Cubic polynomial patches through geodesics

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Abstract

We consider patches that contain any given 3D polynomial curve as a pregeodesic (i.e. geodesic up to reparametrization). A curve is a pregeodesic if and only if its rectifying plane coincides with the tangent plane to the surface, we use this fact to construct ruled cubic patches through pregeodesics and bicubic patches through pairs of pregeodesics. We also discuss the $G^1$ connection of $(1,k)$ patches with abutting pregeodesics.

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1. Preliminaries

The main goal of this paper is to exhibit a simple method to create low degree and in particular cubic polynomial surface patches that contain given curves as geodesics, when reparameterized by arclength.

The method is based on the observation that a curve which lies on a surface is a geodesic (up to parametrization) if and only if its tangent and acceleration vectors are coplanar with the surface normal along the curve. This observation rephrases the statement of O’Neill in his book Elementary Differential Geometry [1, p. 330]: a curve has geodesic curvature zero if and only if its tangent vector and the surface tangential component of the acceleration are collinear.

The problem of finding surface patches that contain a given curve as a geodesic is considered in [2] using the Frenet frame of the curve. They write down a necessary and sufficient condition for a surface to contain a prescribed curve as a pregeodesic, but do not emphasize the case of polynomial patches. The main emphasis in applications is in the shoe industry: finding surfaces that could model the shoe piece with a prescribed girth.

The solution lies in the fact that the shoe piece is fabricated from an approximately flat sheet with a minimum of stretching, so it is nearly isometric to the plane, and hence geodesics correspond to straight lines. Further applications are in textile manufacturing. This is discussed in [2] and the references therein.

Our construction of surface patches that contain prescribed polynomial cubic Bézier curves draws on the necessary relationship between the rectifying plane and the surface tangent plane, avoiding arclength parametrization and the use of Frenet frames.

2. Pregeodesics on patches

A parametrized curve which lies on a given patch is called a geodesic (see [3]) if its acceleration (i.e. the second derivative with respect to its parameter) is orthogonal to the surface along the curve. A nice introduction to geodesics which fits the needs of this paper is given in [4]. It follows that the tangent vector to a geodesic must have constant length, hence it is parametrized by arclength (or arclength times a constant).

A curve being a geodesic depends on its shape as well as on its parametrization. While working with applications for which parametrization is not relevant, it is natural to deal with geometric conditions on the curve to be a geodesic, up to reparametrization. In [1] O’Neill uses the term pregeodesic to refer to such curves. The necessary and sufficient condition for a curve to be a pregeodesic is the coplanarity of the tangent

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vector, the acceleration and the normal to the surface along the curve.

It is easy to construct families of surface patches that contain a given curve as pregeodesic. For example, if \( x(t) \) is any curve in 3D, such that \( x' \times x'' \) does not vanish, then for any \( \alpha \) and \( \hat{\alpha} \neq 0 \) the patch

\[
x(s, t) = x(t) + s(\alpha x'(t) + \hat{\alpha} x'(t) \times x''(t))
\]

contains the isoparametric curve \( x(t) = x(0, t) \) as pregeodesic, since the normal vector \( \hat{n} \) to the patch is orthogonal to the plane generated by \( x'(t) \) and \( \alpha x'(t) + \hat{\alpha} x'(t) \times x''(t) \)

along the curve, so \( \hat{n} \), \( x' \) and \( x'' \) are coplanar, because \( \hat{n} \cdot (x'(t) \times x''(t)) = 0 \) for each \( t \). A simple example of surfaces that contain a given curve as pregeodesic is the family of parabolic hyperboloids obtained in (1), when \( x(t) \) is a quadratic polynomial curve. See Fig. 1. In the special case that \( \alpha = 0 \) and \( \hat{\alpha} = 1 \) this patch is a cylinder.

More generally any Bézier curve \( x(t) \) is a pregeodesic of the ruled patch given by expression (1) when we write \( \alpha = \alpha(t) \) and \( \hat{\alpha} = \hat{\alpha}(t) \), polynomials in \( t \). This expression can also be derived (see [2]) using Frenet frames, but this technique does not emphasize the construction of polynomial patches.

We refer to ruled patches as ribbons. Moreover for any arbitrary polynomial surface \( p(s, t) \)

\[
y(s, t) = x(s, t) + s^2 p(s, t)
\]

is also a polynomial patch that contains \( x(t) \) as a pregeodesic. See Fig. 2 for an example of a ruled and a nonruled patch containing a common parabola as pregeodesic, along which both patches have coincident tangent planes. We refer to ruled patches as ribbons.

