Towards Accurate PWL Approximations of Parameter-Dependent Nonlinear Dynamical Systems With Equilibria and Limit Cycles

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Abstract—This paper deals with piecewise-linear (PWL) approximations of nonlinear dynamical systems dependent on parameters and allowing the presence of few equilibria and/or limit cycles only. A method to derive the parameters of the PWL model is proposed that is based on the minimization of functionals defined to take into account a priori some dynamical features of the systems to be approximated. The method is validated by applying it to two simple dynamical systems, i.e., the cusp bifurcation normal form and the supercritical Hopf bifurcation normal form. The robustness of the approximations is checked, with a view to circuit implementations.

Index Terms—Bifurcation analysis, function approximation, nonlinear dynamical systems, piecewise-linear (PWL) approximation techniques, variational methods.

I. INTRODUCTION

The research context we deal with is the circuit implementation of nonlinear dynamical systems. In particular, this paper focuses on the problem of piecewise-linear (PWL) approximations (for considering structurally stable circuit implementations) [1] of known nonlinear autonomous dynamical systems described by continuous-time state-space models dependent on parameters, i.e., on systems governed by the following set of ordinary differential equations (ODEs):

\[ \dot{x} = f(x(t); p) \] (1)

where \( x(t) \in \mathbb{R}^n \) (state vector), \( p \in \mathbb{R}^p \) (parameter vector), \( f : \mathcal{S} \subset \mathbb{R}^{n+q} \rightarrow \mathbb{R}^n \) is a continuous vector field, \( \mathcal{S} \) is a (hyperrectangular) compact domain, and \( \dot{x} \) denotes the time derivative of \( x(t) \). All the vectors are considered as column vectors. Concerning the class (1), we focus our attention on systems characterized by the presence of a small number of stable equilibria and/or limit cycles only.

Motivations: The long-term target of this research field is the circuit implementation of networks of nonlinear dynamical systems, either:
- interacting, as in the case of biological neuron networks;
- behaving like monads, as in the case of associative memories.

The circuit synthesis of nonlinear dynamical systems admitting equilibria and/or limit cycles is highly desirable whenever one wants to mimic the behaviors of networks of such systems. Circuits governed by the original ODEs (or by good approximations of them) and having the same intrinsically parallel structure as the complex dynamical systems to be simulated can, at least in principle, process large amounts of data in parallel, thus overcoming the bottleneck of serial digital processing.

In particular, one of the main application fields is the circuit implementation of biologically plausible models of neurons. If one varies the parameters, these models (like that by Hodgkin and Huxley or that by Hindmarsh and Rose) can admit the presence of (stable and unstable) equilibria and limit cycles, even coexisting. The main goal of this paper is to take a step in this direction. This general (circuit) modeling problem requires an algorithmic synthesis procedure lying in an approximation method strictly related to a circuit synthesis technique.

State of the Art: Among the many possible approximation methods, we focus our attention on the PWL approach proposed in the last few years for the approximation of continuous functions [2]–[5]. Generally speaking, from a modeling point of view (i.e., in terms of number of approximation parameters required to obtain a reasonably accurate model), the method is not particularly efficient, as compared with other approximation methods like splines, neural networks or other kernel-based methods, but its main advantage lies in its quite direct circuit implementation [6], [7], which can be particularly interesting whenever we aim to emulate the behaviors of dynamical systems made up of a large number of elementary units, e.g., neurons [1], [8] or we need dedicated hardware for real-time, small-size and/or low-power applications (e.g., in smart dust or microcontrol systems). Another advantage of such an approach, not shared, for instance, by the wavelets and prewavelets PWL multi-grid approximations [9], [10], is the simplicity of its theoretical formulation, which allows an easy implementation of a multi-grid resolution approach to functions defined over domains of any (at least in principle) dimensionality [11].

Many PWL models belong to the class of function expansion models

\[ f_i^{\text{PWL}}(\mathbf{y}; N) = \sum_{k=1}^{N} w_k^i(N) \varphi_k(\mathbf{y}; N) \] (2)

where \( f_i^{\text{PWL}} \) is a component of the vector of functions \( f^{\text{PWL}}(\mathbf{y} = (\mathbf{x}^T; \mathbf{p}^T)^T \) is a generic (real) input vector, and \( N \) is the (integer) number of basis functions \( \varphi_k(\mathbf{y}; N) \) whose sum
(weighted by the coefficients $w^f_k(N)$) provides an approximation of a given scalar function $f_i$ (which, in our case, is the $i$-th component of the vector field $f$). This is a very broad-spectrum class of models, including, for instance, kernel estimators based on Bayesian methods [12], [13] or on regularization methods [14]–[16], splines [17] and, in the PWL framework, wavelets and prewavelets [9], [10] and fuzzy models [18], [19].

In the field of dynamical-system approximation, a method based on mixed-integer programming for the simultaneous identification of both the number $N$ of basis functions and the $N$ coefficients has recently been proposed in the context of identification of hybrid control systems [20], [21]. Such a method works well for small values of $N$, for a limited size of the regression set (which is a good feature in the context of real-time black-box identification), and for the approximation of non-wrinkled functions, but in many cases its computational complexity might become critical.

