Output-based Finite Time Control of LTI systems with matched perturbations using HOSM

Marco Tulio Angulo and Leonid Fridman

Abstract—Finite Time Stability of LTI systems with matched perturbations using dynamic output feedback is achieved under the assumptions of strong observability, controllability and known bounds for the perturbations. It is shown that only global controllers are well suited for this task. Two cases are studied: when the relative degree is well-defined and when it is not. Simulation examples are presented in order to illustrate the proposed approach.

I. INTRODUCTION

Motivation. Linear Time Invariant (LTI) systems are the class of systems where the most extensive research has been done, there exists a wide arsenal of tools to accomplish almost any desired control task. However, in real applications, to only consider a simple LTI model for control design automatically implies the need for robustness of the designed controller against unaccounted nonlinearities and perturbations. This way, it is clear that the simpler the model we have chosen the more robust the designed controller must be.

Sliding Mode (SM) Control is a technique that allows the design of robust, or better yet insensitive, controllers against matched perturbations [1]. In addition, it also features exact finite time convergence [1]. This last property seems to be a desired one for controllers in Hybrid or Switching systems, because it provides exact convergence during dwell times, thereby eliminating any error accumulation during successive switchings.

Antecedents. Some results concerning finite time stability of LTI systems can be found, for example, in [4], [5]. However, in the majority of those results, uncertainty is never considered and the whole state is assumed available for feedback. High Order SM (HOSM) controllers as finite time universal controllers for uncertain SISO systems were introduced in [2]. The design methodology is only based on the knowledge of the relative degree of the output and some bounds on the dynamic system. HOSM controllers have found numerous applications, e.g. see [3] for a recent one.

In [6], an output-based Second Order SM controller is presented for MIMO LTI systems with matched perturbations satisfying

- bounded input bounded state (BIBS) for both perturbations and control input,
- the perturbation and its first derivative are bounded by a known constant,
- well-defined relative degree, \( \text{rank}(\dot{C}B) = m \),
- stable invariant zeroes.

Using a step-by-step Second Order SM algorithm for state observation, the authors obtain asymptotic stability of the origin.

Main Contributions. In this paper we make use of global HOSM controllers, recently introduced in [7], to obtain global finite time convergent output controllers for strongly observable LTI systems considering a wider class of matched perturbations. The use of SM controllers of arbitrary order allows us to improve the precision of the implemented controllers with respect to sampling time and noise amplitude. Additionally, a separation criteria to detect the convergence of the HOSM differentiator is introduced.

Paper structure. In section II we outline the problem statement. Section III reviews some tools from HOSM theory and presents a separation property that allows us to properly use the controllers and the differentiator together. Section IV is devoted to analyzing the case when the relative degree is well-defined, showing some relationships with strong observability, and introducing the design methodology. In section V we present the design methodology when the relative degree is not well-defined. Finally, section VI gives some conclusions.

II. PROBLEM FORMULATION

Consider

\[
\Sigma : \begin{cases}
\dot{x} = Ax + B[u + w(t)] \\
y = Cx
\end{cases}
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \), \( w \in \mathbb{R}^m \) are the state, control input, measured output and perturbation signals, respectively. We assume that only the output \( y(t) \) is available for feedback. The control objective is to design \( u \) such that the origin \( x = 0 \) is finite time stable in the presence of bounded perturbations i.e. \( ||w(t)|| \leq W^+ \), \( W^+ \) known and constant.

III. METHODOLOGY

In this section we review some issues about global HOSM controllers and the HOSM differentiator. Then we provide a constructive version of the separation property providing an efficient way to use both tools together.

A. Global HOSM controllers

Global HOSM controllers were recently presented in [7]. Some of their properties are introduced in the following lemma:
Lemma 3.1: Given a system in Canonical Controller Form

\[
\dot{z} = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & a_{n-1} & a_n
\end{bmatrix} z + \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix} [u + w(t)]
\]

with \( z(t) \in \mathbb{R}^n \) available and the unknown input \( w(t) \in \mathbb{R}^1 \) bounded as \( |w(t)| \leq W^+ \), \( W^+ \) a known constant. Then the origin \( z = 0 \) is finite time stabilizable with a proper selection of the control input \( u(t) \).