Under degenerate conditions a ribbon \( x(s, t) \) might fail to have a nonvanishing normal vector along \( x(t) \). In this case \( x(s, t) \) will not be a regular surface in the sense of [11]. This situation will arise if \( \hat{\alpha}(t), x'(t) \) or \( x'(t) \times x''(t) \) have a zero for some \( t \). Throughout this paper we will assume enough conditions on the curve \( x(t) \), its derivatives and \( \hat{\alpha}(t) \) so that the ribbons are regular surfaces (at least near the curve \( x(t) \)), in other words that \( \frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t} \) does not vanish along \( x(t) \).

It is clear from (1) that the degree of \( x(t) \) increases rapidly because of the cross-product term, even when the degrees of \( \alpha(t) \) and \( \hat{\alpha}(t) \) are low. Although in the case that \( x(t) \) is a Bézier cubic the construction yields a ruled cubic patch if we choose \( \alpha \) and \( \hat{\alpha} \) linear. This follows from the observation that \( x'(t) \times x''(t) \) has degree 2.

Generally speaking, the property that guarantees that each of these patches contains \( x(t) \) as pregeodesic is that the tangent plane of the surface along the curve coincides with the rectifying plane of the curve.

In computer-aided geometrical design, among the ruled patches, the developable patches are especially interesting. If

\[
\begin{align*}
\alpha(t) &= \| x'(t) \|^2 (x'(t) \times x''(t)) \cdot x''(t) \\
\hat{\alpha}(t) &= \| x'(t) \times x''(t) \|^2
\end{align*}
\]

the patch

\[
x(s, t) = x(t) + s(\alpha(t)x'(t) + \hat{\alpha}(t)x'(t) \times x''(t))
\]

is developable. For the proof it is enough to check that \( x'(t) \), \( \nu(t) = \alpha x'(t) + \hat{\alpha} x'(t) \times x''(t) \) and \( \nu'(t) \) are coplanar.

As observed in [9] developable surfaces have been widely used in engineering, design and manufacture. Also more recently [10] uses approximately developable patches to decompose meshes.

Developable patches are also important in architectural design, especially in the “paper-like” design of Frank Gehry. Beyond design, the many advantages of these surfaces in constructibility, fabrication and the coherence of physical and digital representation are studied in [6].

3. Patches through pairs of pregeodesics

Given two curves \( x_1(t) \) and \( x_2(t) \) it is possible to write down a patch \( x(s, t) \) such that \( x(0, t) = x_1(t) \) and \( x(1, t) = x_2(t) \) are pregeodesics. For this it is natural to use the cubic Hermite

\[\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t} \neq 0\]
The tangent vectors

The ribbons share a common ruling segment at

The tangent planes at each point of the common segment are

These are ribbons x contains and previous section we assume that x shows a patch through two pregeodesics. As stated in the because both partials of x contains both x(t) since H0(0) = 1, H1(0) = 0 for i ≥ 1 and H3(1) = 1, H4(1) = 0 for i ≤ 2. And x1(t) is a pregeodesic because both partials of x(s, t) for s = 0 are contained in the rectifying plane (generated by x′(t) and x′′(t)). For,

because H′0(0) = H′2(0) = H′4(0) = 0 and H′1(0) = 1.

Similarly, for the tangent plane to x(s, t) along x2(t). Fig. 3 shows a patch through two pregeodesics. As stated in the previous section we assume that x1(t) and x2(t), as well as x2(t) and x3(t) satisfy the necessary conditions for regularity.

Another way of constructing a bicubic patch x(s, t) that contains x1(t) and x2(t) as pregeodesics is to consider cubic ribbons x1(s, t) and x2(s, t) along x1(t) and x2(t), respectively. These are

and we use their control points as the patch’s control points. Namely the control points x10, x11, x12 and x13 of the patch

are those of x1(1, t), the “second curve” of the ribbon x1(s, t). Here the Bγ’s denote the Bernstein polynomials of degree three and x00, x01, x02 and x03 are the control points of x1(t).

If α1(t) = (1 − t)α0 + tα1, γ1(t) = (1 − t)γ0 + tγ1 then the control points x10, x11, x12 and x13 are

x1 = x01 + 2α01Δx1 + α11Δx0 + 6α01(Δx0 × Δx2) + 6α11Δx0 × Δx1

where Δx0 = x01 − x00, Δx1 = x02 − x01, Δx2 = x03 − x02. The control points adjacent to those of x2(t) are determined analogously.