The PWL approach we use is based on an a priori domain partition through a simple type-1 triangulation (or simplicial partition), i.e., a triangulation formed by a rectangular partition plus northeast diagonals (this can also be viewed as a three-directional box spline grid). The rectangular partition is obtained by subdividing each dimensional component $y_k$ of the domain into an integer number $m_k$ of identically sized segments. In this case, $N = \prod_{k=1}^n (m_k + 1)$ can be fixed in a first step by some heuristic criteria, e.g., simply based on function inspection [2]–[5]. The coefficients are determined in a second step by minimizing a proper cost function [3], [5]. In the absence of a priori knowledge, such a method can suffer from the curse of dimensionality [22], since the number of elements of the regression set needed to have an accurate approximation would grow exponentially with the number of dimensions. However, if either the function to be approximated is known or we can sample it arbitrarily, it is possible to fix a reasonable number of subdivisions along each dimensional component of the domain.

Up to now, in most cases PWL approximations have been obtained by a “static” approach, i.e., by focusing attention only on the vector fields defining the considered dynamical systems, without taking into account a priori the expected dynamical behaviors. As a general statement, to obtain accurate dynamical behaviors, the involved nonlinear functions have to be approximated by linear affine functions over very small subregions of the considered domain, thus requiring very large numbers of subregions and then of approximation parameters. In [23], PWL approximations of dynamical systems with fixed parameters have been derived by applying variational criteria and model reduction strategies based on the method proposed in [5], [24] for the PWL “static” approximation of functions.

Goals and methods: In this paper, we propose a variational strategy for the computation of PWL approximations of nonlinear dynamical systems characterized by the presence of a small number of stable equilibria and/or limit cycles. The strategy is based on the minimization of functionals that are tailored to the specific system that should be approximated and are defined to take into account a priori some dynamical features of the systems to be approximated, like their bifurcation diagram or the presence of known invariant sets. These functionals are the sums of two or more terms and, owing to (2) and for a fixed $N$ and vector $\mathbf{q}_k(g, N)$, can also be viewed as cost functions for the coefficients $w^f_k$. A first term is the usual (for “static” approximations) mean square approximation error evaluated over the domain $S$. The minimization of this term provides PWL approximations whose accuracy is almost uniform over $S$. To the first component of the cost function, we add penalty terms to force the approximation to be particularly accurate in a neighborhood of a subset of $S$ (e.g., a bifurcation curve or an invariant set) where the vector field is critical for the behaviors of the dynamical system (1). The balance between the two terms of the cost function is governed by a coefficient that is tuned by minimizing a proper quality factor that measures the approximation quality. Of course, the quality factor as well can be chosen on the basis of the most significant sets of points of the considered system.

Results: The examples given in this paper deal with very simple bifurcation normal forms, i.e., the cusp and the supercritical Hopf. To verify the structural stability—in a given domain—of the original systems to the perturbations induced by the approximations, we carry out bifurcation analysis by resorting to some packages for numerical continuation [25], [26]. Since an essential prerequisite for using such methods is vector field smoothness, we replace a posteriori the PWL vector fields with piecewise-smooth (PWS) versions of them [1]. As already pointed out in [1], such a replacement is not completely “painless,” but does not substantially compromise either the approximation accuracy or the long-term target of the research field concerning the circuit realization of dynamical systems, since the possible circuit realizations of the PWL basis functions are necessarily smooth.

Outline: The rest of the paper is organized as follows. Section II recalls the basic elements of the PWL approximation method, and Section III deals with the specific problem of PWL approximations of dynamical systems. In Section IV and V, the proposed technique is applied to two bifurcation normal forms, i.e., the cusp bifurcation normal form and the supercritical Hopf bifurcation normal form. Finally, some conclusions are drawn in Section VI.

II. BASIC ELEMENTS

We denote by $\mathbf{f}_{\text{PWL}}$ a continuous PWL approximation of the vector field $\mathbf{f}$ over the $(n+q)$-dimensional domain $S$, i.e., $\mathbf{f}_{\text{PWL}} : S \rightarrow \mathbb{R}^n$, where

$$ S \equiv \{ g \in \mathbb{R}^{n+q} : a_i \leq y_k \leq b_i, i = 1, \ldots, n+q \}. \quad (3) $$

Each dimensional component $y_k$ of the domain $S$ can be either a state vector or a parameter vector component, and can be subdivided into $m_k$ subintervals of amplitude $(b_i - a_i)/m_k$. Consequently, $S$ is partitioned into $\prod_{k=1}^{n+q} (m_k + 1)$ hyperrectangles and contains $N = \prod_{k=1}^{n+q} (m_k + 1)$ vertices. This partition is called “boundary configuration $H$ (3)” and depends on the vector $m := (m_1, \ldots, m_{n+q})^T$. By adding the northeast diagonals to the obtained configuration, each hyperrectangle is in turn partitioned into $(n+q)!$ non-overlapping simplices, thus obtaining the simplicial partition $S_H$. The domain associated with a boundary configuration $H$ can be completely described by the triplets $(a_i, b_i, m_k), i = 1, \ldots, n+q$. The class of continuous PWL
functions $f_{\text{PWL}}$ that are linear over each simplex constitutes an $N$-dimensional metric space $\text{PWL}[S_H]$, which is defined by the domain $S$, its simplicial partition $S_H$, and a proper inner product (see [5] for details). In particular, since we work with continuous real functions, we refer to the usual inner product for $L^2$ spaces of real functions, i.e.,

$$ (f, g) = \int_S f(y) g(y) \, ds $$

inducing the norm

$$ ||f||_2 = \sqrt{(f, f)} = \sqrt{\int_S (f(y))^2 \, ds} = \sqrt{\int_S ||f(y)||^2 \, ds} $$

where $|| \cdot ||$ denotes the Euclidean norm in $\mathbb{R}^{n+q}$, and $ds = dy_1 \cdots dy_{n+q}$.

