Correlation Detection and an Operational Interpretation of the Rényi Mutual Information

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Recently, a variety of new measures of quantum Rényi mutual information and quantum Rényi conditional entropy have been proposed, and some of their mathematical properties explored. Here, we show that the Rényi mutual information attains operational meaning in the context of composite hypothesis testing, when the null hypothesis is a fixed bipartite state and the alternate hypothesis consists of all product states that share one marginal with the null hypothesis. This hypothesis testing problem occurs naturally in channel coding, where it corresponds to testing whether a state is the output of a given quantum channel or of a “useless” channel whose output is decoupled from the environment. Similarly, we establish an operational interpretation of Rényi conditional entropy by choosing an alternative hypothesis that consists of product states that are maximally mixed on one system. Specialized to classical probability distributions, our results also establish an operational interpretation of Rényi mutual information and Rényi conditional entropy.

I. INTRODUCTION

In order to distill useful measures of Rényi mutual information and Rényi conditional entropy from a plethora of possible definitions, it is important to find out which definitions correspond to relevant operational quantities. For this purpose, let us consider how efficiently an arbitrary bipartite correlated state $\rho_{AB}$ on systems $A$ and $B$ can be distinguished from product states when the marginal of $\rho_{AB}$ on $A$ is known to be $\rho_A$. This problem can be regarded as the problem of detecting correlations in the state $\rho_{AB}$. Formally, we consider the following binary composite hypothesis testing problem for $n$ copies of such a state:\textsuperscript{1}

**Null Hypothesis:** The state is $\rho_{AB}^{\otimes n}$.

**Alternate Hypothesis:** The state is of the form $\rho_A^{\otimes n} \otimes \sigma_B$, with $\sigma_B$ any state on $n$ copies of $B$.

This hypothesis test figures prominently when analyzing the converse to various channel coding questions in classical as well as quantum information processing.\textsuperscript{2} There, the problem is specified by a description of a channel $E_{A' \rightarrow B}$ and a bipartite state $\rho_{AA'}$ where the system $A$ constitutes an environment of the channel, $A'$ is the channel input, and $B$ its output. We are given an unknown state on $n$ copies of $A$ and $B$ and consider the following two hypotheses.

**Null Hypothesis:** The state is the output of $n$ uses of the channel $E_{A' \rightarrow B}$, namely the state is exactly $\rho_{AB}^{\otimes n}$ where $\rho_{AB} := E_{A' \rightarrow B}[\rho_{AA'}]$.

**Alternate Hypothesis:** The state is the output of a “useless” channel and decoupled from the environment, namely it is of the form $\rho_A^{\otimes n} \otimes \sigma_B$, with $\sigma_B$ any state on $n$ copies of $B$.\textsuperscript{2}

\textsuperscript{1} We want to consider the speed with which the probability that we erroneously support the state $\rho_{AB}^{\otimes n}$ when the actual state is a product state of the form $\rho_A^{\otimes n} \otimes \sigma_B^{\otimes n}$ under a constraint for the opposite error. As is explained later, this problem can be discussed as the Hoeffding bound and Stein’s lemma under this formulation.

\textsuperscript{2} There exists an intimate connection between quantum channel coding and binary hypothesis testing (see, e.g., [21]). This connection is particularly important when analyzing how much information can be transmitted with a single use of a quantum channel [28, 45] or when approximating how much information can be transmitted with finitely many uses of the channel [11, 44]. (See also [19, 37] for the classical case. In particular, Polyanskiy [36, Sec. II] discusses the classical special case of this hypothesis testing problem.)
A hypothesis test for this problem is a binary positive operator-valued measure \( \{Q_{A^nB^n}, 1_{A^nB^n} - Q_{A^nB^n}\} \) on the \( n \) copies of the systems \( A \) and \( B \), determined by an operator \( 0 \leq Q_{A^nB^n} \leq 1_{A^nB^n} \). If the operator \( Q_{A^nB^n} \) “clicks” on our state, we conclude that the null hypothesis is correct, whereas otherwise we conclude that the alternate hypothesis is correct. The error of the first kind, \( \alpha_n(Q_{A^nB^n}) \), is defined as the probability with which we wrongly conclude that the alternate hypothesis is correct even if the state is \( \rho_{AB}^\otimes n \), given by

\[
\alpha_n(Q_{A^nB^n}) = \text{tr}[\rho_{AB}^\otimes n (1_{A^nB^n} - Q_{A^nB^n})],
\]

(1)

Conversely, the error of the second kind, \( \beta_n(Q_{A^nB^n}) \), is defined as the probability with which we wrongly conclude that the null hypothesis is correct even if the state is of the form \( \rho_A^\otimes n \otimes \sigma_B^n \) for some \( \sigma_B^n \), given by

\[
\beta_n(Q_{A^nB^n}) = \max_{\sigma_B^n} \text{tr}[\rho_A^\otimes n \otimes \sigma_B^n Q_{A^nB^n}],
\]

(2)

where the maximum is taken over all states \( \sigma_B^n \) on \( n \) copies of \( B \).

Main Results. The main contribution of this paper is an asymptotic analysis of the fundamental trade-off between these two errors as \( n \) goes to infinity. To investigate this trade-off, we ask the following questions: let us assume that our test is such that \( \beta_n(Q_{A^nB^n}) \leq \exp(-nR) \), what is the minimum value of \( \alpha_n(Q_{A^nB^n}) \) we can achieve? The answer is different depending on whether \( R \) is smaller or larger than the \textit{mutual information} between \( A \) and \( B \), denoted \( I(A:B)_\rho \). If \( R < I(A:B)_\rho \), we show that the minimal error of the first kind vanishes exponentially fast in \( n \). This implies a \textit{quantum Stein’s lemma} [22] for the above composite hypothesis testing problem.

More formally, we define

\[
\hat{\alpha}_n(nR) = \min_{0 \leq Q_{A^nB^n} \leq 1} \left\{ \alpha_n(Q_{A^nB^n}; \rho_{AB}) \mid \beta_n(Q_{A^nB^n}) \leq \exp(-nR) \right\}
\]

(3)

and investigate the exact exponents with which this error vanishes as \( n \) goes to infinity, yielding a \textit{quantum Hoeffding bound} [18, 32] for our composite hypothesis testing problem. We find that the exponents are determined by the \textit{Rényi mutual information}, defined as

\[
I_\alpha(A:B)_\rho = \min_{\sigma_B} D_\alpha(\rho_{AB} \| \rho_A \otimes \sigma_B), \quad \text{for} \quad \alpha \in (0, 1),
\]

(4)

where \( D_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \text{tr} \left[ \sigma^{1-\alpha} \rho^\alpha \sigma^{\frac{1-\alpha}{\alpha}} \right] \) is the Rényi relative entropy first investigated by Petz (see, e.g. [35]) and the minimization is over all states \( \sigma_B \) on \( B \). We obtain

\[
\lim_{n \to \infty} \left\{ \frac{1}{n} \log \hat{\alpha}_n(nR) \right\} = \sup_{\alpha \in (0, 1)} \left\{ \frac{1 - s}{s} (I_\alpha(A:B)_\rho - R) \right\}.
\]

(5)

On the other hand, if \( R > I(A:B)_\rho \), we show that \( \hat{\alpha}_n(R) \) must approach one exponentially fast in \( n \). This implies the strong converse for quantum Stein’s lemma [34] for our problem. We then find the exact exponents (also called \textit{strong converse exponents}, see [17, Ch. 3] and [29, 34]) with which the error of the first kind goes to one as \( n \) goes to infinity and we find that in our case the exponent is determined by the \textit{sandwiched Rényi mutual information} [4, 15], given as

\[
\tilde{I}_\alpha(A:B)_\rho = \min_{\sigma_B} \tilde{D}_\alpha(\rho_{AB} \| \rho_A \otimes \sigma_B), \quad \text{for} \quad \alpha > 1,
\]

(6)

where \( \tilde{D}_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \text{tr} \left[ (\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \right] \) is the (sandwiched) Rényi divergence [31, 46]. We obtain

\[
\lim_{n \to \infty} \left\{ \frac{1}{n} \log \left( 1 - \hat{\alpha}_n(nR) \right) \right\} = \sup_{s > 1} \left\{ \frac{s - 1}{s} \left( R - \tilde{I}_s(A:B)_\rho \right) \right\}.
\]

(7)
Hence, we show that the above composite hypothesis testing problem yields an operational interpretation for different definitions of the Rényi mutual information for the two ranges of $\alpha$, paralleling the observation in [29].

Finally, we also perform a second-order analysis for quantum Stein’s lemma [25, 43] and show that the minimal error of the first kind converges to a constant if $\beta_n(Q_{A^n B^n}) \leq \exp(-n I(A:B)_\rho - \sqrt{nr})$ for some $r \in \mathbb{R}$. Then, for any $r \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \left\{ \hat{\alpha}_n(n I(A:B)_\rho + \sqrt{n} r) \right\} = \Phi \left( \frac{r}{\sqrt{V(A:B)_\rho}} \right),$$

where $\Phi$ is the cumulative standard normal (Gaussian) distribution and

$$V(A:B)_\rho := \text{tr} \left[ \rho_{AB} \left( \log \rho_{AB} - \log \rho_A \otimes \rho_B - I(A:B)_\rho \right)^2 \right].$$

is the mutual information variance.

Analogously, an operational interpretation for conditional Rényi entropies is established by considering the following binary hypotheses testing problem, which is motivated by the task of decoupling of quantum states. The problem is specified by a description of a state $\rho_{AB}$. Given an unknown state on $A$ and $B$, consider the following two hypotheses:

**Null Hypothesis:** The state is the $n$-fold product of $\rho_{AB}$, namely $\rho_{AB}^{\otimes n}$.

**Alternate Hypothesis:** The state is uniform on $A^n$ and decoupled form $B^n$, i.e. it is of the form $\pi_A^{\otimes n} \otimes \sigma_{B^n}$, where $\pi_A$ is the fully mixed state on $A$.

The same analysis as above applied to this problem reveals that the exponents in the quantum Hoeffding bound are determined by the Rényi conditional entropies defined as [42]

$$H^r_\alpha(A|B)_\rho = -\min_{\sigma_B} D_\alpha(\rho_{AB} \| \rho_A \otimes \sigma_B), \quad \text{for} \quad \alpha \in (0, 1),$$

and the strong converse exponents are determined by the sandwiched conditional Rényi entropies [31]

$$\tilde{H}^r_\alpha(A|B)_\rho = -\min_{\sigma_B} \tilde{D}_\alpha(\rho_{AB} \| 1_A \otimes \sigma_B), \quad \text{for} \quad \alpha > 1.$$

**Related Work.** Complementary and concurrent to this work, Cooney et al. [9] investigated the strong converse exponents for a similar hypothesis testing problem when adaptive strategies are allowed — however, they did not treat the case of a composite alternate hypothesis and they also did not analyze the error exponents in the quantum Hoeffding bound.

Our proof of the strong converse exponents parallels the development in a very recent preprint by Mosonyi and Ogawa [30]. There, the authors consider correlated states and use the Gärtner-Ellis theorem of classical large deviation theory in order to investigate the asymptotic error exponents in the presence of correlations. Here, we are not interested in correlated states per se, but our proof technique based on pinching naturally leads us to a classical hypothesis testing problem with correlated distributions, for which the Gärtner-Ellis theorem again provides the right solution.

**Outline.** The remainder of this paper is structured as follows. In Section II we introduce the necessary notation and mathematical preliminaries, and we discuss some properties of the Rényi divergence. We believe that Lemma 2 and Corollary 3 may be of independent interest. In Section III we define the generalized Rényi mutual information (which formally generalizes both Rényi mutual information and Rényi conditional entropy) and discuss various properties, including a duality relation and additivity. Most importantly, in Proposition 7, we show that it can be represented as an asymptotic limit of classical Rényi divergences.
Then, in Section IV we formally define the composite hypothesis test we consider and the required operational quantities. In doing so, we introduce a slightly more general problem that includes the two hypothesis testing problems discussed previously as special cases. In Section V we prove an analogue of the quantum Hoeffding bound that establishes the operational meaning for Rényi mutual information and Rényi conditional entropy for $\alpha < 1$. Moreover, in Section VI we find the strong converse exponents for our problem, yielding an operational meaning for the Rényi mutual information and Rényi conditional entropy for $\alpha > 1$. As in the non-composite case, for $\alpha > 1$ the relevant Rényi divergence is the “sandwiched" Rényi divergence. We conclude our treatment of the problem by considering the second order asymptotics in Section VII. This section is interesting on its own since it provides a new and more intuitive proof of the achievability of the second order that also easily adapts to non-composite hypothesis testing.