**Proof:** Choose \( \sigma := z_1 \). Then the system is equivalent to

\[
\sigma^{(n)} = a^T z + u + w := h(t, z) + u,
\]

where \( a^T := [a_1, a_2, \ldots, a_n] \) and \( h(t, z) := a^T z + w(t) \). Notice that this is the same problem formulated in [7]. This problem is solvable if we can find the so called “gain function” \( \Phi(t, z) \) such that \( \alpha \Phi(t, z) \geq |h(t, z)| \) for a large enough \( \alpha \). We can calculate

\[
|h(t, z)| \leq \|a\|z\| + W^+
\]

and choosing \( \Phi(z) := \varrho_1 z + \varrho_2 \) with \( \varrho_1 > \|a\| \) and \( \varrho_2 > W^+ \), we ensure that \( \alpha \Phi(z) \geq |h(t, z)| \). Then with the use of the controller

\[
u = -\alpha \Phi(z) H_n(\sigma, \dot{\sigma}, \ldots, \sigma^{(n-1)})
\]

where \( H_n(z) \) is the \( n \)-th order sliding mode controller with “gain robust parameters” [7], the following identity is true after finite time [7]

\[
\{\sigma, \dot{\sigma}, \ldots, \sigma^{(n-1)}\} = \{z_1, z_2, \ldots, z_n\} \equiv 0
\]

that is \( z(t) \equiv 0 \), \( \forall t > T \).

It is worth mentioning that “classical” HOSM controllers, [2], [9], can only achieve semi-global stabilization of the origin since they require a bound of the class \( |h(t, z)| \leq C = \text{const} \). For the problem formulated in the last lemma, it is only true for a vicinity of the origin, since the linear term \( a^T z \) is not globally uniformly bounded. On the other hand, it is possible to consider a wider class of perturbations if we can find the proper gain function \( \Phi(z, t) \).

**B. Implementation issues: a separation property**

Throughout this paper we will assume that all the required derivatives of \( y(t) \) are available in real time for feedback. However, since they will be estimated using a HOSM differentiator, the exact estimate is only available after finite time. The more rational solution to this problem is to first obtain the finite time exact estimate of the derivative and then to turn on the controller. But, when has the estimation of the HOSM differentiator converged? This question is answered in the following lemma.

Lemma 3.2: Consider the HOSM differentiator [9] of order \( k \)

\[
\begin{align*}
\dot{\hat{y}} &= v_k = -\alpha_k \left| \hat{y} - y \right|^{\frac{k-1}{2}} \text{sign}(\hat{y} - y) - \hat{y}^{(1)} \\
\dot{\hat{y}}^{(1)} &= v_{k-1} = -\alpha_{k-1} \left| \hat{y}^{(1)} - v_{k-1} \right|^{\frac{k-2}{2}} \text{sign}(\hat{y}^{(1)} - v_k) + y^{(2)} \\
&\vdots \\
\dot{\hat{y}}^{(k-1)} &= v_1 = -\alpha_1 \left| \hat{y}^{(k-1)} - v_2 \right|^{\frac{1}{2}} \text{sign}(\hat{y}^{(k-1)} - v_2) + y^{(k)} \\
\dot{\hat{y}}^{(k)} &= -\alpha_0 \text{sign}(\hat{y}^{(k)} - v_1)
\end{align*}
\]

where \( y(t) \) is the signal to be differentiated and \( \hat{y}^{(i)} \) is the estimation of the true signal \( y^{(i)} \). Assume that \( \{\alpha_0, \ldots, \alpha_k\} \) were properly chosen so the algorithm provides for finite-time estimation, i.e.

\[
[\hat{y}, \ldots, \hat{y}^{(k)}] \equiv [y, \ldots, y^{(k)}], \quad \forall t \geq T
\]

then \( \dot{\hat{y}} \rightarrow y \) after, or at the same instant, that \( \{y^{(1)}, \ldots, y^{(k)}\} \rightarrow \{\hat{y}^{(1)}, \ldots, \hat{y}^{(k)}\} \).