According to (2), each function belonging to $\text{PWL}[S_H]$ can be represented as a sum of $N$ basis functions (organized into a vector by using a given criterion), weighted by an $N$-length coefficient vector $w$. Then, $N$ can be used as an index of the PWL model complexity. For a fixed $m$, the coefficients $w$ determine the shape of $f_{\text{PWL}}$.

The "static" PWL approximation of a given vector field can be obtained by minimizing the following functional [5]:

$$ F_1(f_{\text{PWL}}) = ||f - f_{\text{PWL}}||^2_S = \int_S ||f - f_{\text{PWL}}||^2 \, ds. \tag{6} $$

The functional $F_1$ measures the closeness of $f$ and $f_{\text{PWL}}$ over the domain $S$.

III. PWL APPROXIMATION OF DYNAMICAL SYSTEMS

Our goal is to find a PWL vector field $f_{\text{PWL}}$ expressed as a weighted sum of a limited number $N$ of basis functions, and such that the asymptotic dynamics of the system

$$ \dot{x} = f_{\text{PWL}}(x(t); p) \tag{7} $$

is qualitatively and quantitatively similar to that of the system (1). To attain this objective, we define the following functional:

$$ F(f_{\text{PWL}}, \lambda) = F_1(f_{\text{PWL}}) + \lambda F_2(f_{\text{PWL}}) \tag{8} $$

where $\lambda$ is a positive constant coefficient.

In particular, since we aim to design circuits implementing the PWL approximations, it is important to find accurate approximations with the minimum number $N$ of basis functions.

The regularization parameter $\lambda$ balances the contributions of the two terms, and its value must correspond to the minimum (or maximum) of a function (quality factor) $Q(\lambda)$, which in turn fits the considered original system and takes into account some of its main features. For a fixed domain $S$ and a vector $m$, to obtain the optimal $\lambda$ one could resort to global optimization procedures. As a suboptimal alternative, one could explore a $\lambda$ range $[\lambda_{\text{min}}, \lambda_{\text{max}}]$ for a fixed set of $M$ values $\lambda_i$ by iterating the following three steps.

1) Set $\lambda = \lambda_i$.
2) Minimize the functional $F(f_{\text{PWL}}, \lambda_i)$ and find the $i$-th optimal coefficient vector $w = (w^1, \ldots, w^n)^T$ (where $w^i$ is the vector that determines the shape of the $i$-th component $f_i$ of $f_{\text{PWL}}$).
3) Compute the quality factor $Q(\lambda_i)$.

Then, a more accurate estimate of the optimal value of $\lambda$ can be obtained by local optimization procedures in a neighborhood of the $\lambda_i$ value corresponding to the minimum (or maximum) $Q$. Of course, the algorithm can also be iterated by varying the vector $m$ (and/or the number $N$) to find the minimal simplicial partition that allows one to obtain the desired accuracy. In the examples given in the sections to follow, we set $\lambda_{\text{min}} = 1E-3$ and $\lambda_{\text{max}} = 1E6$, and we iterate the steps of the algorithm for 50 points equally spaced on a logarithmic scale.

A crucial point in the approximation procedure is the choice of the basis functions $\varphi_k(\mathbf{y}, \mathbf{N})$ used in (2). Even if, in principle, there are many possible choices for the basis functions, in the linear space $C[S]$ of the functions continuous over the domain $S$ (of course, $\text{PWL}[S_H]$ is a subspace of $C[S]$), the choice should be as follows:

- Privilege basis functions that are orthonormal with respect to an inner product inducing a metric in the linear space $C[S]$.
- Allow one to express the error functional $F(f_{\text{PWL}}, \lambda)$ as a measure of the distance between the vector field $f(x)$ and its PWL approximation $f_{\text{PWL}}(x)$.

Some examples of basis functions and inner products can be found in [5], [23]. Here we adopt the $\psi$-basis proposed in [5], which is orthonormal with respect to the inner product directly related to the functional $F$ with $\lambda = 0$, i.e., to $F_1$, which is the only invariant term of the proposed method. Whereas other bases of functions (such as the so-called $\alpha$-basis and $\beta$-basis) have quite a direct significance, the $\psi$-basis does not exhibit a particular structure. For instance, Fig. 1 shows three one-dimensional examples of bases (where $\alpha$-basis in the first column, $\beta$-basis in the second column, and $\psi$-basis in the third column) for the domain $[-1, 1]$ partitioned into 4 subdivisions. One can see the local definition of the $\alpha$-functions, the regular structure of the $\beta$-functions, and the irregular shapes of the $\psi$-functions. Generally speaking, the $\psi$-basis is better from a computational point of view, whereas the $\alpha$-basis and the $\beta$-basis are more suitable for circuit implementations (mixed-signal [7] and analog [6], respectively). Whatever basis is used to compute the coefficients, one can easily derive the coefficients of any other basis through a simple matrix product [5].
and the forces the solution to remain close to \( \gamma \) and particularly accurate in a neighborhood of \( \gamma \). In other terms, we aim to con-
strain the PWL approximation to be reasonably accurate all over \( S \) (through \( F_1 \)) and particularly accurate in a neighborhood of.

Fig. 1. Example of three bases of functions for a domain \([-1, 1]\) partitioned into 4 subdivisions: \( \alpha \)-basis (first column), \( \beta \)-basis (second column), and \( \psi \)-basis (third column).