II. NOTATION AND PRELIMINARIES

We model quantum systems, denoted by capital letters (e.g., $A$, $B$), by a finite-dimensional Hilbert spaces (e.g., $\mathcal{H}_A$, $\mathcal{H}_B$). Moreover, $A^n$ denotes a quantum system composed of $n$ copies of the system $A$, modeled by an $n$-fold tensor product of Hilbert spaces, $\mathcal{H}_A^n = \mathcal{H}_A \otimes \cdots \otimes \mathcal{H}_A$. We denote by $U(A)$, $H(A)$ and $P(A)$ the set of unitary, Hermitian, and positive semi-definite operators acting on $\mathcal{H}_A$, respectively. We denote the identity operator on $\mathcal{H}_A$ by $1_A$ and the partial trace by $\text{tr}_A$. Furthermore, we use $|A|$ to denote the dimension of the Hilbert space $\mathcal{H}_A$.

Let $S(A)$ be the set of quantum states, i.e., $S(A) := \{\rho_A \in P(A) \mid \text{tr}[\rho_A] = 1\}$, where $\text{tr}$ denotes the trace. Given a bipartite state $\rho_{AB} \in S(AB)$, we denote by $\rho_A = \text{tr}_B[\rho_{AB}]$ its marginal on $A$. We consequently use subscripts to indicate which physical system an operator acts on. Finally, $\pi_A \in S(A)$ denotes the maximally mixed state given by $\pi_A = 1_A / |A|$.

A. Projectors and Pinching

For two Hermitian operators $L, K \in \mathcal{H}$, we write $L \leq K$ if and only if $K - L \in P$ and we write $L \ll K$ if the support of $L$ is contained in the support of $K$. Moreover, we write \{$L \geq K\} = 1 - \{L < K\}$ for the projector onto the subspace spanned by eigenvectors corresponding to non-negative eigenvalues of $L - K$. By definition we have $(L - K)\{L \geq K\} \geq 0$, and, thus,

$$L\{L \geq K\} \geq K\{L \geq K\}. \tag{12}$$

For any unitary $V$, we further have

$$V\{L \geq K\}V^\dagger = \{VLV^\dagger \geq VKV^\dagger\}. \tag{13}$$

We will also use an inequality by Audenaert et al. [1, Thm. 1], which can be conveniently stated as follows [3, Eq. (24)]. Let $L$ and $K$ be positive semi-definite and $s \in (0, 1)$. Then,

$$\text{tr}[L^sK^{1-s}] \geq \text{tr}[K\{L \geq K\}] + \text{tr}[L\{L < K\}]. \tag{14}$$

For any Hermitian $L$, we write its spectral decomposition as $L = \sum_{\lambda \in \text{spec}(L)} \lambda P^\lambda_L$, where $P^\lambda_L$ are projectors and $\text{spec}(L) \subset \mathbb{R}$ is its spectrum. We denote by $\mathcal{P}_L$ the pinching map for this spectral decomposition, i.e the following completely positive trace-preserving map:

$$\mathcal{P}_L : K \mapsto \sum_{\lambda \in \text{spec}(L)} P^\lambda_L KP^\lambda_L. \tag{15}$$
B. Permutation Invariance and Universal State

We will use the following observation from the representation theory of the group $S_n$ of permutations of $n$ elements. Let $U_{A^n}: S_n \rightarrow U(A^n)$ denote the natural unitary representation of $S_n$ that permutes the subsystems $A_1, A_2, \ldots, A_n$. An operator $L_{A^n}$ is called permutation invariant if it satisfies $U_{A^n}(\pi)L_{A^n}U_{A^n}(\pi)^d = L_{A^n}$ for all $\pi \in S_n$. Similarly, we say that $L_{A^n}$ is invariant under (n-fold) product unitaries if it satisfies $V_{A}^{\otimes n}L_{A^n}V_{A}^{\otimes n} = L_{A^n}$ for all $V_{A} \in U(A)$.

**Lemma 1.** Let $A$ be a system with $|A| = d$. For all $n \in \mathbb{N}$ there exists a state $\omega_{A^n} \in \mathcal{S}(A^n)$, which we call universal state, such that the following holds:

1. For all permutation invariant states $\tau_{A^n} \in \mathcal{S}(A^n)$, we have
   $$\tau_{A^n} \leq g_{n,d} \omega_{A^n}_{A^n} \quad \text{with} \quad g_{n,d} = \binom{n + d^2 - 1}{n} \leq (n + 1)^{d-1}. \quad (16)$$

2. The universal state has the following eigenvalue decomposition:
   $$\omega_{A^n}_{A^n} = \sum_{\lambda \in \Lambda_{n,d}} p_{\lambda} P_{A^n}^\lambda, \quad (17)$$

   where $\Lambda_{n,d}$ is the set of Young diagrams of size $n$ and depth $d$ and satisfies $|\Lambda_{n,d}| \leq (n + 1)^{d-1}$, $\{P_{A^n}^\lambda\}_{\lambda}$ are mutually orthogonal projectors and $\{p_{\lambda}\}_{\lambda}$ is a probability distribution. In particular, $\omega_{A^n}_{A^n}$ is permutation invariant and invariant under product unitaries, and commutes with all permutation invariant states.

Note that a related construction is presented in [7], and we refer the reader to [6] for a thorough discussion of group representation theory in the context of quantum information. A different explicit construction of such a universal state is also proposed in [20, Sec. 3], but the constant given there instead of $g_{n,d}$ is not optimal.

**Proof.** Since $\tau_{A^n}$ is invariant under permutations, it has a purification $\tau_{A^n,A^n}$ in the symmetric subspace of $(\mathcal{H}_A \otimes \mathcal{H}_A)^{\otimes n}$ where $\mathcal{H}_A \equiv \mathcal{H}_A$ are isomorphic (see, e.g., [38, Lem. 4.2.2.]). Let $P_{A^n,A^n}^\text{sym}$ denote the projector onto this symmetric subspace, and its dimension by $g_{n,d}$. Then,

$$\tau_{A^n,A^n} \leq P_{A^n,A^n}^\text{sym}, \quad \text{where} \quad P_{A^n,A^n}^\text{sym} = \frac{1}{|S_n|} \sum_{\pi \in S_n} U_{A^n}(\pi) \otimes U_{A^n}(\pi). \quad (18)$$

Let us now define the universal state as

$$\omega_{A^n}_{A^n} = \frac{1}{g_{n,d}} \text{tr}_{A^n} \left[ P_{A^n,A^n}^\text{sym} \right] = \frac{1}{g_{n,d}|S_n|} \sum_{\pi \in S_n} \text{tr}[U_{A^n}(\pi)] U_{A^n}(\pi) \quad (19)$$

The state clearly has the desired first property, due to (18). Moreover, it is evident from (19) that $\omega_{A^n}_{A^n}$ is invariant under permutations and product unitaries. By the Schur-Weyl duality the natural representation of $S_n \times U(A)$ given by $\pi \times V_{A} \mapsto U_{A^n}(\pi) \cdot V_{A}^{\otimes n}$ decomposes into different irreducible representations labelled by the Young diagrams in $\Lambda_{n,d}$ and Schur’s lemma thus ensures that $\omega_{A^n}_{A^n}$ is of the form given in (17).

The number $|\Lambda_{n,d}|$ is upper bounded by the number of types of strings of length $n$ with $d$ symbols, which in turn is bounded by $(n + 1)^{d-1}$. (See, e.g., [20, Eq. (1)]).

Finally, since the subspaces $P_{A^n}^\lambda$ are spanned by the irreducible representations of $S_n$ of a type $\lambda$, by Schur’s Lemma every permutation invariant state $\tau_{A^n}$ can written in the form

$$\tau_{A^n} = \sum_{\lambda \in \Lambda_{n,d}} \tau_{A^n}^\lambda, \quad (20)$$

where $\tau_{A^n}^\lambda$ is supported in $P_{A^n}^\lambda$. It is thus evident that such states commute with $\omega_{A^n}_{A^n}$. □
C. Rényi Divergence

Let us define the following two families of Rényi divergences for \( \alpha \in (0, 1) \cup (1, \infty) \). For any quantum state \( \rho \in \mathcal{S} \) and positive semi-definite operator \( \sigma \geq 0 \) satisfying \( \rho \ll \sigma \), we define the Rényi relative entropy [35] and the (sandwiched) Rényi divergence [31, 46], respectively, as

\[
D_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{tr} \left[ \sigma^{\frac{1}{2\alpha}} \rho^{\frac{1}{\alpha}} \sigma^{\frac{1}{2\alpha}} \right]
\]

and

\[
\tilde{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{tr} \left[ \left( \sigma^{\frac{1}{2\alpha}} \rho^{\frac{1}{\alpha}} \sigma^{\frac{1}{2\alpha}} \right)^\alpha \right].
\]

The two families of entropies coincide when \( \rho \) and \( \sigma \) commute. For \( \alpha \in \{0, 1, \infty\} \) we define \( D_\alpha(\rho||\sigma) \) and \( \tilde{D}_\alpha(\rho||\sigma) \) as the corresponding limit. The relative entropy emerges when we take the limit \( \alpha \to 1 \) in both cases, namely

\[
\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho||\sigma) = \lim_{\alpha \to 1} D_\alpha(\rho||\sigma) = \text{tr} \left[ \rho(\log \rho - \log \sigma) \right] =: D(\rho||\sigma).
\]

Some special cases of these entropies, in particular \( D_0(\rho||\sigma) \) and \( \tilde{D}_\infty(\rho||\sigma) \) have previously been discussed in [10] and are based on Renner’s min- and max-entropy [38]. A comprehensive overview of other special cases is given in [31].

For the second order analysis we will employ the information variance [25, 43], given as

\[
V(\rho||\sigma) := \text{tr} \left[ \rho(\log \rho - \log \sigma - D(\rho||\sigma))^2 \right].
\]

In particular, we will use the fact that [27, Prop. 35]

\[
\frac{\partial}{\partial \alpha} D_\alpha(\rho||\sigma) \bigg|_{\alpha = 1} = \frac{\partial}{\partial \alpha} \tilde{D}_\alpha(\rho||\sigma) \bigg|_{\alpha = 1} = \frac{V(\rho||\sigma)}{2}.
\]

Let \( |\text{spec}(\sigma)| \) denote the number of mutually different eigenvalues of \( \sigma \). The following property of the sandwiched Rényi divergence is crucial for our derivations:

**Lemma 2.** Let \( \rho \in \mathcal{S} \) and \( \sigma \in \mathcal{P} \). For all \( \alpha \geq 0 \), we have

\[
\tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) \leq \tilde{D}_\alpha(\rho||\sigma) \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + \begin{cases} 
|\text{spec}(\sigma)| & \text{if } \alpha \in [0, 2] \\
2|\text{spec}(\sigma)| & \text{if } \alpha > 2
\end{cases}.
\]

**Proof.** Since \( \mathcal{P}_\sigma(\sigma) = \sigma \), the first inequality is a special case of the data-processing inequality. This special case was first established in [31, Prop. 15].

To derive the upper bound for \( \alpha \in (1, 2] \), we write

\[
\exp \left( (\alpha - 1) \tilde{D}_\alpha(\rho||\sigma) \right) = \text{tr} \left[ (\sigma^{\frac{1}{2\alpha}} \rho^{\frac{1}{\alpha}} \sigma^{\frac{1}{2\alpha}})^{\alpha - 1} \right]
\]

\[
\leq \text{tr} \left[ (\sigma^{\frac{1}{2\alpha}} |\text{spec}(\sigma)| \mathcal{P}_\sigma(\rho) \sigma^{\frac{1}{2\alpha}})^{\alpha - 1} \right]
\]

\[
= |\text{spec}(\sigma)|^{\alpha - 1} \text{tr} \left[ (\sigma^{\frac{1}{2\alpha}} \mathcal{P}_\sigma(\rho) \sigma^{\frac{1}{2\alpha}})^{\alpha - 1} \right]
\]

\[
= |\text{spec}(\sigma)|^{\alpha - 1} \exp \left( (\alpha - 1) \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) \right).
\]

To establish (28), we use [17, Lem. 9] which states that

\[
\rho \leq |\text{spec}(\sigma)| \mathcal{P}_\sigma(\rho)
\]
and, since the function $x \mapsto x^{\alpha-1}$ is operator monotone for $\alpha \in (1, 2)$,
\[
(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha-1} \leq |\text{spec}(\sigma)|^{\alpha-1} \left( \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{P}_\sigma(\rho) \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\alpha-1}.
\] (32)

An analogous argument, with the opposite inequality (28), holds for $\alpha \in (0, 1)$. Thus, for all $\alpha \in (0, 1) \cup (1, 2]$, we conclude that
\[
\tilde{D}_\alpha(\rho||\sigma) \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + \log |\text{spec}(\sigma)|.
\] (33)

To get an upper bound for $\alpha > 2$ we observe that
\[
\exp\left( (\alpha - 1) \tilde{D}_\alpha(\rho||\sigma) \right) = \text{tr} \left[ (\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \right] \leq |\text{spec}(\sigma)|^\alpha \text{tr} \left[ (\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{P}_\sigma(\rho) \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \right]
\] (34)
\[
\tilde{D}_\alpha(\rho||\sigma) \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + \frac{\alpha}{\alpha - 1} \log |\text{spec}(\sigma)| \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + 2 \log |\text{spec}(\sigma)|.
\] (35)

since $A \leq B$ implies $\text{tr}[f(A)] \leq \text{tr}[f(B)]$ for every monotonically increasing function $f$. Thus,
\[
\tilde{D}_\alpha(\rho||\sigma) \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + \frac{\alpha}{\alpha - 1} \log |\text{spec}(\sigma)| \leq \tilde{D}_\alpha(\mathcal{P}_\sigma(\rho)||\sigma) + 2 \log |\text{spec}(\sigma)|.
\] (36)

Finally, note that the inequality thus also holds for the limiting cases $\alpha \in \{0, 1, \infty\}$.

\begin{corollary}
Let $\rho \in \mathcal{S}$ and $\sigma \in \mathcal{P}$. For all $\alpha \geq 0$, we have
\[
\lim_{n \to \infty} \left\{ \frac{1}{n} D_\alpha(\mathcal{P}_{\sigma^\otimes n}(\rho^\otimes n)||\sigma^\otimes n) \right\} = \tilde{D}_\alpha(\rho||\sigma).
\] (37)
\end{corollary}

This extends a prior result by Mosonyi and Ogawa in [29] to all $\alpha \geq 0$.

