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# The three-dimensional dynamics of the die throw

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A three-dimensional model of a die throw which considers the die bounces with dissipation on the fixed and oscillating table has been formulated. It allows simulations of the trajectories for dice with different shapes. Numerical results have been compared with the experimental observation using high speed camera. It is shown that for the realistic values of the initial energy the probabilities of the die landing on the face which is the lowest one at the beginning is larger than the probabilities of landing on any other face. We argue that non-smoothness of the system plays a key role in the occurrence of dynamical uncertainties and gives the explanation why for practically small uncertainties in the initial conditions a mechanical randomizer approximates the random process. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4746038]

**The essential property characterizing random phenomena is the impossibility of predicting individual outcomes. Generally, it is assumed that when we toss a coin, throw a die, or run a roulette ball this condition is fulfilled and all predictions have to be based on the laws of large numbers. In practice, the only thing one can tell with a given degree of certainty is the average outcome after a large number of experiments. On the other hand, it is clear that the dynamics of the coin, die, or roulette ball can be described by the deterministic equations of motion. Knowing the initial condition with a finite accuracy $\epsilon$, the viscosity of the air, the value of the acceleration due to the gravity at the place of experiment, and the friction and elasticity factors of the table one should thus be able to predict the outcome. In real experiment, the predictability is possible only for very small $\epsilon$, i.e., an accuracy which in practice is extremely difficult to implement and that is why the coin toss, die throw, and roulette run can be considered as a random process.**

## I. INTRODUCTION

Dice have been used for gambling for at least a few millennia. Nowadays, the throw of a die is commonly considered as a paradigm for chance, and gambling with the dice is possible in casinos all over the world. Can we predict the result of the coin toss or the throw of the die? For centuries, this question has been asked not only by the gamblers but also philosophers and physicists have asked it when trying to answer the following problem. What is the origin of randomness in physical systems? Currently, there is some kind of duality in answering these questions. A person studying mathematics, even in primary school, is told that the result of the coin toss is random and the probability of heads and tails is equal and equals 1/2 as the coin is symmetrical. Simultaneously, during the physics courses the same student is told that the law of classical mechanics describes the motion of the material body and guarantees the unique dependence of the position and velocity of this body on the initial conditions (initial position and initial velocity). So as the coin is a material body its motion during the toss is fully described and the face on which it falls is determined. The play with the coin toss (simple mechanical experiment) confirms the mathematical point of view, unless one learns some magical tricks which allow predictability and the confirmation of the mechanical laws.

The duality in understanding the coin toss (die throw) results has attracted the attention of a great number of scientists. Gerolano Cardano (1501–1576), Galileo Galilei (1564–1642), and Christiaan Huygens (1629–1695) wrote books about dice games.[1] Blaise Pascal (1623–1662) and Piere-Simon de Fermat (1601–1665) exchanged letters discussing the mathematical analysis of problems related to the dice games. In the XX century, this problem attracted the attention of the pioneers of nonlinear dynamics Henri Poincare (1854–1912)[2] and Eberhard Hopf (1902–1983).[3] The dynamics of the simplified models of the coin or die were studied by Joe Keller[4] and Persi Diaconis *et al.*[5]

The connection between unpredictable behavior of the nonlinear system and random processes such as the coin toss has been discussed 29 years ago by Joseph Ford in his famous paper "How random is a coin toss?"[6] In last decades, there were attempts to explain the problem of randomness in mechanical systems using newly developed chaos theory. One should mention the works of Erik Mosekilde and his coworkers,[7] Vladimir Vulovic and Richard Prange,[8] Tsuyoshi Mizuguchi and Makato Suwashita,[9] Jan Nagler and Peter Richter.[10] In all these studies, simplified two-dimensional models have been considered and the dynamics have been studied through the derived discrete maps.