According to (2), by fixing a priori the parameter \( \lambda \) and the number \( N \), the functional \( F(f_{\text{PWL}}; \lambda) \) becomes a cost function \( F(u) \) dependent on the vector \( u \) of the PWL approximation coefficients. In particular, since we aim to design circuits implementing the PWL approximations, it is important to find accurate approximations with the minimum number \( N \) of basis functions. By varying \( \lambda \) and \( N \) and finding the optimal vector \( u \) for each configuration, we can obtain at least a suboptimal solution with respect to all the involved parameters, at the cost of an increased computational effort. Moreover, the term \( F_2 \) could be extended to a sum of properly scaled integrals, each corre-
sponding to a different dynamical feature. Of course, the larger the number of addends, the heavier the computational cost of the method. Finally, the accurate dynamical behaviors of the approx-
imate system must be checked a posteriori, usually through bifurcation analysis. However, we point out that, once the \( u \) en-
suring the best trade-off between approximation accuracy and model complexity has been found, it is either embedded in circuit parameters [6] or stored in a memory [7], then the off-line computational effort is made only once.

In the next sections, we shall give two examples and show how the functional term \( F_2 \) could be defined to take into ac-
count the dynamical features of the system to be approximated.

IV. EXAMPLE 1: CUSP BIFURCATION NORMAL FORM

The cusp bifurcation normal form is

\[
\dot{x} = f(x; p) = p_1 + p_2 x - x^3 = p_1 + h(x; p_2), \tag{9}
\]

The variables \( p_1 \) and \( p_2 \) are control parameters, and \( x \) is the state variable. Fig. 2(a) shows the bifurcation diagram of the normal form (9), with a region \( \Delta \) corresponding to the presence of three equilibrium points and a region \( B \) with one equilibrium point (see [25], [11] for details).

The two curves \( T_1 \) and \( T_2 \) are saddle-node bifurcation curves, and henceforth we shall denote their union by \( \gamma \). The bifurcation diagram is the projection of the folds of the cusp bifurcation normal form equilibrium manifold (shown in Fig. 2(b)) onto the parameter plane. Owing to the simple structure of (9), we could

Note that the choice is not unique and must be tailored to the considered system. Even if one has only numerical information concerning some invariant sets of the system, the variational strategy can be adopted by simply replacing the integrals with sums extended to the available data sets (further details are pro-
vided in Sections IV and V).

A. Cost Function

The functional considered in this case can be defined as follows [27]:

\[
F(\lambda, h_{\text{PWL}}) = F_1(h_{\text{PWL}}) + \lambda F_2(h_{\text{PWL}})
= \int_S [h(x; p_2) - h_{\text{PWL}}(x; p_2)]^2 \, dx \, dp_2
+ \lambda \int_\gamma [h(x; p_2) - h_{\text{PWL}}(x; p_2)]^2 \, d\gamma \tag{10}
\]

where \( F_2 \) forces the solution to remain close to \( h \) in the \( C^0 \) distance [25] around the curve \( \gamma \). In other terms, we aim to con-
strain the PWL approximation to be reasonably accurate all over \( S \) (through \( F_1 \)) and particularly accurate in a neighborhood of.
the bifurcation curve $\gamma$ (through $F_2$) in order to find a structurally stable approximation even with relatively coarse simplicial partitions. The compromise between the two goals is ruled by the coefficient $\lambda$, which is estimated through the minimization of the quality factor defined in Section IV.B.

For a given $\lambda$ and a fixed domain partition, the functional reduces to a cost function dependent on $w$ only, and the optimal weight vector $w_{\text{opt}}$ can be obtained by imposing $(\partial F)/(\partial w_j) = 0$ for $j = 1, \ldots, N$. With reference to (2), we obtain

$$
\frac{\partial F_1}{\partial w_j} = -2 \int_{\mathcal{S}} h \varphi_j dxdy_2 + 2 \sum_k w_k \int_{\gamma} \varphi_k \varphi_j d\gamma,
$$

$$
\frac{\partial F_2}{\partial w_j} = -2 \int_{\gamma} h \varphi_j d\gamma + 2 \sum_k w_k \int_{\gamma} \varphi_k \varphi_j d\gamma, \quad (11)
$$

Then, we can impose the optimality conditions $(\partial F/\partial w_j) = 0$ in the following form (for $j = 1, \ldots, N$):

$$
-\mathcal{B}_j + \sum_k A_{jk} w_k + \lambda \left\{ -\bar{\mathcal{B}}_j + \sum_k \bar{A}_{jk} w_k \right\} = 0 \quad (12)
$$

where

$$
A_{jk} = \int_{\mathcal{S}} \varphi_j \varphi_k dxdy_2,
$$

$$
B_j = \int_{\gamma} h \varphi_j d\gamma,
$$

$$
\bar{A}_{jk} = \int_{\gamma} \varphi_j \varphi_k d\gamma,
$$

$$
\bar{B}_j = \int_{\gamma} h \varphi_j d\gamma.
$$

Equation (12) can be expressed in compact matrix form as $\mathbf{A} \mathbf{w} = \mathbf{B}$, where $\mathbf{A} = \mathbf{A} + \lambda \mathbf{B}$ and $\mathbf{B} = \mathbf{B} + \lambda \bar{\mathbf{B}}$. Thus, the optimal coefficient vector $\mathbf{w}$ can be directly obtained by numerically solving the system $\mathbf{A} \mathbf{w} = \mathbf{B}$. The matrix $A$ can have a very large condition number, so we have to carefully choose both the integration method and the basis functions in order to obtain reliable results. To attain the results presented in Section IV-C, we resorted to Gauss–Legendre integration formulas with $[(P/m_1)] \times [(P/m_2)]$ collocation points over each elementary rectangle (containing two simplices) of the domain $\mathcal{S}$. The chosen value $P = 2000$ has been numerically proven to ensure reliable results for many configurations of the vector $\mathbf{m}$ corresponding to values of $N$ up to 350.