Finally, note that the first inequality in Lemma 2 (but only the first) also holds if we replace $\tilde{D}_\alpha$ with $D_\alpha$. Taking the asymptotic limit in (37), this yields that $\tilde{D}_\alpha(\rho||\sigma) \leq D_\alpha(\rho||\sigma)$.

\section{Generalized Rényi Mutual Information}

We state our results in a general form that allows us to treat mutual information and conditional entropies at the same time. The usual mutual information and the conditional entropies can then be recovered as special cases.

\subsection{Definitions}

For this purpose, let us define the the \textit{generalized Rényi mutual information} and the \textit{generalized sandwiched Rényi mutual information} for a bipartite state $\rho_{AB} \in \mathcal{S}(AB)$ and any $\tau_A \geq 0$ such that $\tau_A \gg \rho_A$ as follows:
\[
I_\alpha(\rho_{AB}||\tau_A) := \inf_{\sigma_B \in \mathcal{S}(B)} D_\alpha(\rho_{AB}||\tau_A \otimes \sigma_B),
\] (38)
\[
\tilde{I}_\alpha(\rho_{AB}||\tau_A) := \inf_{\sigma_B \in \mathcal{S}(B)} \tilde{D}_\alpha(\rho_{AB}||\tau_A \otimes \sigma_B).
\] (39)

\footnote{If $\sigma$ has $k$ different eigenvalues, the number of different eigenvalues of $\sigma^\otimes n$ is equal to the number of types of sequences of length $n$ with $k$ symbols.}
It is easy to verify that the infimum in the above definitions can be replaced by a minimum since the divergences are continuous in $\sigma_B$ whenever $\sigma_B \ll \rho_B$ and diverges to $+\infty$ otherwise. Specializing to $\alpha = 1$, we find that

$$I(\rho_{AB}\|\tau_A) := \min_{\sigma_B \in S(B)} D(\rho_{AB}\|\tau_A \otimes \sigma_B) = D(\rho_{AB}\|\tau_A \otimes \rho_B),$$

i.e. the minimizer is given by the marginal $\rho_B$ at $\alpha = 1$. We also define

$$V(\rho_{AB}\|\tau_A) := V(\rho_{AB}\|\tau_A \otimes \rho_B).$$

Note that the Rényi mutual information and the sandwiched Rényi mutual information \cite{4, 15} is recovered by choosing $\tau_A = \rho_A$, namely we define\footnote{For $\alpha = \infty$, various variants of the latter definition have been investigated in \cite{8} in the context of the smooth entropy framework (see \cite{41}).}

$$I_\alpha(A:B)_\rho := I_\alpha(\rho_{AB}\|\rho_A), \quad \text{and} \quad \tilde{I}_\alpha(A:B)_\rho := \tilde{I}_\alpha(\rho_{AB}\|\rho_A).$$

Similarly, we define the Rényi conditional entropy \cite{42} and the sandwiched Rényi conditional entropy \cite{31} by choosing $\tau_A = 1_A$. Using the notation of \cite{42}, we have

$$H^\uparrow_\alpha(A|B)_\rho := -I_\alpha(\rho_{AB}\|1_A) = \log |A| - I_\alpha(\rho_{AB}\|\pi_A) \quad \text{and} \quad \tilde{H}^\uparrow_\alpha(A|B)_\rho := -\tilde{I}_\alpha(\rho_{AB}\|1_A) = \log |A| - \tilde{I}_\alpha(\rho_{AB}\|\pi_A).$$

### B. Duality Relation for $\tilde{I}_\alpha$

We will take advantage of the following duality relation for the mutual information:

**Lemma 4.** Let $\rho_{AB} \in S(AB)$, $\tau_A \geq 0$ such that $\tau_A \gg \rho_A$ and $\alpha, \beta \geq \frac{1}{2}$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 2$. Then, for any purification $\rho_{ABC}$ of $\rho_{AB}$, we have

$$\tilde{I}_\alpha(\rho_{AB}\|\tau_A) = -\tilde{I}_\beta(\rho_{AC}\|\tau_A^{-1}),$$

where the inverse is taken on the support of $\tau_A$.

This result is a rather straightforward generalization of the duality relation for the conditional Rényi entropy that was recently established independently in \cite{31} and \cite{4}. We provide a proof in Appendix B for completeness.

### C. Additivity of $I_\alpha$ and $\tilde{I}_\alpha$

We are interested in the additivity of the mutual informations $I_\alpha$ and $\tilde{I}_\alpha$ defined above. For $I_\alpha$ we can find the state $\sigma_B$ that minimizes the Rényi divergence using Sibson’s identity.

**Lemma 5.** For all $\alpha \geq 0$ and any states $\rho_{AB}$, $\rho_{A'B'}$, $\tau_A$, and $\pi_A'$, we have

$$I_\alpha(\rho_{AB} \otimes \omega_{A'B'}\|\tau_A \otimes \pi_A') = I_\alpha(\rho_{AB}\|\tau_A) + I_\alpha(\omega_{A'B'}\|\pi_A').$$
Moreover, the convergence is uniform in \( \alpha \) where we used [31, Prop. 5] in the last step and set \( d \) which implies that
\[
\tilde{I}_\alpha := \frac{1}{\alpha - 1} \log \operatorname{tr} \left\{ \frac{1}{\alpha^2} \frac{1}{\pi} \right\}. \tag{48}
\]

Furthermore, as an immediate consequence of the positive definiteness of \( D_\alpha(\sigma_B^*(\alpha)||\sigma_B) \), we find that \( \arg \min_{\sigma_B \in S(B)} D_\alpha(\rho_{AB}||\sigma_B \otimes \sigma_B) = \sigma_B^*(\alpha) \) is unique. In particular,
\[
I_\alpha(\rho_{AB}||\tau_A) = \frac{1}{\alpha - 1} \log \operatorname{tr} \left\{ \frac{1}{\alpha^2} \frac{1}{\pi} \right\} \tag{49}
\]
which implies that \( I_\alpha \) is additive and concludes the proof.

**Proof.** The following quantum Sibson’s identity is adapted from [39, Lem. 3 in Suppl. Mat.]. Let \( \rho_{AB} \in S(AB), \tau_A \in S(A), \) and \( \sigma_B \in S(B) \). For any \( \alpha > 0 \), we have
\[
D_\alpha(\rho_{AB}||\tau_A \otimes \sigma_B) = D_\alpha(\rho_{AB}||\tau_A^* \otimes \sigma_B^*(\alpha)) + D_\alpha(\sigma_B^*(\alpha)||\sigma_B), \tag{47}
\]
where
\[
\sigma_B^*(\alpha) := \frac{\operatorname{tr}_A \left\{ \frac{1}{\alpha^2} \frac{1}{\pi} \right\}}{\operatorname{tr} \left\{ \frac{1}{\alpha^2} \frac{1}{\pi} \right\}}. \tag{48}
\]

We note that a trivial extension of [4, Th. 10 and 11] establishes that \( \tilde{I}_\alpha \) is additive for \( \alpha \geq \frac{1}{2} \). However, the result also directly follows from the duality relation in Lemma 4.5.

**Lemma 6.** For all \( \alpha \geq \frac{1}{2} \) and any states \( \rho_{AB}, \rho_{AB}', \tau_A, \) and \( \pi_A' \), we have
\[
\tilde{I}_\alpha(\rho_{AB} \otimes \omega_{AB}'||\tau_A \otimes \pi_A') = \tilde{I}_\alpha(\rho_{AB}||\tau_A) + \tilde{I}_\alpha(\omega_{AB}'||\pi_A'). \tag{50}
\]

**D. Uniform Asymptotic Achievability of \( \tilde{I}_\alpha \)**

The following result forms the core of our proof for the achievability of the strong-converse exponent. It establishes that the mutual information \( \tilde{I}_\alpha(\rho_{AB}||\tau_A) \) can be expressed as a limit of classical Rényi divergences. More precisely, we have the following.

**Proposition 7.** Let \( \rho_{AB} \in S(AB) \) and \( \tau_A \in S(A) \) such that \( \tau_A \gg \rho_A \). For any \( \alpha \geq \frac{1}{2} \), we have
\[
\frac{1}{n} D_\alpha(\mathcal{P}_{\tau_A^n \otimes \omega_B^n}(\rho_{AB}^n) \bigg| \bigg| \tau_A^n \otimes \omega_B^n) = \tilde{I}_\alpha(\rho_{AB}||\tau_A) + O \left( \frac{\log n}{n} \right). \tag{51}
\]

Moreover, the convergence is uniform in \( \alpha \).

**Proof.** For any \( \sigma_B \in S(B) \), employing the data-processing inequality we find
\[
D_\alpha(\mathcal{P}_{\tau_A^n \otimes \omega_B^n}(\rho_{AB}^n) \bigg| \bigg| \tau_A^n \otimes \omega_B^n) \leq D_\alpha(\rho_{AB}^n || \tau_A^n \otimes \omega_B^n) \leq \tilde{D}_\alpha(\rho_{AB}^n || \tau_A^n \otimes \omega_B^n) + \log g_{n,d}, \tag{52}
\]
where we used [31, Prop. 5] in the last step and set \( d = \max\{|A|,|B|\} \). And thus, in particular,
\[
\frac{1}{n} D_\alpha(\mathcal{P}_{\tau_A^n \otimes \omega_B^n}(\rho_{AB}^n) \bigg| \bigg| \tau_A^n \otimes \omega_B^n) \leq \min_{\sigma_B \in S(B)} \tilde{D}_\alpha(\rho_{AB} || \tau_A \otimes \sigma_B) + \frac{\log g_{n,d}}{n} \tag{54}
\]
\[
= \tilde{I}_\alpha(\rho_{AB}||\tau_A) + \frac{\log g_{n,d}}{n}. \tag{55}
\]

\(^5\) To see this, note that the inequality trivially holds in one direction by definition — the other direction then follows by applying the duality relation on both sides for a product purification.
The upper bound then follows by taking the limit $n \to \infty$. For the lower bound, we first invoke Lemma 2, which yields
\begin{equation}
\tilde{D}_\alpha(\mathcal{P}_{\hat{\tau}^n_A \otimes \hat{\omega}^n_B}(\rho_{AB}^n) \| \tau^\otimes_n \otimes \omega^\otimes_n) \geq \tilde{D}_\alpha(\rho_{AB}^n \| \tau^\otimes_n \otimes \omega^\otimes_n) - \log(v_{r_A^\otimes n \otimes \omega^\otimes_n}) \geq \tilde{I}_\alpha(\rho_{AB}^n \| \tau^\otimes_n) - \log(v_{r_A^\otimes n \otimes \omega^\otimes_n}).
\end{equation}
where we use the shorthand notation $v_{\sigma} = |\text{spec}(\sigma)|$. Next, we recall that Lemma 6 establishes the additivity of $\tilde{I}_\alpha$, in particular $\tilde{I}_\alpha(\rho_{AB}^n \| \tau^\otimes_n) = n \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$. Thus, we have
\begin{equation}
\frac{1}{n} \tilde{D}_\alpha(\mathcal{P}_{\hat{\tau}^n_A \otimes \hat{\omega}^n_B}(\rho_{AB}^n) \| \tau^\otimes_n \otimes \omega^\otimes_n) \geq \tilde{I}_\alpha(\rho_{AB} \| \tau_A) + \frac{\log(v_{r_A^\otimes n \otimes \omega^\otimes_n})}{n}.
\end{equation}
Finally, note that $v_{r_A^\otimes n \otimes \omega^\otimes_n} \leq v_{r_A^\otimes n} v_{\omega^\otimes_n} \leq (n + 1)^{2(d-1)}$ to conclude the proof. 

It is important that the correction terms in the above derivation are of the order $o(n^{-\frac{1}{2}})$. This allows for the following corollary.