Note that, as expected, the points x10 and x13 have to lie on the rectifying planes of x1(t) at t = 0 and t = 1, respectively.

Similarly, the control points x20, x21, x22 and x23 can be expressed in terms of the control points of the ribbon x2(s, t) along x2(t).

Now it is easy to join (1, 3) and (3, 3) patches with tangent plane continuity along pregeodesics. See Fig. 4.

4. G1 joining of pregeodesic curves

In the previous section we considered G1 joining of patches along common pregeodesics. In this section we will look at G1 contact of ribbons along common rulings.

The G1 connection of two ribbons containing G1 abutting pregeodesics, as illustrated in Fig. 5, might be useful in applications because it offers a wider family of G1 patches containing G1 curves as piecewise pregeodesics. We will establish precisely the constraints on the position of the control points and the polynomials α(t), γ(t), β(t) and δ(t) associated to each ribbon.

Two ribbons

that satisfy y(0) = y(1) are connected G1 along a common generator at this point if and only if the following three conditions are satisfied:

1. The tangent vectors x′(0) and y′(1) are parallel.
2. The ribbons share a common ruling segment at x(0) = y(1).
3. The tangent planes at each point of the common segment are equal for both patches.

The most interesting case arises when the degree is low. We will look more closely at the case when the degree of x(t) and y(t) are k = 3, 4.
If \( k = 3 \) and

\[
\alpha(t) = (1 - t)a_0 + t\alpha_1, \quad \beta(t) = (1 - t)b_0 + t\beta_1,
\]
then \( x(s, t) \) and \( y(s, t) \) are \((1, 3)\) tensor products.

If \( k = 4, \hat{a}_0 = \hat{a}_1 \) and \( \hat{b}_0 = \hat{b}_1 \) then the ribbons \( x(s, t) \) and \( y(s, t) \) are \((1, 4)\) tensor products.

More generally, assume that \( x(s, t) \) and \( y(s, t) \) have degree \( k \), 

\[\hat{a}_0 = a_1 \quad \text{and} \quad \hat{b}_0 = b_1 \]

so that the ribbons \( x(s, t) \) and \( y(s, t) \) join \( G^1 \) along a common generator.

Condition 1 says that \( x'(0) \) and \( y'(1) \) are parallel at \( s(0) = y(1) = p \) (see Fig. 6, for the labeling of the control points in the case that \( x(t) \) and \( y(t) \) are cubics) and Condition 2 implies the equality \( a_0x'(0) + \hat{a}_0y'(0) \times x''(0) = q_1 = \beta_1y'(1) + \beta_1y'(1) \times y''(1) \). So it follows that the tangent planes of \( x(s, t) \) and \( y(s, t) \) at \( p \) coincide. Hence

\[
a_0x' = \beta_1y' \quad \text{and} \quad \hat{a}_0x' \times x'' = \beta_1y' \times y''
\]

where \( x', x'' \) stands for \( x'(0), x''(0) \) and \( y', y'' \) for \( y'(1), y''(1) \), respectively. This equations will be useful to write Condition 3 in terms of the derivatives of \( x \) and \( y \) and the \( \alpha' \)s and \( \beta' \)s. In fact, Condition 3 is satisfied if and only if the vectors \((1 - s)(\hat{a}_0 - y_L) + s(a_0 - y_L), (1 - s)(\hat{a}_0 - y_L) + s(a_0 - y_L)\) and \(a_0 - \hat{a}_0\) are colinear for each \( s \) (see [7]), i.e.

\[
\det(a_0 - \hat{a}_0, (1 - s)(\hat{a}_0 - y_L) + s(a_0 - y_L), (1 - s)(\hat{a}_0 - y_L) + s(a_0 - y_L))
\]

is the zero polynomial.