Let us now suppose for a while that the analytical expression of $\gamma$ is unknown, but a set of samples $\{\gamma_s\} (s = 1, \ldots, S)$ of such a curve is available for $S$ pairs $(p_{1s}, p_{2s})$, for instance as the result of the bifurcation analysis of the original system performed by resorting to numerical tools such as AUTO [28] or MATCONT. In this case, the term $F_2$ of (10), as well as the terms $\bar{A}_{jk}$ and $\bar{B}_j$, can be numerically computed through discretization techniques that are directly induced by the available data set.

### B. Quality Factor for the $\lambda$ Estimation

The fold bifurcation condition on the two branches $T_1$ and $T_2$ of the curve $\gamma$ is

$$
\frac{\partial h}{\partial x}_\gamma = p_2 - 3x_2^2 |\gamma| = 0. \quad (13)
$$

Then, in order to estimate the optimal value also for $\lambda$, we define the following quality factor:

$$
Q(\lambda) = \int_{\gamma} \left( \frac{\partial h}{\partial x} - \frac{\partial h_{\text{PWL}}}{\partial x} \right)^2 d\gamma = \int_{\gamma} \left( \frac{\partial h_{\text{PWL}}}{\partial x} \right)^2 d\gamma, \quad (14)
$$

This function measures the distance between the true system and the approximated one over the regions of the main features by using the norm $(5)$. This choice takes into account that the function $h_{\text{PWL}}$ is expressed as a linear combination of basis functions that are orthonormal with respect to norm $(5)$. Other possible choices, such as an infinite norm, would lead to less accurate results.

Roughly speaking, such a function tends to its minimum when the solution—which for a fixed $\lambda$ has been determined to be close to $\mathbf{h}$ in the $C^0$ distance around the curve $\gamma$—tends to remain close to $\mathbf{h}$ in the $C^1$ distance as well. According to (13), the curve $\gamma$ can be parameterized by $x$, by setting $p_2 = 3x_2^2$ (thus, $x = \pm \sqrt{p_2/3}$ and $dp_2 = 6x_2 dx$), with $p_2 \in [a_2, b_2]$. Then, if we expand $h_{\text{PWL}}$ in terms of the basis functions weighted by the optimal vector $\mathbf{w}_{\text{opt}}(\lambda)$, the quality factor can be finally expressed as follows:

$$
Q(\lambda) = \int_{-\sqrt{|a_2|/3}}^{\sqrt{|b_2|/3}} \left[ \sum_{k=1}^{N} \mathbf{w}_k \frac{\partial \varphi_k(x, 3x_2^2)}{\partial x} \right]^2 \sqrt{1 + 36x_2^2} dx, \quad (15)
$$

where $x_2 = -\sqrt{|a_2|/3}$ and $x_2 = \sqrt{|b_2|/3}$.

The optimal $\lambda$ corresponds to the absolute minimum of the quality factor. In this case, the second step of the algorithm for the computation of the optimal $\lambda$ becomes: 2) compute the matrix $\bar{\mathbf{A}}$ and the vector $\bar{\mathbf{B}}$ and find the $\lambda$-th optimal coefficient vector $\mathbf{w}$.

### C. Results

Once we have found the optimal values for both $\lambda$ and the weights $\mathbf{w}$, we have to check the structural stability of the system (9), with $\lambda$ replaced by the obtained PWL approximation $h_{\text{PWL}}$. Actually, since we want to obtain a reliable bifurcation diagram by applying numerical continuation methods [25] that require smoothness of the vector field, $h_{\text{PWL}}$ must be previously smoothed, as described in [1]. Since any considered PWL approximation can be expressed in terms of the absolute value function, the smoothing is obtained by replacing the absolute value with the following function:

$$
y(x) = \frac{2x}{\pi} \arctan(ax) \quad (16)
$$

where the parameter $a$ controls the degree of smoothness. Of course, the smoothed (PWS) versions of the $\psi$-functions still form a basis, provided that the parameter $a$ is not too small (in our continuations, we fixed $a = 40$).
Fig. 3. Approximations of the cusp bifurcation normal form (9). Results obtained for \( m_1 = 5, m_2 = 24 \), and \( \lambda = 3.39E4 \) (first column), \( \lambda = 0 \) (second column). (a,b) Equilibrium manifolds \( x_{eq}(P_1, P_2) \). (c,d) Bifurcation curves. (e,f) Surfaces \( \hat{h}(x; p_2) \rightarrow h_{\text{PWL}}(x; p_2) \) together with the black curve \( p_2 = 3x^2 = 0 \) plotted on the plane \( h = h_{\text{PWL}} \).

Fig. 3(a) shows the equilibrium manifold for the approximate system obtained by setting \( m_1 = 5 \) and \( m_2 = 24 \) (i.e., \( N = 150 \)). The minimum of the quality factor \( Q \) (\( Q_{\min} \approx 0.00963 \)) has been obtained for \( \lambda = 3.39E4 \). For comparison, Fig. 3(b) shows the equilibrium manifold for the approximate system obtained for \( \lambda = 0 \), i.e., by using the cost function \( F_1(\mathbf{w}) \).

Fig. 3(c) and (d) shows the bifurcation diagrams that are the projections of the equilibrium manifolds of Fig. 3(a) and (b), respectively, onto the parameter plane \((p_1, p_2)\). The comparison of the two approximations points out that in the first column the equilibrium manifold is approximated reasonably well, and the bifurcation curves are very well approximated. In the second column, vice versa, the approximation of the bifurcation curves is less accurate.