In this review, we reconsider this problem on the example of the die throw. We give evidence that the dynamics of the dice game can be fully described in terms of the deterministic nonlinear dynamics. We consider a full 3-dimensional model of the die developed in our previous works,[11–14] derived equations of motion which consider the energy dissipation at the collision of the die with the fixed and oscillating

 

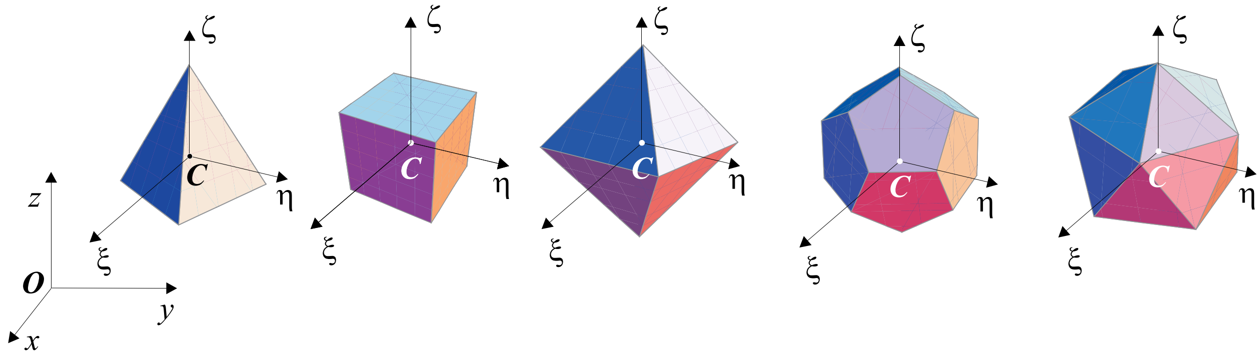


FIG. 1. Examples of typical dice: tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

table. Supplementing our previous studies,[11–14] we add the comparison of the numerical simulations with the experimental observation using high speed camera and consider the dynamics of the die on the oscillating table. Our studies show that the results of the toss are predictable when one sets initial conditions with an appropriate accuracy. This accuracy increases with the increase of the number of die bounces on the table. In practice, when the initial conditions are set at random the results can be considered as random but the probability that the die lands on the side or face which is the lowest at the beginning of throw is higher than the probability that the die lands on the other side or face. We also find that in the limiting theoretical case when the energy is not dissipated during the bounces on the table or the table is oscillating the dynamics of the die is chaotic.

## II. IS THE GAME OF DICE FAIR?

The die is usually a cube of homogeneous material. The symmetry suggests that such a die has the same chance of landing on each of its six faces after a vigorous roll thus it is considered to be fair. Generally, a die with a shape of convex polyhedron (with $n$ faces) is considered fair by symmetry if and only if it is symmetric with respect to all its faces, i.e., each face must have the same relationship with all other faces, and each face must have the same relationship with the center of the mass. The polyhedra (for example, tetrahedron, octahedron, dodecahedron, and icosahedron shown in Figure 1) with this property have an even number of faces and are called the isohedra.[15] Diaconis and Keller[16] suggested that there are non-symmetric polyhedra which are fair by continuity. As an example, consider the duality of $n$-prism which is a double pyramid with $2n$ identical triangular faces from which two tips have been cut with two planes parallel to the base and equidistant from it as shown in Figures 2(a)–2(c). If the cuts are close to the tips (Figure 2(b)), the solid has a very small probability of landing on one of two tiny new faces. However, if the cuts are near the base (Figure 2(c)), the probability of landing on them is high. Therefore, by continuity there must be the cuts for which new and old faces have an equal probability. It has been suggested that the locations of these cuts depend upon the mechanical properties of the die and the table and can be found either experimentally or by the analysis based on the classical mechanics.

However, these definitions do not consider the dynamics of the die motion during the throw. This dynamics is described by perfectly deterministic laws of classical mechanics which map initial conditions (position, configuration, momentum, and angular momentum of the die) at the beginning of the motion into one of the final configurations defined by the number on the face on which the die lands. The initial condition–final configuration mapping is strongly nonlinear, so one can expect the complex structure of the basin boundaries between the basins of different final configurations. The analysis of the structure of the basin boundaries allows us to identify the condition under which the die throw is predictable and fair by dynamics.[12,14]