**Corollary 8.** For any $t \in \mathbb{R}$, we have
\begin{equation}
\lim_{n \to \infty} \left\{ \frac{t}{\sqrt{n}} \left( D_{1+\frac{t^2}{\sqrt{n}}}(\mathcal{P}_{\hat{\tau}^n_A \otimes \hat{\omega}^n_B}(\rho_{AB}^n) \| \tau^\otimes_n \otimes \omega^\otimes_n) - n \tilde{I}(\rho_{AB} \| \tau_A) \right) \right\} = \frac{t^2}{2} V(\rho_{AB} \| \tau_A).
\end{equation}

**Proof.** According to Proposition 10 below, the Taylor expansion of $\tilde{I}_\alpha(\rho_{AB} \| \tau_A)$ for $\alpha$ close to 1 is
\begin{equation}
\tilde{I}_\alpha(\rho_{AB} \| \tau_A) = D(\rho_{AB} \| \tau_A \otimes \rho_B) + \frac{\alpha - 1}{2} V(\rho_{AB} \| \tau_A \otimes \rho_B) + O((\alpha - 1)^2).
\end{equation}
Substituting this into (51) with $\alpha = 1 + \frac{t}{\sqrt{n}}$ yields the desired limit. \hfill \Box

From the fact that the function $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$ is a point wise limit of a sequence of classical Rényi divergences and the convergence is uniform in $\alpha$, we can immediately deduce the following:

**Corollary 9.** The function $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$ is continuous and monotonically increasing. Moreover, the function $t \mapsto t\tilde{I}_1(\rho_{AB} \| \tau_A)$ is continuous and convex.

### E. Differentiability of $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$

However, the argument used to derive Corollary 9 does not suffice to establish differentiability of the sandwiched Rényi mutual information.

**Proposition 10.** Let $\rho_{AB} \in \mathcal{S}(AB)$ and $\tau_A \in \mathcal{S}(A)$ such that $\tau_A \triangleright \rho_A$. Then, the function $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$ is continuously differentiable for $\alpha \geq \frac{1}{2}$. Moreover,
\begin{equation}
\frac{\partial}{\partial \alpha} \tilde{I}_\alpha(\rho_{AB} \| \tau_A) \bigg|_{\alpha = 1} = \frac{1}{2} V(\rho_{AB} \| \tau_A).
\end{equation}

Let us remark that continuity of $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB} \| \tau_A)$ also directly follows from the fact that $\alpha \mapsto D_\alpha(\rho_{AB} \| \tau_A \otimes \sigma_B)$ is continuous and the duality relation. However, due to the optimization over $\sigma_B$ involved in the definition of $\tilde{I}_\alpha(\rho_{AB} \| \tau_A)$, it is not at all clear that the function is differentiable. We show this in Appendix C by first characterizing the optimal state $\sigma_B^*(\alpha)$ for each $\alpha$ and then showing that $\alpha \mapsto \sigma_B^*(\alpha)$ is continuously differentiable.

Here, we present the proof for the second statement and establish the derivative at $\alpha = 1$. 

Proof. First, note that
\[
\limsup_{h \to 0} \left\{ \frac{1}{\hbar} \left( \tilde{I}_{1+h}(\rho_{AB}\|\tau_A) - I(\rho_{AB}\|\tau_A) \right) \right\} 
\leq \limsup_{h \to 0} \left\{ \frac{1}{\hbar} \left( \tilde{D}_{1+h}(\rho_{AB}\|\tau_A \otimes \rho_B) - D(\rho_{AB}\|\tau_A \otimes \rho_B) \right) \right\} 
= \frac{\partial}{\partial h} \tilde{D}_{1+h}(\rho_{AB}\|\tau_A \otimes \rho_B) \bigg|_{h=0} = \frac{1}{2} V(\rho_{AB}\|\tau_A \otimes \rho_B). 
\]

On the other hand, using Lemma 4, we find
\[
\tilde{I}_{1+h}(\rho_{AB}\|\tau_A) - I(\rho_{AB}\|\tau_A) = I(\rho_{AC}\|\tau_A^{-1}) - \tilde{I}_{1-f(h)}(\rho_{AC}\|\tau_A^{-1}),
\]
where \( f : h \mapsto \frac{h}{1+2h} \) satisfies \( f(0) = 0 \) and \( f'(0) = 1 \). Using this, we find
\[
\liminf_{h \to 0} \left\{ \frac{1}{\hbar} \left( \tilde{I}_{1+h}(\rho_{AB}\|\tau_A) - I(\rho_{AB}\|\tau_A) \right) \right\} 
= \liminf_{h \to 0} \left\{ \frac{1}{\hbar} \left( I(\rho_{AC}\|\tau_A^{-1}) - \tilde{I}_{1-f(h)}(\rho_{AC}\|\tau_A^{-1}) \right) \right\} 
\geq \liminf_{h \to 0} \left\{ \frac{1}{\hbar} \left( D(\rho_{AC}\|\tau_A^{-1} \otimes \rho_C) - \tilde{D}_{1-f(h)}(\rho_{AC}\|\tau_A^{-1} \otimes \rho_C) \right) \right\} 
= -\frac{\partial}{\partial h} \tilde{D}_{1-f(h)}(\rho_{AC}\|\tau_A^{-1} \otimes \rho_C) \bigg|_{h=0} = \frac{V(\rho_{AC}\|\tau_A^{-1} \otimes \rho_C)}{2}. 
\]

It is easy to verify that \( V(\rho_{AC}\|\tau_A^{-1}) = V(\rho_{AB}\|\tau_A) \), which concludes the proof. \( \square \)

IV. PROBLEM DEFINITION AND OPERATIONAL QUANTITIES

We define a more general hypothesis testing problem that allows us to treat both problems discussed in the introduction together.

Let \( \rho_{AB} \in S(AB) \) be a bipartite quantum state on systems \( A \) and \( B \) and let \( \tau_A \in S(A) \) be a state on system \( A \). We are interested in the following composite hypothesis testing problem:

null hypothesis: \( \) state is \( \rho_{AB} \) \hspace{1cm} (70) 
alternate hypothesis: \( \) state is \( \tau_A \otimes \sigma_B \), for any state \( \sigma_B \in S(B) \). \hspace{1cm} (71)

We consider arbitrary bipartite hypothesis tests, given by an operator \( 0 \leq Q_{AB} \leq 1 \) on \( AB \) and define the type-I error and type-II error, respectively, as follows:

\[
\alpha(Q_{AB};\rho_{AB}) := \text{tr}[\rho_{AB} - Q_{AB}], \hspace{1cm} \text{and} \hspace{1cm} \beta(Q_{AB};\tau_A) := \max_{\sigma_B \in S(B)} \text{tr}[Q_{AB}(\tau_A \otimes \sigma_B)]. 
\]

(72) \hspace{1cm} (73)

It is convenient to define the quantity \( \hat{\alpha}(\mu;\rho_{AB},\tau_A) \) as the minimum type-I error when the type-II error is below \( \varepsilon \), i.e. we consider the following optimization problem:

\[
\hat{\alpha}(\mu;\rho_{AB}\|\tau_A) := \min_{0 \leq Q_{AB} \leq 1} \left\{ \alpha(Q_{AB};\rho_{AB}) \left| \beta(Q_{AB};\tau_A) \leq \mu \right. \right\} 
\]

(74)
and note that this quantity can trivially be bounded as
\[
\hat{\alpha}(\mu; \rho_{AB} \| \tau_A) \geq \min_{\sigma_B \in \mathcal{S}(B)} \left\{ \min_{0 \leq Q_{AB} \leq 1} \frac{1}{n} \sum_{\tau_A \in S} \log \left( \frac{1}{n} \mathbb{E}_{s \sim \mathcal{S}} \left[ Q_{AB} \rho_{AB} \right] \right) \right\}.
\]

The quantity \( \hat{\alpha}(\mu; \rho_{AB} \| \tau_A) \) is the object of our study here. More precisely, for any fixed \( n \in \mathbb{N} \), we consider the following \( n \)-fold extension of this composite hypothesis testing problem:

- **null hypothesis:** state is \( \rho_{AB}^n \)
- **alternate hypothesis:** state is \( \tau_A^n \otimes \sigma_B^n \), for any state \( \sigma_B^n \in \mathcal{S}(B^n) \).

Here, it is important to note that \( \sigma_B^n \) is an arbitrary state in \( \mathcal{S}(B^n) \), and not restricted to product or permutation invariant states. We are interested in the asymptotic behavior of \( \hat{\alpha}(\mu_n; \rho_{AB}^n \| \tau_A^n) \) for suitably chosen sequences \( \{\mu_n\}_n \) for large \( n \).

\[\text{V. HOEFFDING BOUND}\]

Our first result considers the case where the error of the second kind goes to zero exponentially with a rate below the mutual information \( I(\rho_{AB} \| \tau_A) \). In this case, we find that the error of the first kind converges to zero exponentially fast, and the exponent is determined by the generalized Rényi mutual information, \( I_\alpha(\rho_{AB} \| \tau_A) \), for \( \alpha < 1 \).

**Theorem 11.** Let \( \rho_{AB} \in \mathcal{S}(AB) \) and \( \tau_A \in \mathcal{S}(A) \). Then, for any \( R > 0 \), we have

\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \hat{\alpha} \left( \exp(-nR); \rho_{AB}^n \| \tau_A^n \right) \right\} = \sup_{s \in (0,1)} \left\{ \frac{1-s}{s} (I_\alpha(\rho_{AB} \| \tau_A) - R) \right\}.
\]

Note that if \( R \geq I(\rho_{AB} \| \tau_A) \) the right hand side of (79) evaluates to zero, revealing that in this case the error of the first kind will decay slower than exponential in \( n \). Furthermore, if \( R < I_\alpha(\rho_{AB} \| \tau_A) \), we find that the right hand side of (79) diverges to \( +\infty \) indicating that the decay is faster than exponential in \( n \). This includes the case where the error of the first kind is identically zero for sufficiently large \( n \), e.g. in zero-error channel coding.

We also consider the following two special cases:

**Corollary 12.** Let \( \rho_{AB} \in \mathcal{S}(AB) \). Then, for any \( R > 0 \), we have

\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \hat{\alpha} \left( \exp(-nR); \rho_{AB}^n \| \rho_A^\otimes n \right) \right\} = \sup_{s \in (0,1)} \left\{ \frac{1-s}{s} (I(A:B)_\rho - R) \right\},
\]

\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \hat{\alpha} \left( \exp(-nR); \rho_{AB}^n \| \tau_A^n \right) \right\} = \sup_{s \in (0,1)} \left\{ \frac{1-s}{s} (I(A:B)_\rho - R) \right\}.
\]

This corollary establishes an operational interpretation of the Rényi mutual information, \( I_\alpha(A:B)_\rho \), as well as the Rényi conditional entropies, \( H_\alpha(A|B)_\rho \), for \( 0 \leq \alpha \leq 1 \).

In the following, we treat the proof of the achievable and optimality in Theorem 11 separately.

---

\(^6\) In fact, we will see in Theorems 13 and 17, respectively, that the error of the first kind will converge to 1 exponentially fast in \( n \) if \( R > I(\rho_{AB} \| \tau_A) \), and that it will converge to \( \frac{1}{2} \) if \( R = I(\rho_{AB} \| \tau_A) \).
A. Proof of Achievability

The achievability is shown using a quantum Neyman-Pearson test comparing $\rho_{AB}^{\otimes n}$ with $\tau_{A}^{\otimes n} \otimes \omega_{B}^{n}$, where $\omega_{B}^{n}$ is the universal state defined in Lemma 1. The analysis follows the lines of the proof of the direct part of the quantum Hoeffding bound given in [3, Sec. 5.5] and further hinges on the additivity of the mutual information expressed in Lemma 5.