Fig. 3(e) and (f) shows the surfaces \( \hat{h}(x; p_2) \rightarrow h_{\text{PWL}}(x; p_2) \) for \( \lambda = 3.39E4 \) and \( \lambda = 0 \), respectively. A comparison of the two cases points out that, in the case (e), the black curve \( \gamma \) lies almost perfectly on the surface. This means that the approximation is particularly accurate for the curve \( \gamma \), as expected.

Table I summarizes some of the obtained results. For each considered simplicial partition, the values of both the optimum \( \lambda \) and the corresponding quality factor are given.

It is reasonable to have qualitatively similar bifurcation diagrams for similar values of \( Q \), even if obtained by different simplicial partitions. This is confirmed by the first column in Fig. 4, which shows the bifurcation diagrams obtained for three boundary configurations (\( \mathbf{m} = (5,9)^T \), \( \mathbf{m} = (9,14)^T \), and \( \mathbf{m} = (24,9)^T \)) in the first, second, and third rows, respectively). Sharing almost the same quality factor and characterized by quite different values of \( N \) (see Table I). The quality of the three diagrams is similar, as expected. The second column of figures shows the bifurcation diagrams obtained for the same boundary configurations and for \( \lambda = 0 \), pointing out the improvement in the quality obtained for \( \lambda = \lambda_{\text{opt}} \).

The quality factor \( Q \) is defined to take into account the gradient of the function \( \hat{h} \) on the bifurcation curve. As can be easily inferred from the table, if \( N \) increases, the accuracy of the “static” approximation improves, but we are not guaranteed that the quality factor decreases in turn, since it is not directly related to the “static” approximation.

### D. Robustness of the Approximations

The robustness of the obtained results has been tested with a view to circuit implementations of the PWL dynamical system where the coefficients \( \mathbf{w} \) are embedded in circuit parameters (such as the geometric dimensions of some transistors [6]) whose target values could be affected by phenomena that can modeled, at least in principle, as Gaussian noise. In this perspective, a rough sensitivity analysis has been made by randomly varying the weights according to the following:

\[
\delta_k = w_k(1 + \varepsilon) \tag{17}
\]

where \( \varepsilon \) is a Gaussian random variable with zero mean and standard deviation \( \sigma \). Starting from the best \( \mathbf{w} \) obtained for the par-
tition $\mathbf{m} = (5,24)^T$, we performed tens of simulations for three different values of $\sigma$ ($1E-4, 1E-3$, and $1E-2$). The obtained bifurcation diagrams show that up to $\sigma = 1E-3$ the results are robust, whereas for $\sigma = 1E-2$ the bifurcation diagram can significantly change (e.g., see the bifurcation diagram in Fig. 5).

V. EXAMPLE 2: SUPERCritical HOPF BIFURCATION NORMAL FORM

As a further step towards the circuit implementation of biologically plausible neuron models, the type of reasoning introduced into Section III can be applied to dynamical systems whose state portraits may contain one stable limit cycle. It is sufficient to add to $F_1(\mathbf{f}_{\text{PWL}})$ a proper line integral $F_2(\mathbf{f}_{\text{PWL}})$ weighted by a parameter $\lambda$ and evaluated over the limit cycle (which is an invariant curve), so as to force the PWL approximate dynamical system to have a limit cycle qualitatively and quantitatively similar to the original one. Of course, also a proper quality factor must be defined. In this section, the presence of a known limit cycle is induced in the approximation by properly imposing the integral phase condition [25].

The supercritical Hopf bifurcation normal form (in Cartesian coordinates) is

$$\begin{cases} \dot{x}_1 &= f_1(x_1,x_2,p) = px_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= f_2(x_1,x_2,p) = x_1 + px_2 - x_2(x_1^2 + x_2^2). \end{cases} \tag{18}$$

The variables $x_1$ and $x_2$ are state variables, and $p$ is the control parameter. For $p < 0$, the only asymptotic invariant set of the system (18) is the stable equilibrium point $(0,0)^T$, whereas, for $p < 0$, the equilibrium point becomes unstable and a stable limit cycle of radius $\sqrt{p}$ appears (see [25] for details).

We focus on the PWL approximations of the vector field (with $n = 2$ and $q = 1$) over the domain $S = \{ \mathbf{z} = (x_1,x_2,p)^T \in \mathbb{R}^3 : a_i \leq y_k \leq b_i, i = 1,2,3 \}$, with $a_1 = a_2 = a_3 = -1.5$ and $b_1 = b_2 = b_3 = 1.5$. The rectangular partition is obtained by fixing a priori the vector $\mathbf{m}$, and the coefficients $w_k$ are determined in a second step by minimizing the cost function defined in the next subsection.

A. Cost Function

We consider the following functional [29]:

$$F(\mathbf{f}_{\text{PWL}}, \lambda) = F_1(\mathbf{f}_{\text{PWL}}) + \lambda F_2(\mathbf{f}_{\text{PWL}}) =$$

$$= \int_S \| \mathbf{f}(\mathbf{x},p) - \mathbf{f}_{\text{PWL}}(\mathbf{x},p) \|^2 dx_1 dx_2 dp +$$

$$+ \lambda \int_0^{T_p} \int_0^T (\mathbf{x}, \dot{\mathbf{x}}_{\text{PWL}})^2 dt dp \tag{19}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual (Euclidean) inner product for vectors and $T(= 2\pi)$ is the period of the stable cycle in the Hopf system. $F_2(\mathbf{f}_{\text{PWL}})$ imposes the integral phase condition for positive values of the control parameter, i.e., it constrains the
vector $\dot{p}_{\text{PWL}}(=f_{\text{PWL}})$ to be orthogonal to the asymptotic solution $\mathbf{x}(t)$, which is a stable cycle. In other terms, for $p > 0$, we aim to constrain the PWL approximation to be reasonably accurate all over $S$ and particularly accurate in the generation of the limit cycles in order to find, also in this case, a structurally stable approximation even by relatively coarse simplicial partitions. The compromise between the two goals is ruled again by the coefficient $\lambda$, which is estimated through the minimization of the quality factor defined in Section V-B.