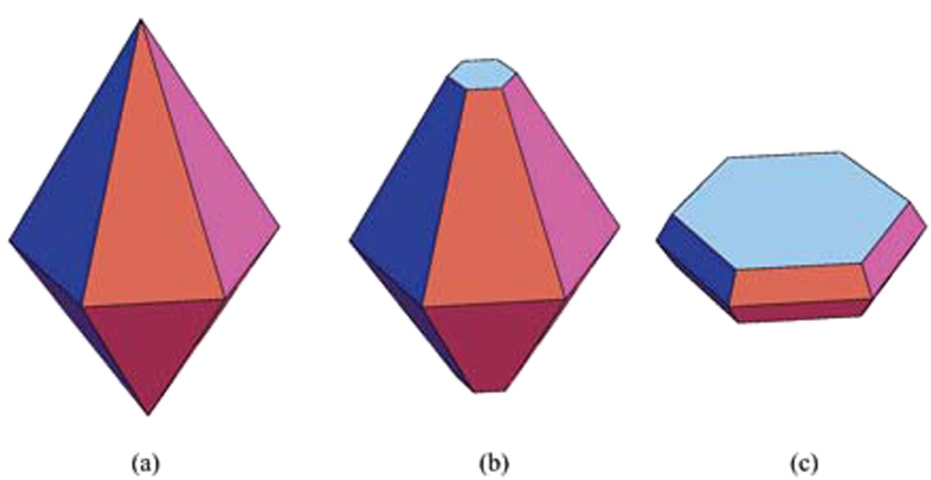


FIG. 2. (a) $n$-prism which is a di-pyramid with $2n$ identical triangular faces from which two tips have been cut with two planes parallel to the base and equidistant from it, (b) the cuts are close to the tips, (c) the cuts are near the base.

*Definition 1. The die throw is predictable if for almost all initial conditions $x_0$ there exists an open set $U$ ($x_0 \in U$) which is mapped into the given final configuration.*

Assume that the initial condition $x_0$ is set with the inaccuracy $\epsilon$. Consider a ball $B$ centered at $x_0$ with a radius $\epsilon$. Definition 1 implies that if $B \subset U$ then randomizer is predictable.

*Definition 2. The die throw is fair by dynamics if in the neighborhood of any initial condition leading to one of the n final configurations $F_1, ..., F_i, ..., F_n$, where $i = 1, ..., n$, there are sets of points $\beta(F_1), ..., \beta(F_i), ..., \beta(F_n)$, which lead to all other possible configurations and measures of sets $\beta(F_i)$ are equal.*

Definition 2 implies that for the infinitely small inaccuracy of the initial conditions all final configurations are equally probable.

## III. MODEL

Let us consider the die model introduced in Refs. 12 and 14. In this model, the die is a rigid body of a homogeneous material, i.e., the center of the mass and the geometrical center are located in the same point $C$. We consider the dice with isohedral shape, sharp edges and corners and neglect the influence of the air resistance. Precise casino dice have their pips drilled, and then filled with the flush with paint of the same density as the acetate used for the dice, so they remain in balance. They also have sharp edges and corners. In our previous work, it has been shown that the influence of the air resistance on the motion of the small rigid body like a die or a coin is very small. We assume that during the throw from the height $z_0$ the die rotates around the $\xi, \eta, \zeta$ axis (see Figures 3(a) and 3(b)). Using the laws of classical mechanics, we have derived the equations of motion which allow the determination of the position and velocity of any die point in the frame $x, y, z$. Details of our model and the analyses of the dynamics it produces have been presented in a recent book.[14]

At the die-table collision, a portion of the energy is dissipated. The level of dissipation is characterized by the restitution coefficient $\chi$ $(0 < \chi < 1)$. When the die lands on the soft surface without bouncing $\chi = 0$, and if the energy is not dissipated at the collision $\chi = 1$. Of course, the second case cannot be carried out experimentally. For the typical die colliding with the wooden table this coefficient is approximately equal to 0.5. In the current study, we show that the probability of the die landing on the face, which is the lowest one, at the beginning is larger than on any other face. In dice games, the top face usually counts. We shall refer to the lowest face (the face on which a die lands) as one of the considered die shapes is tetrahedron which has no top face. Previous work on die tossing[7] has demonstrated that friction has a significant effect on the dynamics of the simplified die model. In particular, friction controls the transition from sliding to non-sliding collisions and can change the outcome of the throw. However, consideration of the stick-slip friction in the full three-dimensional model of the die leads to the complicated relations and required further studies so in the present discussion we neglected the friction between the die and the table.