Note that the expression on the right hand side of (79) is zero if $R \geq I(\rho_{AB} \| \tau_{A})$ and the inequality thus trivially holds. In the following we show that, for any $0 < R < I(\rho_{AB} \| \tau_{A})$,

$$\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \delta \left( \exp(-nR); \rho_{AB}^{\otimes n} \bigg\| \tau_{A}^{\otimes n} \right) \right\} \geq \sup_{s \in (0,1)} \left\{ \frac{1-s}{s} \left( I_{s}(\rho_{AB} \| \tau_{A}) - R \right) \right\},$$

(82)

**Proof of Achievability in Theorem 11.** Let us fix $s \in (0,1)$ and let $\{\lambda_{n}\}_{n \in \mathbb{N}}$ be real numbers to be specified later and define the sequence of tests

$$Q_{A^{n}B^{n}}^{n} := \{ \rho_{AB}^{\otimes n} \geq \exp(\lambda_{n}) \tau_{A}^{\otimes n} \otimes \omega_{B}^{n} \},$$

(83)

where $\omega_{B}^{n}$ is the universal state introduced in Lemma 1. First, note that the natural representation of $S_{n}$ decomposes as $U_{A^{n}B^{n}}(\pi) = U_{A^{n}}(\pi) \otimes U_{B^{n}}(\pi)$ and that $Q_{A^{n}B^{n}}^{n}$ is permutation invariant as a direct consequence of Eq. (13). Thus, we have

$$\beta(Q_{A^{n}B^{n}}^{n}; \tau_{A}^{\otimes n}) = \max_{\sigma_{B}^{n} \in \mathcal{S}(B^{n})} \text{tr} \left[ Q_{A^{n}B^{n}}^{n} \left( \tau_{A}^{\otimes n} \otimes \sigma_{B}^{n} \right) \right]$$

(84)

$$= \max_{\sigma_{B}^{n} \in \mathcal{S}(B^{n})} \frac{1}{|S_{n}|} \sum_{\pi \in S_{n}} \text{tr} \left[ U_{A^{n}B^{n}}(\pi) Q_{A^{n}B^{n}}^{n} (\tau_{A}^{\otimes n} \otimes \sigma_{B}^{n}) U_{A^{n}B^{n}}(\pi)^{\dagger} \right]$$

(85)

$$= \max_{\sigma_{B}^{n} \in \mathcal{S}(B^{n})} \text{tr} \left[ Q_{A^{n}B^{n}}^{n} \left( \tau_{A}^{\otimes n} \otimes \sigma_{B}^{n} \right) \right],$$

(86)

where $\sigma_{B}^{n} := \frac{1}{|S_{n}|} \sum_{\pi \in S_{n}} U_{B^{n}}(\pi) \sigma_{B^{n}} U_{B^{n}}(\pi)^{\dagger}$ is permutation invariant. Lemma 1 then yields

$$\beta(Q_{A^{n}B^{n}}^{n}; \tau_{A}^{\otimes n}) \leq g_{n,d} \text{tr} \left[ Q_{A^{n}B^{n}}^{n} \left( \tau_{A}^{\otimes n} \otimes \omega_{B}^{n} \right) \right],$$

(87)

where we set $d = |B|$. Furthermore, using Audenaert et al.’s inequality (14) we find

$$\beta(Q_{A^{n}B^{n}}^{n}; \tau_{A}^{\otimes n}) \leq g_{n,d} \exp(-s\lambda_{n}) \text{tr} \left[ \left( \rho_{AB}^{\otimes n} \right)^{s} (\tau_{A}^{\otimes n} \otimes \omega_{B}^{n})^{1-s} \right]$$

(88)

$$= g_{n,d} \exp(-s\lambda_{n}) \exp \left( -sD_{s}(\rho_{AB}^{\otimes n} \| \tau_{A}^{\otimes n} \otimes \omega_{B}^{n}) \right)$$

(89)

$$\leq g_{n,d} \exp(-s\lambda_{n}) \exp \left( -sI_{s}(\rho_{AB}^{\otimes n} \| \tau_{A}^{\otimes n}) \right)$$

(90)

Observing that $I_{s}(\rho_{AB}^{\otimes n} \| \tau_{A}^{\otimes n}) = n I_{s}(\rho_{AB} \| \tau_{A})$ due to the additivity of the mutual information established in Lemma 5, we find that the choice

$$\lambda_{n} = \frac{1}{s} \left( \log g_{n,d} + n \left( R - (1-s)I_{s}(\rho_{AB} \| \tau_{A}) \right) \right)$$

(91)

achieves the desired bound $\beta(Q_{A^{n}B^{n}}^{n}; \tau_{A}^{\otimes n}) \leq \exp(-nR)$.

On the other hand, again using (14) and Lemma 5, we find

$$\alpha(Q_{A^{n}B^{n}}^{n}; \rho_{AB}^{\otimes n}) = \text{tr} \left[ \{ \rho_{AB}^{\otimes n} < \exp(\lambda_{n}) \tau_{A}^{\otimes n} \otimes \omega_{B}^{n} \} \rho_{AB}^{\otimes n} \right]$$

(92)

$$\leq \exp \left( (1-s)\lambda_{n} \right) \text{tr} \left[ \left( \rho_{AB}^{\otimes n} \right)^{s} (\tau_{A}^{\otimes n} \otimes \omega_{B}^{n})^{1-s} \right]$$

(93)

$$\leq \exp \left( (1-s)\lambda_{n} - n(1-s)I_{s}(\rho_{AB} \| \tau_{A}) \right)$$

(94)
Substituting (91) for $\lambda_n$, we thus find
\[
\hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes_A \| \tau_A^\otimes\right) \leq \alpha(Q_{n,B^n} \| \rho_{AB}^\otimes_A) \leq \exp\left(\frac{1-s}{s} \left(\log g_{n,d} + nR - nI_s(\rho_{AB} \| \tau_A)\right)\right).
\] (95)

Since $g_{n,d} = O(\log n)$, taking the limit $n \to \infty$ yields
\[
\liminf_{n \to \infty} \left\{-\frac{1}{n} \log \hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes \| \tau_A^\otimes\right)\right\} \geq \frac{1-s}{s} (I_s(\rho_{AB}, \tau_A) - R). \quad (96)
\]

Finally, since this holds for all $s \in (0, 1)$, we established the direct part. \qed

### B. Proof of Optimality

To show optimality, we will directly employ the converse of the quantum Hoeffding bound established in [32] together with a minimax theorem derived in Appendix A.

Recall that it remains to show that for any $R > 0$, we have
\[
\limsup_{n \to \infty} \left\{-\frac{1}{n} \log \hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes \| \tau_A^\otimes\right)\right\} \leq \sup_{s \in (0,1)} \left\{\frac{1-s}{s} (I_s(\rho_{AB} \| \tau_A) - R)\right\}. \quad (97)
\]

**Proof of Optimality in Theorem 11.** We fix $\sigma_B \in \mathcal{S}(B)$ and note that
\[
\hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes_A \| \tau_A^\otimes\right) \geq \hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes \| (\tau_A \otimes \sigma_B)^\otimes\right)
\] (98)

At this point we can apply the converse of the quantum Hoeffding bound [32] to the expression on the right-hand side, which yields
\[
\limsup_{n \to \infty} \left\{-\frac{1}{n} \log \hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes \| \tau_A^\otimes\right)\right\} \leq \limsup_{n \to \infty} \left\{-\frac{1}{n} \log \hat{\alpha}\left(\exp(-nR); \rho_{AB}^\otimes \| (\tau_A \otimes \sigma_B)^\otimes\right)\right\} \leq \sup_{s \in (0,1)} \left\{\frac{1-s}{s} (D_s(\rho_{AB} \| \tau_A \otimes \sigma_B) - R)\right\}. \quad (100)
\]

Since this holds for all $\sigma_B \in \mathcal{S}(B)$, the limit in (99) is in fact upper bounded by
\[
\inf_{\sigma_B \in \mathcal{S}(B)} \sup_{s \in (0,1)} \left\{\frac{1-s}{s} (D_s(\rho_{AB} \| \tau_A \otimes \sigma_B) - R)\right\}. \quad (102)
\]

It remains to observe that the functional
\[
(s, \sigma_B) \mapsto (1-s)D_s(\rho_{AB} \| \tau_A \otimes \sigma_B) = -\log \text{tr} \left[\rho_{AB}^s(\tau_A \otimes \sigma_B)^{1-s}\right]
\] (103)
is convex in $\sigma_B$ for $s \in (0, 1)$ due to Lieb’s theorem [26] and concave in $s$ as was shown in [2, Lem. 2.1]. Hence, the minimax theorem (Proposition 18) in Appendix A applies to (102). This, together with the definition of $I_{\alpha}$ in (38), concludes the proof. \qed
VI. STRONG CONVERSE EXPONENT

Our second result considers the case where the error of the second kind goes to zero exponentially with a rate exceeding the mutual information $I(\rho_{AB}\|\tau_A)$. In this case, we find that the error of the first kind converges to 1 exponentially fast, and the exponent is determined by the sandwiched Rényi mutual information, $\bar{I}_s(\rho_{AB}\|\tau_A)$, with $s > 1$.

**Theorem 13.** Let $\rho_{AB} \in S(AB)$ and $\tau_A \in S(A)$. Then, for any $R > 0$, we have

$$\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^\otimes_n \| \tau_A^\otimes_n \right) \right) \right\} = \sup_{s > 1} \left\{ \frac{s-1}{s} \left( R - \bar{I}_s(\rho_{AB}\|\tau_A) \right) \right\}.$$  \hspace{1cm} (104)

Again, we are interested in the following two special cases:

**Corollary 14.** Let $\rho_{AB} \in S(AB)$. Then, for any $R > 0$, we have

$$\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^\otimes_n \| \rho_A^\otimes_n \right) \right) \right\} = \sup_{s > 1} \left\{ \frac{s-1}{s} \left( R - \bar{I}_s(A:B)_\rho \right) \right\}, \hspace{1cm} (105)$$

$$\lim_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^\otimes_n \| \pi_{AB}^\otimes_n \right) \right) \right\} = \sup_{s > 1} \left\{ \frac{s-1}{s} \left( R - \log |A| + \bar{H}_s(A|B)_\rho \right) \right\}. \hspace{1cm} (106)$$

This corollary establishes an operational interpretation of the sandwiched Rényi mutual information, $\bar{I}_s(A:B)_\rho$, as well as the sandwiched Rényi conditional entropies, $\bar{H}_s(A|B)_\rho$, for $\alpha \geq 1$.

Before we commence with the proof, we will discuss the classical Neyman-Pearson test we use and some results from classical large deviation theory. Following this, we treat the proof of the achievability and optimality in Theorem 13 separately.

A. Classical Neyman-Pearson Test

To show the direct part we employ a classical Neyman-Pearson test for the pinched state $\mathcal{P}_{\tau_A^\otimes_n \otimes \omega_{B^n}^n}(\rho_{AB}^\otimes_n)$ and the state $\tau_A^\otimes_n \otimes \omega_{B^n}^n$. The idea to use a classical Neyman-Pearson test on the pinched state goes back to [16]. We start by discussing some properties of this test.

**Lemma 15.** Let $\rho_{AB} \in S(AB)$, $\tau_A \in S(A)$, $n \in \mathbb{N}$ and $\mu_n \in \mathbb{R}$. Consider the test

$$Q_{AB^n}^n(n):= \left\{ \mathcal{P}_{\tau_A^\otimes_n \otimes \omega_{B^n}^n}(\rho_{AB}^\otimes_n) \geq \exp(\mu_n) \tau_A^\otimes_n \otimes \omega_{B^n}^n \right\}. \hspace{1cm} (107)$$

Let $\{\phi_{x_n}\}_{x_n}$ be a common orthonormal eigenbasis of $\mathcal{P}_{\tau_A^\otimes_n \otimes \omega_{B^n}^n}(\rho_{AB}^\otimes_n)$ and $\tau_A^\otimes_n \otimes \omega_{B^n}^n$, and define the probability distributions

$$P_n(x_n) = \langle \phi_{x_n} | \mathcal{P}_{\tau_A^\otimes_n \otimes \omega_{B^n}^n}(\rho_{AB}^\otimes_n) | \phi_{x_n} \rangle, \text{ and } Q_n(x_n) = \langle \phi_{x_n} | \tau_A^\otimes_n \otimes \omega_{B^n}^n | \phi_{x_n} \rangle. \hspace{1cm} (108)$$

Then, with $X_n$ distributed according to the law $P_n$ and $X'_n$ according to the law $Q_n$, we have

$$\alpha(Q_{AB^n}^n;\rho_{AB}^\otimes_n) = \Pr\left[ P_n(X_n) < \exp(\mu_n)Q_n(X_n) \right], \hspace{1cm} (109)$$

$$\beta(Q_{AB^n}^n;\tau_A^\otimes_n) \leq g_{n,d} \Pr\left[ P_n(X'_n) \geq \exp(\mu_n)Q_n(X'_n) \right]. \hspace{1cm} (110)$$

where $d = |B|$.
Proof. It is easy to verify that the pinched quantity is permutation invariant, and thus $Q_{A^nB^n}^n$ is permutation invariant as well. Let us evaluate

$$\beta(Q_{A^nB^n}^n; \tau_A^{\otimes n}) = \max_{\sigma_{B^n} \in S(B^n)} \operatorname{tr} [Q_{A^nB^n}^n (\tau_A^{\otimes n} \otimes \sigma_{B^n})]$$

(111)

$$= \max_{\sigma_{B^n} \in S(B^n)} \frac{1}{|S_n|} \sum_{\pi \in S_n} \operatorname{tr} [U_{A^nB^n}(\pi) Q_{A^nB^n}^n (\tau_A^{\otimes n} \otimes \sigma_{B^n})^U_{A^nB^n}(\pi)^\dagger]$$

(112)

$$= \max_{\sigma_{B^n} \in S(B^n)} \operatorname{tr} [Q_{A^nB^n}^n (\tau_A^{\otimes n} \otimes \tilde{\sigma}_{B^n})],$$

(113)

where $\tilde{\sigma}_{B^n} := \frac{1}{|S_n|} \sum_{\pi \in S_n} \sigma_{B^n} U_{B^n}(\pi) \sigma_{B^n} U_{B^n}(\pi)^\dagger$ is permutation invariant. Lemma 1 then yields

$$\beta(Q_{A^nB^n}^n; \tau_A^{\otimes n}) \leq g_{n,d} \operatorname{tr} [Q_{A^nB^n}^n (\tau_A^{\otimes n} \otimes \omega_{B^n}^n)]$$