For a given $\lambda$ and a fixed domain partition, the functional reduces once more to a cost function dependent on $\mathbf{w}$ only, and the optimal weight vector $\mathbf{w} = [(w^1)^T, (w^2)^T]^T$ (where 1 and 2 refer to the components of the vector field $f_{\text{PWL}}$) can be obtained by imposing $(\partial F_i/\partial w^j_i) = 0$ for $j = 1, \ldots, N$ and $i = 1, 2$. With reference to (2), we obtain for $F_2$ (the equation for $F_1$ is a straightforward generalization of the cusp case)

$$\frac{\partial F_2}{\partial w^1_j} = 2 \sum_k w^1_k \int_0^T x_1^2 \varphi_j \varphi_j dtdp +$$

$$+ 2 \sum_k w^2_k \int_0^T x_1 x_2 \varphi_j \varphi_j dtdp$$

$$\frac{\partial F_2}{\partial w^2_j} = 2 \sum_k w^1_k \int_0^T x_1 x_2 \varphi_j \varphi_j dtdp +$$

$$+ 2 \sum_k w^2_k \int_0^T x_2^2 \varphi_j \varphi_j dtdp. \quad (20)$$

Then, we can impose the optimality conditions in compact matrix form as $A\mathbf{w} = B$, where the matrix

$$A = \begin{bmatrix}
A + \lambda A^1 & \lambda A^2 \\
\lambda A^2 & A + \lambda A^3
\end{bmatrix} \quad (21)$$

is defined in terms of submatrices whose elements are given by the following integrals:

$$A_{jk} = \int_S \varphi_j \varphi_k dx_1 dx_2 dp$$

$$\hat{A}^1_{jk} = \int_0^{t_3} \int_0^T x_1^2 \varphi_j \varphi_k dtdp$$

$$\hat{A}^2_{jk} = \int_0^{t_3} \int_0^T x_1 x_2 \varphi_j \varphi_k dtdp$$

$$\hat{A}^3_{jk} = \int_0^{t_3} \int_0^T x_2^2 \varphi_j \varphi_k dtdp \quad (22)$$

where the vector $B = [(B^1)^T (B^2)^T]^T$ collects the terms

$$B^1_j = \int_S f_1 \varphi_j dx_1 dx_2 dp.$$

Thus, the optimal coefficient vector $\mathbf{w}$ can be directly obtained by numerically solving the system $A\mathbf{w} = B$. To obtain the results presented in Section V-C, we resorted to Gauss–Legendre integration formulas with $400 \times 400 \times 400$ collocation points over the domain $S$ to compute $A$ and $B$, and with $1024 \times 1024$ points over the domain $[0, t_3] \times [0, T]$ to compute $\hat{A}^1, \hat{A}^2, \text{ and } \hat{A}^3$.

Of course, in this simple example, we know the analytical expression for the cycle. In more common cases, the cycle can be available only numerically, but this does not substantially change the validity of the proposed procedure. Similar results could be obtained by applying refined numerical techniques, such as those exploited in numerical continuation tools (see, e.g., [25]) in order to accurately “follow” a cycle by varying one or more parameters. In this case, a proper set of samples of the state variables could be available and then the double integral in (19) (and hence the terms $\hat{A}^1_{jk}, \hat{A}^2_{jk}, \text{ and } \hat{A}^3_{jk}$) could be evaluated with discretization techniques that are directly induced by the available data set.

B. Quality Factor for the $\lambda$ Estimation

In order to estimate the optimal value also for $\lambda$, we define the following quality factor:

$$Q(\mathbf{w}, \lambda) = \frac{1}{R} \sum_{i=1}^{2} \sum_{r=1}^{R} (L_i(p_r) - \hat{L}_i(\mathbf{w}, \lambda; p_r))^2 \quad (23)$$

where $R$ is the number of sample values of $p \in [0, b_3]$, $L_i(p_r)(i = 1, 2)$ are the Lyapunov exponents of the original system’s cycle for $p = p_r (r = 1, \ldots, R)$, whereas $\hat{L}_i(\mathbf{w}, \lambda; p_r)$ are the corresponding Lyapunov exponents of the PWL system’s cycle. In our simulations, we fixed 50 linearly spaced values $p_r \in [0.001, \ldots, 1.5](r = 1, \ldots, 50)$. Roughly speaking, this quality factor tends to its minimum when the PWL system’s trajectories tend to coincide asymptotically locally with the original system’s ones, i.e., to converge to the cycle following the same paths and at the same convergence rate. In the considered case, the original system has $L_1(p) = 0$ and $L_2(p) = -2p$, but, for generic systems, the Lyapunov exponents can be numerically computed as well as the $\hat{L}_i$’s [30].