**Proof:** By induction. For \( k = 1 \) we have

\[
\begin{align*}
\dot{\hat{y}} &= v_1 = -\alpha_1 \left| \hat{y} - y \right|^{\frac{1}{2}} \text{sign}(\hat{y} - y) + y^{(1)} \\
\dot{\hat{y}}^{(1)} &= -\alpha_0 \text{sign}(\hat{y}^{(1)} - v_1)
\end{align*}
\]

so \( \dot{\hat{y}} = y \) implies \( \dot{\hat{y}} = \hat{y}^{(1)} \). But, since \( \dot{\hat{y}} = y, \forall t \in [t_0, t_1] \) a non-zero measure time interval also implies that \( \dot{\hat{y}} = y \) in the same interval. Using this last two conditions, we obtain that \( y^{(1)} \equiv \hat{y} = y^{(1)} \).

Now suppose that this is true for each integer less or equal that \( k-1 \). Then \( \hat{y}^{(k-1)} \equiv y^{(k-1)} \) and \( v_2 = \hat{y}^{(k-2)} \). But, since \( k-2 < k-1 \) the induction hypothesis also applies so

\[
\hat{y}^{(k-2)} \equiv y^{(k-2)} \Rightarrow \hat{y}^{(k-2)} = y^{(k-2)} \equiv y^{(k-1)}
\]

so \( v_2 = \hat{y}^{(k-1)} \), and then, by the structure of the differentiator

\[
\dot{\hat{y}}^{(k-1)} = y^{(k)}
\]

but since \( \hat{y}^{(k-1)} = y^{(k-1)} \) in a non-zero measure time interval, then \( \dot{\hat{y}}^{(k-1)} = y^{(k)} \) in the same interval and obtain that \( y^{(k)} \equiv \hat{y}^{(k)} \).

Using this fact, we have an easy way to check if the differentiator has converged by verifying if \( \|y - \hat{y}\| \leq \varepsilon \), in some non-zero time interval. The constant \( \varepsilon \) is the error produced by implementation and depends on the differentiator gain \( L \), discretization step \( h \) and noise magnitude [7]. For example, for a differentiator of \( k \)-th order and no noise, the asymptotic error can be calculated as \( \varepsilon = \gamma L h^{k} \) [7] where \( \gamma \) is a constant that can be estimated trough simulation. In fact, given that \( \gamma \) may be large or small, it is natural to tune this parameter using simulations based on the initial calculation. This way, we propose to build the overall control in the following form

\[
u(t) = \begin{cases}
0 & \text{if } \|y - \hat{y}\| \geq \varepsilon, \forall t \in [t - \delta, t] \\
\bar{u} & \text{otherwise}
\end{cases}
\]
for \( \delta > 2\epsilon \) and where \( \tilde{u} \) is the control calculated using the estimated derivatives. So the estimation of the derivative is executed first, with the important distinction that \( [y^{(r)}] \) does not have an additional component due to \( u \). Then, when the estimation of the derivatives have converged in a time interval, the actual control \( \tilde{u}(t) \) is applied.

IV. CONTROL DESIGN: WELL-DEFINED RELATIVE DEGREE CASE

In this section we assume \( p = m \). Let us introduce the following concepts:

**Definition 1 (Isidori, 1996):** The output \( y = Cx \) has well-defined vector relative degree \( (r_1, \ldots, r_m) \) if

\[
c_iA^kB = 0, \quad k = 0, 1, \ldots, r_k - 2, i = 1, 2, \ldots, m
\]

and

\[
\text{rank}(Q) = m, \quad Q := \begin{bmatrix}
    c_1A^{r_1-1}B \\
    c_2A^{r_2-1}B \\
    \vdots \\
    c_mA^{r_m-1}B
\end{bmatrix} \in \mathbb{R}^{m \times m}
\]

**Definition 2 (Hautus, 1983):** System \( \Sigma \) (1), or equivalently the triplet \((A, B, C)\), is said to be strongly observable if \( y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \), \( \forall (u + w(t), x_0) \).