(114)

$$= g_{n,d} \operatorname{tr} \left[ \left\{ \mathcal{P}_{\tau_A^{\otimes n} \otimes \omega_{B^n}^n} (\rho_{AB}^n) \geq \exp(\mu_n) \tau_A^{\otimes n} \otimes \omega_{B^n}^n \right\} (\tau_A^{\otimes n} \otimes \omega_{B^n}^n) \right].$$

(115)

The two operators $\mathcal{P}_{\tau_A^{\otimes n} \otimes \omega_{B^n}^n} (\rho_{AB}^n)$ and $\tau_A^{\otimes n} \otimes \omega_{B^n}^n$ in (115) commute. Let $\{\{\phi_x\}_x\}_{x}$ be a common orthonormal eigenbasis for these operators and define the probability distributions in (108) as well as the corresponding random variables. Then we can simplify (115) by noting that

$$\operatorname{tr} \left[ \left\{ \mathcal{P}_{\tau_A^{\otimes n} \otimes \omega_{B^n}^n} (\rho_{AB}^n) \geq \exp(\mu_n) \tau_A^{\otimes n} \otimes \omega_{B^n}^n \right\} (\tau_A^{\otimes n} \otimes \omega_{B^n}^n) \right] = \Pr [P_n(X') \geq \exp(\mu_n)Q_n(X')],$$

(116)

which yields (110). Finally, it is easy to verify that

$$\alpha(Q_{A^nB^n}^n; \rho_{AB}^n) = \operatorname{tr} \left[ \left\{ \mathcal{P}_{\tau_A^{\otimes n} \otimes \omega_{B^n}^n} (\rho_{AB}^n) < \exp(\mu_n) \tau_A^{\otimes n} \otimes \omega_{B^n}^n \right\} \mathcal{P}_{\tau_A^{\otimes n} \otimes \omega_{B^n}^n} (\rho_{AB}^n) \right]$$

(117)

$$= \Pr [P_n(X_n) < \exp(\mu_n)Q_n(X_n)].$$

(118)

\[\square\]

B. Classical Large Deviation Theory

Our proof will rely on a variant of the Gärtner-Ellis theorem of large deviation theory (see, e.g., [12, Sec. 2 and Sec. 3.4] for an overview), which we recall here. Given a sequence of random variables $\{Z_n\}_{n \in \mathbb{N}}$ we introduce its asymptotic cumulant generating function as

$$\Lambda_Z(t) := \lim_{n \to \infty} \left\{ \frac{1}{n} \log \left( \mathbb{E} [\exp(ntZ_n)] \right) \right\},$$

(119)

if it exists. For our purposes it is sufficient to use the following variant of the Gärtner-Ellis theorem due to Chen [5, Thm. 3.6] (see also [30, Lem. B.2]).

Proposition 16. Let us assume that $t \mapsto \Lambda_Z(t)$ as defined in (119) exists and is differentiable in some interval $(a,b)$. Then, for any $z \in (\lim_{t \to a^-} \Lambda_Z(t), \lim_{t \to b^+} \Lambda_Z(t))$, we have

$$\limsup_{n \to \infty} \left\{ \frac{1}{n} \log \Pr[Z_n \geq z] \right\} \leq \sup_{t \in (a,b)} \{zt - \Lambda_Z(t)\}.$$  

(120)

Finally, in order to evaluate the asymptotic cumulant generating function, we will employ the asymptotic achievability in Proposition 7, namely the fact that

$$\lim_{n \to \infty} \frac{1}{n} D(\Pr_n \| Q_n) = \tilde{I}_A(\rho_{AB}||\tau_A).$$

(121)
C. Proof of Achievability

We are now ready to present the proof of achievability, namely we show that
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right) \right\} \leq \sup_{s > 1} \left\{ \frac{s - 1}{s} \left( R - \tilde{I}_s(\rho_{AB} \| \tau_A) \right) \right\}. \tag{122}
\]

We restrict our attention to the case where \( I(\rho_{AB} \| \tau_A) < R < \tilde{I}_\infty(\rho_{AB} \| \tau_A) \), for which we provide a novel proof. For general \( R > 0 \) we refer the reader to a recent analysis of the strong converse exponent by Mosonyi and Ogawa [30] that can be adapted to cover the situation at hand here. (See also [33] for an earlier discussion of this issue in classical hypothesis testing.)

**Proof of Achievability in Theorem 13.** First, we introduce the function
\[
f : (s,t) \mapsto t(R - s\tilde{I}_{1+t}(\rho_{AB} \| \tau_A)) + (s - 1)\tilde{I}_s(\rho_{AB} \| \tau_A). \tag{123}
\]
As is shown later, we have
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right) \right\} \leq \sup_{t > 0} \frac{f(s,t)}{s} \tag{124}
\]
for all \( s \in (1, \infty) \). Taking the infimum for \( s \), we find that
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right) \right\} \leq \inf_{s > 1} \sup_{t > 0} \frac{f(s,t)}{s}, \tag{125}
\]
It is straightforward to verify that \( f(s,t) \) is concave in \( t \) and convex in \( s \) since \( t \mapsto t\tilde{I}_{1+t}(\rho_{AB} \| \tau_A) \) is convex (cf. Corollary 9). Thus, by Proposition 18 in Appendix A, we have
\[
\inf_{s > 1} \sup_{t > 0} \frac{f(s,t)}{s} = \sup_{t > 0} \inf_{s > 1} \frac{f(s,t)}{s} \leq \sup_{t > 0} \frac{f(t + 1, t)}{t + 1}, \tag{126}
\]
where we simply chose \( s = t + 1 \) in the last step. The latter term corresponds to the right hand side of (122) by a suitable change of variable. Then, we obtain (122). Hence, it suffices to show (124) for all \( s \in (1, \infty) \).

Given an arbitrary fixed \( s \in (1, \infty) \), we choose a sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) of real numbers as
\[
\mu_n = \frac{1}{s} \left( \log g_{n,d} + nR + (s - 1)D_s(\rho_{AB} \| Q_n) \right). \tag{127}
\]
Consider the sequence of tests given by Lemma 15. Then, due to (110), we have
\[
\beta(Q^n_{A^nB^n}; \tau_A^{\otimes n}) \leq g_{n,d} \Pr \left[ P_n(X') \geq \exp(\mu_n)Q_n(X') \right] \tag{128}
\]
\[
\leq g_{n,d} \exp(-s\mu_n) \sum P_n(x)^s Q_n(x)^{1-s} \tag{129}
\]
\[
= g_{n,d} \exp(-s\mu_n) \exp((-s - 1)D_s(\rho_{AB} \| Q_n)). \tag{130}
\]
Hence, the requirement that \( \beta(Q^n_{A^nB^n}; \tau_A^{\otimes n}) \leq \exp(-nR) \) can then be satisfied by the choice (127). Let us now take a closer look at error of the first kind in (109). We find
\[
1 - \alpha(Q^n_{A^nB^n}; \rho_{AB}^{\otimes n}) = \Pr \left[ P_n(X_n) \geq \exp(\mu_n)Q_n(X_n) \right] = \Pr \left[ Z_n \geq R \right], \tag{131}
\]
where we introduced the sequence of random variables \( \{Z_n\}_{n \in \mathbb{N}} \) with
\[
Z_n(X_n) := \frac{1}{n} \left( \log P_n(X_n) - \log Q_n(X_n) - \mu_n \right) + R
\]
\[
= \frac{1}{n} \left( \log P_n(X_n) - \log Q_n(X_n) - \log g_{n,d} \right) + \frac{s-1}{s} \left( R - \frac{1}{n} D_s(P_n || Q_n) \right).
\]

Since \( \beta(Q^\oplus_{A^*B^*}; \tau^\otimes_{A}) \leq \exp(-nR) \) holds for our test, (131) yields
\[
1 - \hat{\alpha} \left( \exp(-nR); \rho^\ominus_{AB || \tau^\otimes_{A}} \right) \geq \Pr \{ Z_n \geq R \},
\]
which implies that
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho^\ominus_{AB || \tau^\otimes_{A}} \right) \right) \right\} \leq \limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( \Pr \{ Z_n \geq R \} \right) \right\}.
\]

Now, we want to tackle the asymptotics of this quantity using the Gärtner-Ellis theorem. We therefore calculate the asymptotic cumulant generating function, as in (119), for \( t \geq -\frac{1}{2} \) as follows:
\[
\Lambda_Z(t) = \lim_{n \to \infty} \left\{ -\frac{1}{n} \log \left( \mathbb{E} \left[ \exp(ntZ_n) \right] \right) \right\}
\]
\[
= \lim_{n \to \infty} \left\{ -\frac{1}{n} \log \mathbb{E} \left[ \frac{P_n(x)^t}{Q_n(x)^t} \right] - \frac{t \log g_{n,d}}{sn} + \frac{t(s-1)}{s} \left( R - \frac{1}{n} D_s(P_n || Q_n) \right) \right\}
\]
\[
= \lim_{n \to \infty} \left\{ -\frac{1}{n} D_{1+t}(P_n || Q_n) + \frac{s-1}{s} \left( R - \frac{1}{n} D_s(P_n || Q_n) \right) \right\}
\]
\[
= t \left( \tilde{I}_{1+t}(\rho_{AB} || \tau_A) + \frac{s-1}{s} \left( R - \tilde{I}_s(\rho_{AB} || \tau_A) \right) \right).
\]

We used Proposition 7 in the form of (121) twice to establish (139). Now, it is easy to verify that \( \Lambda_Z(t) \) is continuously differentiable for \( t \geq -\frac{1}{2} \) due to Proposition 10. Furthermore, we have
\[
\lim_{t \to 0} \Lambda'_Z(t) = I(\rho_{AB} || \tau_A) + \frac{s-1}{s} \left( R - \tilde{I}_s(\rho_{AB} || \tau_A) \right)
\]
\[
\leq I(\rho_{AB} || \tau_A) + \frac{s-1}{s} \left( R - I(\rho_{AB} || \tau_A) \right) < R,
\]
where we used that \( R > I(\rho_{AB} || \tau_A) \) and \( \frac{s-1}{s} < 1 \) in the last step. On the other hand, using the convexity of \( t \mapsto t\tilde{I}_{1+t}(\rho_{AB} || \tau_A) \) (cf. Corollary 9), we find\(^7\)
\[
\lim_{t \to \infty} \Lambda'_Z(t) \geq \tilde{I}_{\infty}(\rho_{AB} || \tau_A) + \frac{s-1}{s} \left( R - \tilde{I}_s(\rho_{AB} || \tau_A) \right)
\]
\[
\geq \tilde{I}_{\infty}(\rho_{AB} || \tau_A) + \frac{s-1}{s} \left( R - \tilde{I}_{\infty}(\rho_{AB} || \tau_A) \right) > R,
\]
where we used that \( R < \tilde{I}_{\infty}(\rho_{AB} || \tau_A) \). Hence, we may apply Proposition 16, which yields
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \left( \Pr \{ Z_n \geq R \} \right) \right\} \leq \sup_{t \geq 0} \{ tR - \Lambda_Z(t) \} = \sup_{t \geq 0} \frac{f(s,t)}{s}.
\]
Therefore, the combination of (144) and (135) yields (124), which concludes the proof. \( \square \)

\(^7\) More precisely, recall that \( \phi(t) := t\tilde{I}_{1+t}(\rho_{AB} || \tau_A) \) is convex and \( \phi(0) = 0 \). This implies that \( \phi(\lambda t) \leq \lambda \phi(t) \) for all \( \lambda \in (0, 1) \). Moreover, \( \phi'(t) = \lim_{\lambda \to 1} \frac{\phi(t) - \phi(\lambda t)}{t(1-\lambda)} \geq \frac{\phi(t)}{t} \).
D. Proof of Optimality

It remains to show that, for all $R > 0$,
\[
\liminf_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right) \right\} \geq \sup_{s > 1} \left\{ \frac{s-1}{s} \left( R - \bar{I}_s(\rho_{AB} \| \tau_A) \right) \right\}.
\]
(145)

Proof of Optimality in Theorem 13. Analogous to the optimality proof for Theorem 11, we first fix $\sigma_B \in S(B)$ and this time apply the converse bound in [29, Thm. IV.9]. This yields
\[
\liminf_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right) \right\} \geq \liminf_{n \to \infty} \left\{ -\frac{1}{n} \log \left( 1 - \hat{\alpha} \left( \exp(-nR); \rho_{AB}^{\otimes n} \| (\tau_A \otimes \sigma_B)^{\otimes n} \right) \right) \right\} \geq \sup_{s > 1} \left\{ \frac{s-1}{s} \left( R - \bar{D}_s(\rho_{AB} \| \tau_A \otimes \sigma_B) \right) \right\}.
\]
(146)

Since this holds for all $\sigma_B \in S(B)$, we may maximize the expression in (148) with regards to $\sigma_B$, yielding the desired result. \hfill \Box

VII. SECOND ORDER

For completeness, we also investigate the second order behavior, namely we investigate the error of the first kind when the error of the second kind vanishes as $\exp\left( -nI(\rho_{AB} \| \tau_A) - \sqrt{n} r \right)$. This analysis takes a step beyond the quantum Stein’s lemma [22, 34]. Paralleling the results in [25, 43] for simple hypothesis tests, we find that the error of the first kind converges to a constant in this case.