C. Results

After finding the optimal $\lambda$ and $\mathbf{w}$, we checked the structural stability of the system (18) with the vector field $f$ replaced by the smoothed version of the PWL approximation $f_{\text{PWL}}$ obtained for $m_1 = m_2 = 4$ and $m_3 = 3$ (i.e., $N = 100$). The tool used for numerical continuations was MATCONT [26]. Fig. 6 shows the continuation of the equilibrium of the approximate system with $\lambda = 35.1119$ (gray curve) and $\lambda = 0$ (black curve). The top-left zoom shows a detail corresponding to the black box.
the cycle appears for the stable limit cycle is evident, even if it is smaller, the equilibrium position is almost exact, and remains very small in the gray curve; ii) after

A comparison shows that, corresponding to (a) and (b). The comparison diagram obtained for (18) is the transparent gray surface. the supercritical Hopf bifurcation, the equilibrium remains unstable in both cases.

Fig. 7 shows two bifurcation diagrams for limit cycles in the control space for the approximate system, together with the bifurcation diagram of the original system. Fig. 7(a) is the bifurcation diagram obtained for \( \lambda = 35.1119 \), corresponding to the minimum of the quality factor \( Q \) \( (Q_{\text{min}} \approx 0.12) \), whereas Fig. 7(b) has been obtained for \( \lambda = 0 \). A comparison shows that for \( \lambda = 0 \) the cycle appears for \( p < 0 \) and remains very small up to about \( p = 0.5 \). Around \( p = 0.5 \), the system undergoes two fold bifurcations of cycles and, for a narrow range of values of \( p \), it admits two coexisting stable cycles. For \( \lambda = 35.1119 \), the cycles are more similar to the original ones and do not undergo any bifurcations.

Fig. 8 shows four state portraits (for \( p = -0.5, 0.2, 0.5, \) and 0.7, respectively) of the approximate systems obtained by \( \lambda = 35.1119 \) (first column) and \( \lambda = 0 \) (second column). A comparison of the two approximations points out once more that, if one adds the term \( F_2 \) properly weighted, the approximation of the limit cycles improves, at the cost of worse approximations of both the equilibrium and the transient trajectories. For instance, for \( p = -0.5 \) the equilibrium position is almost exact, for \( p = 0.2 \) the stable limit cycle is evident, even if it is smaller than the cycle in the original system, for \( p = 0.5 \) and \( p = 0.7 \) the shape of the limit cycle is more similar to that of the original cycle.

Fig. 9 show the surfaces \( f_1(x) - f_{1, \text{PWL}}(x) \) obtained for \( p = 0.7 \) and with \( \lambda = 35.1119 \) (a) and \( \lambda = 0 \) (b). The compar-
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VI. CONCLUDING REMARKS

The method presented in this paper constitutes a step towards the circuit synthesis of networks of nonlinear dynamical systems. This method has been applied to two very simple dynamical systems, but could be quite simply tailored to other dynamical systems with equilibria and/or limit cycles. In the case of coexisting invariant sets, the term \( F_2 \) could contain more integrals, one for each invariant set, at the cost of heavier computations. The cost function term(s) weighted by the parameter \( \lambda \) can force the approximation to be particularly accurate in the neighborhood of either a bifurcation curve in the parameter space or an invariant set in the control space.

Even if the error around the bifurcation points or curves might seem excessively large also for \( \lambda = \lambda_{\text{cr}} \), we point out that usually one is interested in finding good approximations outside a belt across the bifurcation point [1] in order to ensure structurally stable behaviors of the approximated systems with respect to the original one when the system works far away from the bifurcation point. In this perspective, the approximations obtained by the example are widely acceptable. Of course, the approximation accuracy can be further improved by increasing \( N \).

The chosen functionals and quality factors are not necessarily the best in absolute terms, but are quite simple and allowed us to show the validity of the proposed approach. The main limit of such an approach lies in its computational heaviness, which increases with the number of variables involved in the optimization process (in the worst case, \( w, \lambda, N \) and/or \( m \)) and with the limitations on the knowledge of the system to be approximated. The latter issue becomes particularly significant when the reference invariant sets or bifurcation curves are known only numerically. The most expensive part of the algorithm is the numerical computation of the integrals. For instance, on a personal computer with a 2.6GHz processor and a 512 MB RAM, such computations for the second example take less than one hour. Once the matrices \( A \) and \( \tilde{A}(\ell = 1, 2, 3) \) and the vector \( B \) have been stored in the memory, the weight vector \( w \) is computed in a few seconds for any value of \( \lambda \). If we consider the functional \( F_1 \) only (i.e., we set \( \lambda = 0 \)), we have to compute the matrix \( A \) and the vector \( B \) only, and such computations take about 55 minutes. Then the influence of the added functional term on the whole computation time is negligible. Of course, the larger the number of addends, the heavier the computational cost of the method. However, the computational effort is made once and off-line and the obtained coefficients are embedded in a circuit or used for software simulations of the dynamical system. The complexity of the PWL dynamical system emulation depends on the number \( N \) of coefficients only.
Another limit concerns the class of systems that can be approximated. For instance, the definition of ad hoc functionals seems to be quite difficult for systems with chaotic attractors. In such cases, only fine partitions would ensure reasonably accurate approximations [1].

The main advantage of this approach is that it could allow one to obtain accurate and not excessively complex PWL approximations of dynamical systems by focusing on the most important dynamical features of such systems, with a view to their circuit implementations. The proposed examples are just a first step in this direction.

ACKNOWLEDGMENT

The authors wish to thank Dr. M. Bergami and Dr. A. Carlavaro for their valuable help in performing the simulations and Dr. Oscar De Feo and Prof. Mauro Parodi for constructive comments and discussions.

REFERENCES


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