A. Preliminaries

Consider the auxiliary system \( \Sigma_I \)

\[
\Sigma_I: \begin{cases}
    \dot{x} = Ax + B[u + w(t)] \\
    \dot{y} = \tilde{C}x
\end{cases}
\]

where

\[
\tilde{C} := \begin{bmatrix}
    c_1T^r, \ldots, (c_1A^{r_1-1})T, \\
    \vdots \\
    , c_mT^r, \ldots, (c_mA^{r_m-1})T
\end{bmatrix}^T.
\]

Now the following is true.

**Lemma 4.1:** \( \Sigma \) is strongly observable if, and only if, \( \Sigma_I \) is strongly observable.

**Lemma 4.2:** [12] If \( \Sigma \) has well-defined relative degree, then \( \text{rank} \tilde{C} = \sum_{i=1}^{m} r_i := \rho^* \).

**Theorem 4.1:** If \( \Sigma \) has well-defined relative degree then it is strongly observable if, and only if, \( \sum_{i=1}^{m} r_i = n \).

**Proof:** (\( \Rightarrow \)) Assume well-defined relative degree and \( \sum r_i = n \). If the relative degree is well defined then, by Lemma 4.2, obtain that \( \text{rank} \tilde{C} = \sum r_i \), but \( \sum r_i = n \), then \( \text{rank} \tilde{C} = n \). Since \( \tilde{C} \in \mathbb{R}^{r \times n} \) then necessarily \( \rho^* = n \) and \( \tilde{C} \) is a full rank square matrix. Then, since \( \tilde{y} = \tilde{C}x \), \( x = 0 \) if and only if \( \tilde{y} = 0 \), that means strong observability for \( \Sigma_I \) and, by Lemma 4.1, system \( \Sigma \) is strongly observable too.

(\( \Leftarrow \)) Assume well-defined relative degree and strong observability. By contradiction. Suppose that \( \text{rank} \tilde{C} < n \) then we can find \( x_0 \neq 0 \) such that \( x_0 \in \ker(\tilde{C}) \) and \( \tilde{y}(0) = 0 \). Then differentiate to obtain

\[
\dot{\tilde{y}} = \tilde{C}Ax + \tilde{C}Bu, \quad \tilde{C}B \in \mathbb{R}^{m \times m}
\]

since

\[
\text{rank}(\tilde{C}B) = \text{rank} \begin{bmatrix}
    c_1A^{r_1-1}B \\
    c_2A^{r_2-1}B \\
    \vdots \\
    c_mA^{r_m-1}B
\end{bmatrix} = m
\]

there exists an input, namely \( u = (\tilde{C}B)^{-1}(-\tilde{C}A)x \), such that \( \dot{\tilde{y}} = 0 \) and then \( \tilde{y}(t) \equiv 0 \), \( \forall t \geq 0 \) meanwhile \( x(t) \) is not identically zero, leading to contradiction. So \( \text{rank} \tilde{C} = n \) and, as shown above, this conditions implies strong observability of \( \Sigma \).

Notice that in addition, if the conditions of the last theorem hold, then

\[
x = M \begin{bmatrix}
    y_{r,1} \\
    y_{r,2} \\
    \vdots \\
    y_{r,p}
\end{bmatrix} := MY_r, \quad Y_r := \begin{bmatrix}
    y_1 \\
    \vdots \\
    y_{r-1}
\end{bmatrix}
\]

for \( M = \tilde{C} \) and the state \( x \) can be represented as a linear combination of the output and its derivatives.

B. Control design

Let us consider the following two assumptions

**Assumption A4.1:** System \( \Sigma \) has well-defined relative degree \( (r_1, \ldots, r_p) \) with respect to \( u + w \).