Theorem 17. Let $\rho_{AB}$ be a bipartite quantum state on $AB$ and let $\tau_A$ a quantum state on $A$. Then, for any $r \in \mathbb{R}$, we have
\[
\lim_{n \to \infty} \left\{ \hat{\alpha} \left( \exp\left( -nI(\rho_{AB} \| \tau_A) - \sqrt{n} r \right); \rho_{AB}^{\otimes n} \| \tau_A^{\otimes n} \right) \right\} = \Phi \left( \frac{r}{\sqrt{V(\rho_{AB} \| \tau_A)}} \right),
\]
(149)

where $\Phi$ is the cumulative standard normal (Gaussian) distribution.

The proof of the optimality follows directly from the bound in Eq. (76) and the second order expansion for binary quantum hypothesis testing independently established in [25] and [43], and we omit it here. Second-order achievability is proven using the hypothesis test of Section VI together with Corollary 8 in the following.

A. Proof of Achievability

Proof of Achievability in Theorem 17. We again use the test in Lemma 15 and set $\mu_n = nI(\rho_{AB} \| \tau_A) + \sqrt{n} r + \log g_{n,d}$. Then, Eq. (110) yields
\[
\beta(Q^n_{A^nB^n}; \tau_A^{\otimes n}) \leq g_{n,dB} \Pr \left[ P_n(X'_n) \geq \exp(\mu_n)Q_n(X'_n) \right]
\leq g_{n,dB} \exp(-\mu_n) \Pr \left[ P_n(X_n) \geq \exp(\mu_n)Q_n(X_n) \right]
\leq \exp \left( -nI(\rho_{AB} \| \tau_A) - \sqrt{n} r \right).
\]
(150)
Moreover, using (109), we find
\[ \alpha(Q^n_{A^n B^n}; \rho_{AB}^{\otimes n}) = \Pr \left[ \log P_n - \log Q_n < nI(\rho_{AB} \parallel \tau_A) + \sqrt{n} r + \log g_{n,d} \right] \]
(153)
\[ = \Pr \left[ Y_n < r \right]. \]
(154)
where we defined the following sequence of random variables:
\[ Y_n(X_n) := \frac{1}{\sqrt{n}} (\log P_n(X_n) - \log Q_n(X_n) - nI(\rho_{AB} \parallel \tau_A) - \log g_{n,d}). \]
(155)
with \( X_n \) distributed according to the law \( P_n \) as usual.

Now, note that the cumulant generating function of the sequence \( \{Y_n\}_n \) converges to
\[ \Lambda_Y(t) := \lim_{n \to \infty} \left\{ \log \mathbb{E} [\exp(tY_n)] \right\} \]
(156)
\[ = \lim_{n \to \infty} \left\{ \frac{t}{\sqrt{n}} (D_{1+\frac{1}{\sqrt{n}}} (P_n || Q_n) - nI(\rho_{AB} || \tau_A)) \right\} - \lim_{n \to \infty} \left\{ \frac{t}{\sqrt{n}} \log g_{n,d} \right\} \]
(157)
\[ = \frac{t^2}{2} V(\rho_{AB} \parallel \tau_A). \]
(158)
In the last step we used Corollary 8 to evaluate the first term and the fact that \( \log g_{n,d} = O(\log n) \) to evaluate the second term. Hence, by Lévi’s continuity theorem (see, e.g., [14, Ch. 14, Thm. 21]), the sequence of random variable \( \{Y_n\}_n \) converges in distribution to a random variable \( Y \) with cumulant generating function \( \Lambda_Y(t) \), i.e., a Gaussian random variable with zero mean and variance \( V(\rho_{AB} \parallel \tau_A) \). In particular, this yields
\[ \lim_{n \to \infty} \Pr \left[ Y_n < r \right] = \Pr \left[ Y < r \right] = \Phi \left( \frac{r}{\sqrt{V(\rho_{AB} \parallel \tau_A)}} \right). \]
(159)
Finally, due to (152) we have
\[ \limsup_{n \to \infty} \left\{ \hat{\alpha} \left( \exp \left( - nI(\rho_{AB} \parallel \tau_A) - \sqrt{n} r \right); \rho_{AB}^{\otimes n} \right) \right\} \leq \lim_{n \to \infty} \alpha(Q^n_{A^n B^n}; \rho_{AB}^{\otimes n}). \]
(160)
Combining this with (154) and (159) concludes the proof.

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**Appendix A: A Minimax Theorem**

Here we show a useful minimax theorem, that is essentially a corollary of König’s minimax theorem [24]. First, we need to introduce a weaker notion of concavity and convexity. A function \( f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is 1/2 concavelike in \( \mathcal{X} \) if, for every \( x_1, x_2 \in \mathcal{X} \), there exists \( x_3 \in \mathcal{X} \) such that
\[ f(x_3, y) \geq \frac{1}{2} (f(x_1, y) + f(x_2, y)) \quad \text{for every} \quad y \in \mathcal{Y}. \]
(A1)
Analogously, $f$ is 1/2 convexlike in $\mathcal{Y}$ if, for every $y_1, y_2 \in \mathcal{Y}$, there exists $y_3 \in \mathcal{Y}$ such that
\[
f(x,y_3) \leq \frac{1}{2}(f(x,y_1) + f(x,y_2)) \quad \text{for every } x \in \mathcal{X}.
\] (A2)

König's minimax theorem now reads as follows [24] (see also [23]). Let $\mathcal{Y}$ be a compact Hausdorff space and let $f(x, \cdot)$ be lower-semicontinuous for every $x \in \mathcal{X}$. Moreover, let $f$ be 1/2 concavelike in $\mathcal{X}$ and 1/2 convexlike in $\mathcal{Y}$. Then, we have
\[
\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(x,y) = \inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(x,y).
\] (A3)

We sacrifice a bit of generality to state the following result.

**Proposition 18.** Let $\mathcal{X} \subseteq \mathbb{R}^+$ be convex and let $\mathcal{Y}$ be a convex compact Hilbert space. Further, let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function that is convex in $\mathcal{Y}$ and concave in $\mathcal{X}$. Then,
\[
\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \frac{f(x,y)}{x} = \inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} \frac{f(x,y)}{x}.
\] (A4)

**Proof.** We just need to show that $\frac{f(x,y)}{x}$ satisfies the conditions required for (A3) to hold. First, since $f$ is a convex function on the convex set $\mathcal{Y}$, it is in particular 1/2 concavelike and lower-semicontinuous in $\mathcal{Y}$. It is also 1/2 convexlike in $\mathcal{X}$ due to the following argument. Let $x_1, x_2 \in \mathcal{X}$ with $x_1 < x_2$ and $y \in \mathcal{Y}$ be arbitrary. We have
\[
\frac{1}{2} \left( \frac{f(x_1,y)}{x_1} + \frac{f(x_2,y)}{x_2} \right) = \frac{x_1 + x_2}{2x_1x_2} \left( \frac{x_2}{x_1 + x_2} f(x_1,y) + \frac{x_1}{x_1 + x_2} f(x_2,y) \right)
\] (A5)
\[
\leq \frac{x_1 + x_2}{2x_1x_2} f \left( \frac{2x_1x_2}{x_1 + x_2}, y \right)
\] (A6)
by the concavity of $f(\cdot,y)$. Thus, choosing $x_3 = \frac{2x_1x_2}{x_1 + x_2} \in [x_1, x_2] \subseteq \mathcal{X}$, we see that $\frac{f(x_3,y)}{x_3}$ is indeed 1/2 concavelike according to (A1).

---

**Appendix B: Proof of Lemma 4**

**Proof.** By symmetry it is sufficient to proof the statement for $\alpha \in \left[ \frac{1}{2}, 1 \right]$ so that $\beta > 0$. First, recall that
\[
\exp \left( -\gamma \tilde{I}_a(\rho_{AB}\|\tau_\mathcal{A}) \right) = \sup_{\sigma_B \in \mathcal{S}(B)} \left\| (\tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma) \times \rho_{AB} \right\|_\alpha
\] (B1)
where we set $\gamma := \frac{1}{1-a} \in (0,1]$ and use the notation $L \times R$ to denote the Hermitian operator $L^{\frac{1}{2}} RL^{\frac{1}{2}}$. By introducing the purification $\rho_{ABC}$, we see that
\[
\sup_{\sigma_B \in \mathcal{S}(B)} \left\| (\tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma) \times \rho_{AB} \right\|_\alpha = \sup_{\sigma_B \in \mathcal{S}(B)} \left\| \text{tr}_C \left[ (\tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma \otimes I_\mathcal{C}) \times \rho_{ABC} \right] \right\|_\alpha
\] (B2)
\[
= \sup_{\sigma_B \in \mathcal{S}(B)} \left\| \text{tr}_{AB} \left[ (\tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma \otimes I_\mathcal{C}) \times \rho_{ABC} \right] \right\|_\alpha
\] (B3)
\[
= \sup_{\sigma_B \in \mathcal{S}(B)} \inf_{\sigma_C \in \mathcal{S}(C)} \inf_{\sigma_C > 0} \left\| \text{tr} \left[ (\tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma \otimes \sigma_\mathcal{C}^\gamma) \times \rho_{ABC} \right] \right\|_\alpha
\] (B4)
\[
= \sup_{\sigma_B \in \mathcal{S}(B)} \inf_{\sigma_C \in \mathcal{S}(C)} \inf_{\sigma_C > 0} \left\| \left( \tau_\mathcal{A}^\gamma \otimes \sigma_\mathcal{B}^\gamma \otimes \sigma_\mathcal{C}^\gamma \right) \rho_{ABC} \right\|_\alpha.
\] (B5)
where we employed [31, Lm. 12] to establish (B4). Now, it is easy to verify that $\mathcal{S}(B)$ is convex compact, the set of strictly positive elements of $\mathcal{S}(C)$ is convex and the function $\text{tr}[\tau_A^\gamma \otimes \sigma_B^\gamma \otimes \sigma_C^{-\gamma}]\rho_{ABC}$ is concave in $\sigma_B$ and convex in $\sigma_C$. Thus, Sion’s minimax theorem [40] applies and yields the following alternative expression:

\[
\exp\left(-\gamma \tilde{I}_\alpha(\rho_{AB} \| \tau_A)\right) = \inf_{\sigma_C \in \mathcal{S}(C)} \sup_{\sigma_B \in \mathcal{S}(B)} \text{tr} \left[ (\tau_A^\gamma \otimes \sigma_B^\gamma \otimes \sigma_C^{-\gamma}) \rho_{ABC} \right] 
\]

(B6)

\[
= \inf_{\sigma_C > 0} \sup_{\sigma_B \in \mathcal{S}(B)} \text{tr} \left[ \sigma_B^\gamma \text{tr}_{AC} \left( (\tau_A^\gamma \otimes \sigma_C^{-\gamma}) \times \rho_{ABC} \right)\right] 
\]

(B7)

\[
= \inf_{\sigma_C > 0} \left\| \text{tr}_{AC} \left( (\tau_A^\gamma \otimes \sigma_C^{-\gamma}) \times \rho_{ABC} \right) \right\|_\beta 
\]

(B8)

\[
= \inf_{\sigma_C > 0} \left\| (\tau_A^\gamma \otimes \sigma_C^{-\gamma}) \times \rho_{AC} \right\|_\beta. 
\]

(B9)

We again used [31, Lm. 12] to establish (B8) and note that $\beta = \frac{\alpha}{2\alpha - 1} = \frac{1}{1-\gamma}$. Substituting for $\gamma = -\frac{\beta}{\alpha}$ in (B9) establishes the desired equality.  

\[\square\]

**Appendix C: Differentiability of the Rényi Mutual Information**

1. **Directional Derivatives**

We define the following directional derivatives. For two density operators $\sigma, \omega \in \mathcal{S}$, define

\[
\partial_\omega f(\sigma) = \lim_{s \searrow 0} \left\{ \frac{\partial}{\partial s} f((1-s)\sigma + s\omega) \right\} \quad \text{and} \quad \partial_\omega^2 f(\sigma) = \lim_{s \searrow 0} \left\{ \frac{\partial^2}{\partial s^2} f((1-s)\sigma + s\omega) \right\}. 
\]

(C1)

First, we establish the following property.

**Lemma 19.** For $\gamma \in (0, 1)$, we have

\[
\partial_\omega f(\sigma)^\gamma = \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma-1} \partial_\omega f(\sigma)(t1 + f(\sigma))^{-1} 
\]

(C2)

\[
\partial_\omega^2 f(\sigma)^\gamma = \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma-1} \left( (t1 + f(\sigma))^{-1} \partial_\omega^2 f(\sigma)(t1 + f(\sigma))^{-1} \right. 
\]

\[- 2(t1 + f(\sigma))^{-1} \partial_\omega f(\sigma)(t1 + f(\sigma))^{-1} \partial_\omega f(\sigma)(t1 + f(\sigma))^{-1} \left. \right) \cdot 
\]

(C3)

Furthermore, the same relations hold if $\gamma \in (-1, 0)$ and $f(\sigma) > 0$.

