$ |
| $\langle 2k, l \rangle$-strip | $\arctan \left( \frac{1}{2} \left( \frac{1}{k(2+\sqrt{3})+\frac{3}{2}+1} \right) \right)$ | $\frac{1}{4} \left( k(2+\sqrt{3})+\frac{3}{2}+1 \right)$ |

Furthermore, with $k$ and $l$ ranging over all positive integers this gives a complete irredundant classification of possible full components of $T^*$.

REFERENCES