**Assumption A4.2:** The well-defined relative degree vector satisfies \( \sum r_i = n \).

Since by Theorem 4.1, A4.1 and A4.2 are equivalent to the strong observability of the system, then it is possible to reformulate the control problem as to design \( u \) such that

\[
y(t) \equiv 0, \quad \forall t \geq T
\]

The following theorem introduces the design methodology for this case:

**Theorem 4.2:** Consider system \( \Sigma \) with \( p = m \), assumptions A4.1, A4.2 and \( \|w(t)\| \leq W^+ \), \( W^+ \) known and constant. Then system \( \Sigma \) is finite time stabilizable to \( x = 0 \) with the use of the controller

\[
v_i = -\alpha_i\Phi_i(Y_r)H_r(y_1, y_2, \ldots, y_{r-1}), \quad i = 1, \ldots, m,
\]

where \( v_i \) is the \( i \)-th row of \( v = Qu \), \( \alpha_i \) is a large enough constant \( \Phi_i(Y_r) := k_{i,1}\|Y_r\| + k_{i,2}, \quad i = 1, \ldots, m \), and \( k_{i,1} > \|c_1A^{r_1}\|, k_{i,2} > \|c_mA^{r_m-1}\|W^+ \).

**Proof:** Differentiate each output \( y_i \) until an input appears and group them together

\[
y_1^{(r_1)} = c_1A^{r_1}x + c_1A^{r_1-1}B[u + w(t)]
\]

\[
y_m^{(r_m)} = c_mA^{r_m}x + c_mA^{r_m-1}B[u + w(t)]
\]

By assumption A4.1, matrix \( Q \) is square and full rank. Introduce the following control input transformation \( u = Q^{-1}v \), so then

\[
y_1^{(r_1)} = c_1A^{r_1}x + v_1 + c_1A^{r_1-1}Bw
\]

\[
y_m^{(r_m)} = c_mA^{r_m}x + v_m + c_mA^{r_m-1}Bw
\]

Each output \( y_i \) satisfies the problem formulation of Lemma 3.1 with

\[
h_i := c_iA^{r_i}x + c_iA^{r_i-1}Bw, \quad i = 1, \ldots, m,
\]
Bounds for this last equation are easy to obtain as 
\[ |h_i| \leq \|c_i A^{r_i} \| \|M\| \|Y_r\| + \|c_i A^{r_i-1} B\| W^+ \]
Choosing
\[ \Phi_i(Y_r) := k_{i,1} \|Y_r\| + k_{i,2} \]
and selecting, for example, \( k_{i,1} > \|c_i A^{r_i}\|, k_{i,2} > \|c_i A^{r_i-1} B\| W^+ \) and \( \alpha_i \) large enough (to compensate for \( M \)) we have \( \alpha_i \Phi_i > |h_i| \). By Lemma 3.1, designing
\[ v_i = -\alpha_i \Phi_i(Y_r) H_{r,i}(y_i, \dot{y}_i, \ldots, y_i^{(r_i-1)}) \]
the following equality is obtained after finite time
\[ \{y_i, \dot{y}_i, \ldots, y_i^{(r_i-1)}\} = 0, \quad \forall t \geq T \]
for \( i = 1, \ldots, m \), that is \( \{y_1, \ldots, y_m\} = 0, \forall t \geq T \). In turn, by assumption A4.1, A4.2 and Theorem 4.1, system \( \Sigma \) is strongly observable and, by definition, it implies \( x(t) \equiv 0, \forall t \geq T \).

**Remark.** Notice that the control signal \( u \) can be made arbitrary smooth if enough integrators are added \( u^{(k)} = \bar{u} \) provided that \( w(t) \) is uniformly bounded.

V. CONTROL DESIGN: PARTIAL RELATIVE DEGREE CASE

This section presents the design methodology for the case when the relative degree is not well-defined. In this situation we propose to first observe the state \( x \) and then to control the system. Let us introduce the following concept:

**Definition 3 (Davila et.al, 2008):** The vector output \( y = Cx \) has **partial vector relative degree** \( (r_1, \ldots, r_p) \) if
\[ c_i A^k B = 0, \quad k = 0, 1, \ldots, r_i - 2, i = 1, 2, \ldots, p \]
and \( c_i A^{r_i-1} B \neq 0 \). If \( k > n \) for some \( i \), the corresponding coordinate of the partial relative degree vector is \( \infty \).