**Proof.** For $\gamma \in (0, 1)$, let $\sigma(s)$ be an arbitrary function of $s$, and $\dot{\sigma}(s)$ and $\ddot{\sigma}(s)$ its first and second derivative with regards to $s$, respectively. For $\gamma \in (0, 1)$, we have the following integral representation:

\[
\sigma(s)^\gamma = \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma-1} \sigma(s)(t1 + \sigma(s))^{-1}. 
\]

(C5)

We also recall that

\[
\frac{\partial}{\partial s}(t1 + \sigma(s))^{-1} = -(t1 + \sigma(s))^{-1} \dot{\sigma}(s)(t1 + \sigma(s))^{-1}. 
\]

(C6)
This allows us to evaluate
\[
\frac{\partial}{\partial s} \sigma(s)^\gamma = \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma-1} \left( \sigma(s)(t1 + \sigma(s))^{-1} - \sigma(t1 + \sigma(s))^{-1} \sigma(s)(t1 + \sigma)^{-1} \right)
\] (C7)
\[
= \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma} (t1 + \sigma(s))^{-1} \sigma(s)(t1 + \sigma(s))^{-1}
\] (C8)
\[
\frac{\partial^2}{\partial s^2} \sigma(s)^\gamma = \frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma} \left( (t1 + \sigma(s))^{-1} \sigma(s)(t1 + \sigma(s))^{-1}
\right.
\]
\[
- 2(t1 + \sigma(s))^{-1} \sigma(s)(t1 + \sigma(s))^{-1} \sigma(s)(t1 + \sigma(s))^{-1} \right).
\] (C9)

Finally, for \( \gamma \in (-1, 0) \), multiplying (C5) by \( \sigma(s)^{-1} \), we find
\[
\sigma(s)^\gamma = \sigma(s)^{-1} \sigma(s)^{\gamma+1} = \frac{\sin(\pi (\gamma + 1))}{\pi} \int_0^\infty dt \ t^{\gamma} (t1 + \sigma(s))^{-1}
\]
\[
= -\frac{\sin(\pi \gamma)}{\pi} \int_0^\infty dt \ t^{\gamma} (t1 + \sigma(s))^{-1},
\] (C12)
and the result follows by taking the derivative.

\[\square\]

2. Characterization of the Optimal Marginal State

The following lemma characterizes the marginal state that minimizes the sandwiched mutual information.

**Lemma 20.** Let \( \rho_{AB} \in S(AB) \) and \( \tau_A \in S(A) \) and let \( \alpha \in (\frac{1}{2}, 1) \). Consider the functional
\[
\sigma_B \mapsto \chi(\sigma_B) := \text{tr} \left[ \left( (\tau_A \otimes \sigma_B)^{1-\frac{\alpha}{2}} \rho_{AB}(\tau_A \otimes \sigma_B)^{1-\frac{\alpha}{2}} \right)^{\alpha} \right].
\] (C13)
Then, \( \sup_{\sigma_B \in S(B)} \chi(\sigma) \) is uniquely achieved by the fixed-point of the following non-linear map:
\[
\sigma_B \mapsto \chi_{\alpha}^*(\sigma_B) := \frac{1}{\chi(\sigma)} \text{tr}_A \left[ \left( (\tau_A \otimes \sigma_B)^{1-\frac{\alpha}{2}} \rho_{AB}(\tau_A \otimes \sigma_B)^{1-\frac{\alpha}{2}} \right)^{\alpha} \right].
\] (C14)

We focus here on the case where \( \alpha \in (\frac{1}{2}, 1) \) since this is sufficient for our purposes. However, the same characterization is also expected to hold for \( \alpha > 1 \). In fact, our proof below shows that any fixed point of the map \( \chi_{\alpha} \) achieves \( \sup_{\sigma_B \in S(B)} \chi(\sigma) \) for all \( \alpha \geq \frac{1}{2} \); however, we do not show uniqueness for \( \alpha > 1 \).

**Proof.** First, we derive a sufficient condition for a state \( \sigma_B \) to a maximum of the above optimization as follows. Let \( \alpha \in (\frac{1}{2}, 1) \cup (1, \infty) \) and set \( \gamma = \frac{1-\alpha}{\alpha} \in (0, 1) \). Moreover, set
\[
X = \gamma / \rho^{1/2}, \quad \text{such that} \quad \chi(\sigma) = \text{tr} \left[ (X^\dagger \sigma^\gamma X)^{\alpha} \right].
\] (C15)
where we omitted the identity symbol and dropped the subscripts to make the presentation more concise in the following. Since \( \chi \) is concave [13], a sufficient condition for \( \sigma \) to be a maximizer is given if \( \partial_\omega f(\sigma) = 0 \) for all \( \omega_B \in S(B) \).

Using the cyclicity of the trace (see also Lemma 19), we find that
\[
\partial_\omega f(\sigma) = \alpha \text{tr} \left[ (X^\dagger \sigma^\gamma X)^{\alpha-1} \cdot \partial_\omega (X^\dagger \sigma^\gamma X) \right]
\] (C16)
\[
= \alpha \text{tr} \left[ \sigma^{\gamma/2} X (X^\dagger \sigma^\gamma X)^{\alpha-1} X^\dagger (X^\dagger \sigma^\gamma X)^{\alpha-2} \cdot \sigma^{-\gamma/2} (\partial_\omega \sigma^\gamma) \sigma^{-\gamma/2} \right]
\] (C17)
\[
= \alpha \text{tr} \left[ \text{tr}_A \left[ \left( \sigma^{\gamma/2} X X^\dagger \sigma^{\gamma/2} \right)^{\alpha} \cdot \sigma^{-\gamma/2} (\partial_\omega \sigma^\gamma) \sigma^{-\gamma/2} \right] \right]
\] (C18)
Now let us check the condition $\partial_\omega f(\bar{\sigma}) = 0$ for an arbitrary fixed-point $\bar{\sigma}$ of the map $X_\alpha$. Such a state always exists by Brouwer’s fixed-point theorem, and by definition satisfies
\[
\chi(\sigma) \cdot \bar{\sigma} = \text{tr}_A \left[[\gamma/2 X X^\dagger \gamma/2]^{\sigma}\right].
\] (C19)
Substituting this into (C18) yields
\[
1 \over \alpha \chi(\sigma) \partial_\omega f(\bar{\sigma}) = \text{tr} \left[\bar{\sigma}^{1-\gamma} \cdot \partial_\omega \bar{\sigma}^{\gamma}\right] = \gamma \text{tr} \left[\partial_\omega \bar{\sigma}\right] = \gamma \text{tr} \left[\omega \cdot \bar{\sigma}\right] = 0.
\] (C20)
Here, we again took advantage of the cyclicity of the trace to evaluate the derivative $\partial_\omega \bar{\sigma}^\gamma$. Hence, we have established that $\bar{\sigma}$ is indeed a maximizer.

In the following, we will show that $\bar{\sigma}$ is unique. Since $\chi$ is concave, in order to establish uniqueness it is sufficient to prove that $\partial_\omega^2 f(\bar{\sigma}) < 0$ for all $\omega \neq \bar{\sigma}$ for one maximizer $\bar{\sigma}$. Using Lemma 19, we find
\[
\partial_\omega^2 f(\sigma) = \frac{\sin(\pi \alpha)}{\pi} \int_0^\infty \, dt \, t^\alpha \left(\text{tr} \left[(t\mathbf{1} + Q(\sigma))^{-1} (\partial_\omega^2 Q(\sigma))(t\mathbf{1} + Q(\sigma))^{-1}\right] - 2 \text{tr} \left[(t\mathbf{1} + Q(\sigma))^{-1} (\partial_\omega Q(\sigma))(t\mathbf{1} + Q(\sigma))^{-1}(\partial_\omega Q(\sigma))(t\mathbf{1} + Q(\sigma))^{-1}\right]\right),
\] (C21)
where we set $Q(\sigma) = X^\dagger \sigma^\gamma X$. Due to the cyclicity of the trace, we can integrate the first term. Furthermore, we note that the second trace term is never negative, and thus
\[
\partial_\omega^2 f(\sigma) \leq \alpha \text{tr} \left[(X^\dagger \sigma^\gamma X)^{\alpha-1} \cdot \partial_\omega^2 (X^\dagger \sigma^\gamma X)\right] = \alpha \text{tr} \left[\text{tr}_A \left[[\gamma/2 X X^\dagger \gamma/2]^{\sigma}\right] \cdot \sigma^{-\gamma/2} (\partial_\omega^2 \sigma^\gamma) \sigma^{-\gamma/2}\right] = \alpha \text{tr} \left[[\gamma/2 X X^\dagger \gamma/2]^{\sigma}\right] \cdot \sigma^{-\gamma/2} (\partial_\omega^2 \sigma^\gamma) \sigma^{-\gamma/2}
\] (C22)
Specializing this to a fixed-point $\bar{\sigma}$ satisfying (C19) yields
\[
\partial_\omega^2 f(\bar{\sigma}) \leq \alpha \chi(\bar{\sigma}) \text{tr} \left[[\gamma/2 X X^\dagger \gamma/2]^{\bar{\sigma}}\right] \cdot \bar{\sigma}^{-\gamma/2} (\partial_\omega^2 \bar{\sigma}^\gamma) \bar{\sigma}^{-\gamma/2}.
\] (C23)
Furthermore, evaluating the derivative using Lemma 19 and using the fact that $\partial_\omega \sigma = \omega - \bar{\sigma}$ and $\partial_\omega^2 \sigma = 0$, we find that the trace term on the right hand side evaluates to
\[
\text{tr} \left[\bar{\sigma}^{1-\gamma} \cdot \partial_\omega^2 \sigma^\gamma\right] = \frac{-2 \sin(\pi \gamma)}{\pi} \int_0^\infty dt \, t^\gamma \text{tr} \left[\bar{\sigma}^{1-\gamma} (t\mathbf{1} + \sigma)^{-1} (t\mathbf{1} + \sigma)^{-1} (\omega - \bar{\sigma})(t\mathbf{1} + \sigma)^{-1}\right] = \frac{-2 \sin(\pi \gamma)}{\pi} \int_0^\infty dt \, t^\gamma \text{tr} \left[\bar{\sigma}^{1-\gamma} (t\mathbf{1} + \sigma)^{-2} (\omega - \bar{\sigma})^2\right],
\] (C24)
where we used that $(t\mathbf{1} + \sigma)^{-1} \geq (t + 1)^{-1} \mathbf{1}$ and the cyclicity of the trace in the last step. We can now integrate this operator to get
\[
Z(\bar{\sigma}) = \frac{-2 \sin(\pi \gamma)}{\pi} \int_0^\infty dt \, \frac{t^\gamma}{t + 1} \bar{\sigma}^{1-\gamma} (t\mathbf{1} + \sigma)^{-2} = 2(1 - \bar{\sigma})^{-2} (\bar{\sigma}^{1-\gamma} - \gamma (1 - \bar{\sigma}) - \bar{\sigma})
\] (C25)
Since $t \mapsto t^{1-\gamma}$ is strictly concave, taking the tangent at $t = 1$ yields $\bar{\sigma}^{1-\gamma} \leq 1 - (1 - \gamma)(1 - \bar{\sigma})$, where the inequality is strict unless $\bar{\sigma}$ has a unit eigenvalue. Hence, we find $Z(\bar{\sigma}) < 0$ and, collecting the results starting from (C25), we have
\[
\partial_\omega^2 f(\bar{\sigma}) \leq \alpha \chi(\bar{\sigma}) \text{tr} \left[Z(\bar{\sigma})(\omega - \bar{\sigma})^2\right] < 0
\] (C26)
for all $\omega \neq \bar{\sigma}$. □
3. Proof of Proposition 10

It remains to show that $\alpha \mapsto \tilde{I}_\alpha(\rho_{AB}\|\tau_A)$ is differentiable for $\alpha \geq \frac{1}{2}$.

Proof. For $\alpha \in (\frac{1}{2}, 1)$ we recall Lemma 20 which establishes that $\tilde{I}_\alpha(\rho_{AB}\|\tau_A) = \tilde{D}_\alpha(\rho_{AB}\|\tau_A \otimes \bar{\sigma}_B(\alpha))$, where $\bar{\sigma}_B(\alpha)$ is the unique zero of the functional $X_\alpha(\bar{\sigma}_B) - \bar{\sigma}_B$. Since $X_\alpha$ is continuously differentiable in $\alpha$, its zero $\bar{\sigma}_B(\alpha)$ is continuously differentiable in $\alpha$ as well by the implicit function theorem. Thus, by the continuity of $\tilde{D}_\alpha(\rho_{AB}\|\tau_A \otimes \bar{\sigma}_B(\alpha))$ in $\alpha$ and $\bar{\sigma}_B$, this implies that $\tilde{I}_\alpha(\rho_{AB}\|\tau_A)$ is continuously differentiable.

For $\alpha > 1$, the same result follows due to the duality relation in Lemma 4. \qed


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