So we need that the coefficients of \((1 - s)^2, (1 - s)s, s^2\), namely: \( \det(a_0 - \hat{a}_0, \hat{a}_0 - y_L, \hat{a}_0 - \hat{a}_0) \), \( \det(a_0 - \hat{a}_0, \hat{a}_0 - y_L, \hat{a}_0 - \hat{a}_0) \), \( \det(a_0 - \hat{a}_0, \hat{a}_0 - \hat{a}_0, \hat{a}_0 - \hat{a}_0) \), and \( \det(a_0 - \hat{a}_0, \hat{a}_0 - \hat{a}_0, \hat{a}_0 - \hat{a}_0) \) to vanish.
\[+(y^{12} \times y') \cdot x^{13} \tilde{\beta}_1 \tilde{\omega}_0 (1 + \beta_1 - \beta_0)\]
\[+(y^{12} \times y'') \cdot x' \tilde{\beta}_1 \beta_1 (1 + \alpha_1 - \alpha_0)\]
\[+(y^{12} \times y^3) \cdot x' \tilde{\alpha}_1 (1 + \alpha_1 - \alpha_0) = 0.\]  

Note that the quadratic part of (4), as a polynomial in the \(\alpha,\beta,\tilde{\alpha}\) and \(\tilde{\beta}\)'s, vanishes due to (3). So, substituting \(\alpha_0, \tilde{\alpha}_0, \beta_1\) and \(\tilde{\beta}_1\), in terms of \(\omega, \alpha_1, \tilde{\alpha}_1, \beta_0\) and \(\tilde{\beta}_0\), hence we can solve for any of them in terms of the other three and \(\omega\).

Therefore the \(G^1\) connection of the ribbons \(x(s, t)\) and \(y(s, t)\) along their common generator depends on the choice of four homogeneous parameters. These parameters can be easily adjusted because, once a choice has been made for \(\alpha_0, \tilde{\alpha}_0, \beta_1\) and \(\tilde{\beta}_1\), the remaining parameters, \(\alpha_1, \tilde{\alpha}_1, \beta_0\) and \(\tilde{\beta}_0\), satisfy a linear equation. Fig. 5 illustrates the \(G^1\) joining of two \((1, 3)\) ribbons along a common generator.

A salient fact in the \(G^1\) joining of the ribbons \(x(s, t)\) and \(y(s, t)\) is that the linear homogeneous equations (2) and (3) force a proportionality relation between \(\alpha_0\) and \(\tilde{\alpha}_0\) and also between \(\beta_1\) and \(\tilde{\beta}_1\). The proportionality factors depend on the derivatives up to degree 3 of \(x(t)\) and \(y(t)\) at \(p\). This suggests that it is easy to extend the above procedure to join more ribbons, which will then contain a piecewise \(G^1\) pregeodesic.

For example, let \(z(t)\) be another curve such that \(z(1) = y(0)\) and suppose that its rectifying plane at this point coincides with the rectifying plane of \(y(t)\), then the ribbon
\[
z(s, t) = z(t) + s[(((1 - t)\gamma_0 + t\gamma_1)z'(t) + ((1 - t)\tilde{\gamma}_0 + t\tilde{\gamma}_1)z'(t) \times z''(t)]
\]
will connect \(G^1\) to \(y(s, t)\) along a common generator if \(\beta_0 y'(0) = \gamma_1 z'(1)\) and \(\beta_0 y'(0) \times y'(0) = \tilde{\gamma}_1 z'(1) \times z''(1)\), and two homogeneous equations analogous to (2) and (3) for \(x(s, t)\) and \(y(s, t)\) above, are satisfied. They are obtained by substitution of \(x', x'', x'''\) by \(y'(0), y''(0), y'''(0)\) and of \(y', y'', y'''\) by \(z'(1), z''(1), z'''(1)\).

The new set of four equations forces proportionality relationships between \(\beta_0\) and \(\tilde{\beta}_0\) and also between \(\gamma_1\) and \(\tilde{\gamma}_1\), which depend on the derivatives of \(y(t)\) and \(z(t)\) of degree up to 3 at their joint.

The resulting \(G^1\) surface which consists of three ribbons depends on four homogeneous parameters. This process can be extended to any number of ribbons.

5. Special situations

When joining two ribbons: \(x(s, t)\) depending on \(\alpha_0, \alpha_1, \tilde{\alpha}_0\) and \(\tilde{\alpha}_1\) and \(y(s, t)\) on \(\beta_0, \beta_1, \tilde{\beta}_0\) and \(\tilde{\beta}_1\) additional constraints can be imposed on these parameters or on the curves \(x(t)\) and \(y(t)\) themselves — this might arise in some specific applications and would simplify the calculations.

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4 The quadratic part of (4) is the sum of the quadratic monomials of the last four summands of this expression.
to the ribbons along the curves $x(t)$ and $y(t)$ are orthogonal to $\pi$. A possibly useful application is the $G^1$ joining along the piecewise pregeodesic curve of a $G^1$ patch with a cylinder orthogonal to $\pi$. This is illustrated in Fig. 9, where the large solid dot denotes the point $q$.

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