A. State observation

Finite time observation of strongly observable systems with bounded unknown inputs was recently presented in [13]. The reader is encouraged to see that reference for a more appropriate discussion of this topic. In review, the procedure is based, for the problem formulation of this article, on introducing a Luenberger observer and then using the robust differentiator to obtain a finite time estimation of the Luenberger observation error. First, let us introduce the following assumption

**Assumption A5.1:** The triplet \((A, B, C)\) is strongly observable.

Note that the requirement \( r_i \leq n \) will be met assuming controllability of the system (assumption A5.2, to be introduced later). Introducing the Luenberger observer
\[ \dot{\hat{x}} = A \hat{x} + Bu + L(y - \bar{y}), \quad \dot{\bar{y}} = C \hat{x} \]
define \( e = x - \hat{x} \) to obtain \( \dot{e} = (A - LC)e + Bw, \bar{y} = Ce \). Design \( L \) such that \( \bar{A} := A - LC \) is Hurwitz. Then, for any bounded \( w \) the Luenberger error \( e \) remains bounded. Differentiate each output \( \bar{y}_i \) until \( r_i - 1 \) to form
\[ U_2 e := \begin{bmatrix} U_{21} \\ \vdots \\ U_{2p} \end{bmatrix}, U_2 e = \begin{bmatrix} \bar{y}_i \\ \vdots \\ \bar{y}_i^{(r_i - 1)} \end{bmatrix} = \bar{Y}_{r,i} \]
if \( r_1 + \cdots + r_p < n \), then there exists some subspace that is not spanned by \( U_2 \). That subspace is obtained through a modification of Molinari’s algorithm [14]. Define
\[ \bar{U}_{r,i} := \begin{bmatrix} c_1 A^{r_i - 1} \\ \vdots \\ c_p A^{r_i - 1} \end{bmatrix}, \bar{Y}_{r,i} := \begin{bmatrix} \bar{y}_i^{(r_i)} \\ \vdots \\ \bar{y}_i^{(r_p)} \end{bmatrix} \]
le \( M_0 = 0_{n \times n}, \rho_0 = 0_{n \times 1} \). The algorithm is given as the following recursion [13]
\[ \hat{\rho}_1 := \begin{bmatrix} M_1 B \\ \hat{U}_r B \end{bmatrix} \begin{bmatrix} \hat{\rho}_1 \\ \hat{Y}_{r,i} \end{bmatrix} \]
\[ M_{i+1} := \begin{bmatrix} M_i B \\ \hat{U}_r B \end{bmatrix} \begin{bmatrix} \hat{\rho}_i \\ \hat{Y}_{r,i} \end{bmatrix}, \rho_{i+1} := \hat{\rho}_i \]
As shown in [13], given A5.1 the recursion is repeated until for some \( i \), \( M_i \) completes the missing rank of \( U_2 \), that is, until
\[ \text{rank} \begin{bmatrix} U_2^T & M_i^T \end{bmatrix}^T = n \]
Let rank(\( U_2 \)) := \( n_2 \), then choose \( n - n_2 \) linearly independent rows of \( M_i \) orthogonal to \( U_2 \) to form the matrix \( M_d \) and the corresponding rows of \( \rho_i \) to form \( \rho_d \). Finally the state estimation is built as
\[ \dot{x} := \xi + \begin{bmatrix} U_2 \\ M_d \end{bmatrix}^{-1} \begin{bmatrix} \hat{Y}_{r,i} \\ \rho_d \end{bmatrix} \]
and given A5.1, \( \dot{x}(t) \equiv x(t), \forall t \geq T \) is ensured.

B. Control design: full state feedback

First we assume,

**Assumption A5.2:** The pair \((A, B)\) is controllable and rank(\( B \)) = \( m \).

and let us introduce the following result:

**Lemma 5.1 ([15]):** Consider system \( \Sigma \) with rank(\( B \)) = \( m \) and the pair \((A, B)\) controllable. Then, there always exist a state transformation \( z = Tx \) and input transformation \( u = Gv \) such that the transformed systems is decomposed as \( m \)-single input systems in canonical controller form possible coupled through the last line of each block.

Both matrices \( T \) and \( G \) can be easily computed based on the knowledge of \( A \) and \( B \) (see [15] or [16] for example), but for reasons of economy the required computations are not presented here. The following theorem introduces the design methodology for this case

**Theorem 5.1:** Consider system \( \Sigma \), assumptions A5.1, A5.2, the whole state \( x \) available and \( \|w(t)\| \leq W^+ \), \( W^+ \) a
known constant. Then the system is finite time stabilizable to $x = 0$ with the use of the controller

$$
\dot{v}_i = -\alpha_1 \Phi_1(z) H_{\mu_i}(z_{\mu_i-1+1}, \ldots, z_{\mu_i-1})
$$

$i = 1, \ldots, m$, where $\mu_i$ is the $i$-th controllability index, $z = Tx$, $u = Gv$ and $\Phi_i(z) := k_{i,1} ||z|| + k_{i,2}$ with $\alpha_i$, $k_{i,1}$ and $k_{i,2}$ large enough.

**Proof:** Introduce the state transformation $z = Tx$ and input transformation $u = Gv$. Note that in this way the system is decomposed as $m$ single input systems of the form

$$
\begin{align*}
\dot{z}_{\mu_1} & = A_{11} z_{\mu_1} + A_{12} z_{\mu_2} + B_{c,1} u + B_{c,1} w + B_{c,2} w \\
\dot{z}_{\mu_2} & = A_{22} z_{\mu_2} + B_{c,2} u + B_{c,2} w
\end{align*}
$$

and it is not well defined. The Luenberger gain $w$ strongly observable. The relative degree is $\text{rank} \left[ \begin{array}{c} U_{22}^T \\ M_1 \end{array} \right] = 4$

Finally the finite time estimation is

$$
\dot{x} = \xi + \left[ \begin{array}{c} U_{22}^T \\ U_{22}^T M_1 \end{array} \right]^{-1} \left[ \begin{array}{c} \tilde{y}_1 \\ \tilde{y}_2 \end{array} \right]
$$

where $\tilde{y}_i := y_i - \hat{y}_i$ and $\tilde{x}(t) \equiv x(t), \forall t \geq T$. Now matrix $S := [b_1, b_2, Ab_1, Ab_2]$ and $\mu_1 = \mu_2 = 2$. The state transformation is given by

$$
T := \left[ \begin{array}{c} U_{22}^T A \\ l_1 u \\ l_4 A \end{array} \right] = \left[ \begin{array}{ccc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{array} \right]
$$

and the transformed system $z = Tx$ is

$$
\begin{align*}
\dot{z} & = \left[ \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] z + \left[ \begin{array}{c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] [u + w]
\end{align*}
$$

The input transformation is $G = I$, so it is ommited. The controllers are designed as

$$
\begin{align*}
\tilde{u}_1 & := -\alpha_1 \Phi_1(z) H_2(z_1, z_2), & \tilde{u}_2 & := -\alpha_2 \Phi_2(z) H_2(z_3, z_4)
\end{align*}
$$

with $\alpha_1 = \alpha_2 = 10$, $\lambda_1 = \lambda_2 = 1$, $\Phi_1(z) = ||z||^2 + 0.8$, $\Phi_2(z) = ||z||^2 + 1$ and the differentiators gains as $L = 500$ for the first one and $L = 100$ for the second one. The control is finally designed as (2) with $\epsilon = 10^{-3}$ and $\delta = 0.0561$. The integration was carried out using Euler integration with time step of $10^{-4}$ seconds. The state $x(t)$ is shown in Figure 3, the observation error is presented in Figure 4 and Figure 5 shows the control signal.
VI. CONCLUSIONS

Finite time convergence of the whole state $x$ to zero using only output measurements is achieved under the assumptions of strong observability, controllability and bounded unknown inputs. In the case when the relative degree is well-defined, it is shown that strong observability is equivalent to the condition that the sum of relative degrees is $n$. This result supports the well known conclusion about zero dynamics, since if $r_1 + \cdots + r_m = n$ then there is no orthogonal subspace to $\Sigma_f$ and no zero dynamics. However, this condition regarding the relative degree is not necessary for strong observability if the relative degree is not well-defined. In that case, we make use of a finite time observer and a state and input transformation to perform the control task. The control design procedure for both cases presents a MIMO design algorithm for HOSM controllers in the case of linear systems.

REFERENCES