A die is modeled as a rigid body of homogeneous material, i.e., the center of mass and the geometrical center are located in the same point $C$. We consider the dice with isohedral shape and sharp edges and corners. The typical examples of such dice are tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

To describe the motion of the die in 3-dimensional space, we introduce the body embedded frame $(C\xi\eta\zeta)$. It is possible to show that the axis $\xi, \eta$ and $\zeta$ as well as any other axes passing through point $C$ are principal axis so the considered die models are spherical tops.[17] Their moments of inertia for any axes passing through $C$ are equal to $J$ (i.e., $J_\xi = J_\eta = J_\zeta = J$) and deviation moments $J_{\xi\eta}, J_{\eta\zeta}$, and $J_{\xi\zeta}$ are equal to zero. Using the numerical procedures for moments and products of inertia of polyhedron bodies proposed in Ref. 18 we get $J = ma^2/20$ for tetrahedron, $J = ma^2/6$ for cube, $J = ma^2/10$ for octahedron, $J = \frac{ma^2}{300}(95 + 39\sqrt{5})$ for dodecahedron, and $J = \frac{ma^2}{20}(3 + \sqrt{5})$ for icosahedron, where $a$ is the length of the dice edge.

Consider the die motion in 3-dimensional space described by the fixed frame $0xyz$ as shown in Figure 3(a). Neglecting the influence of the air resistance one obtains Newton-Euler equations of motion in the following form:

$$m\ddot{x} = 0\,, \quad m\ddot{y} = 0\,, \quad m\ddot{z} = -mg\,, \tag{1}$$

$$J\frac{d\omega_\xi}{dt} = 0\,, \quad J\frac{d\omega_\eta}{dt} = 0\,, \quad J\frac{d\omega_\zeta}{dt} = 0\,, \tag{2}$$

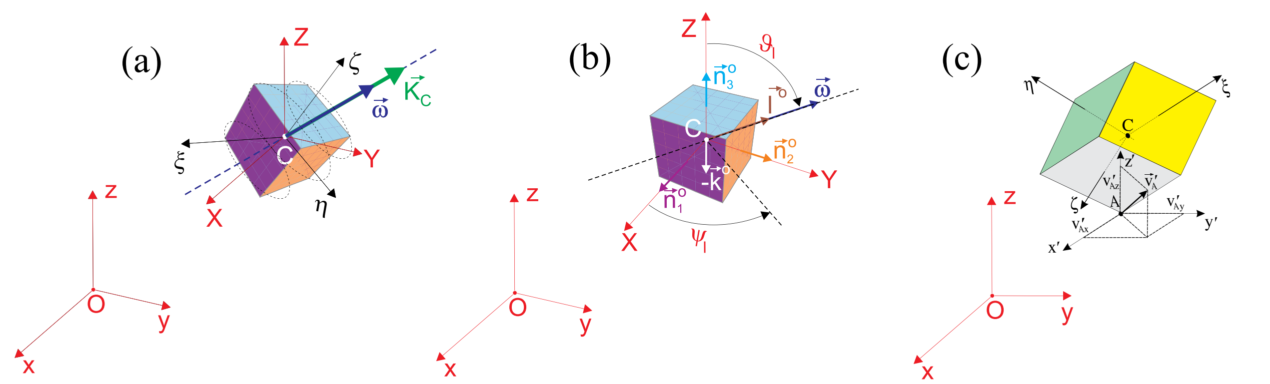


FIG. 3. Die as a homogeneous cube: (a) rotation axis $\xi, \eta, \zeta$, angular velocity $\tilde{\omega}$ and angular momentum $\tilde{\mathbf{K}}_C$, (b) orientation of die rotation axis $\tilde{\mathbf{l}}^0$, angular velocity vector, and face orientation vectors $\tilde{\mathbf{n}}_i^o$, (c) velocity vector $v'_{Az}$ of point $A$ after the collision and its scalar components $v'_{Ax}, v'_{Ay}, v'_{Az}$.

where $(\omega_\xi, \omega_\eta, \omega_\zeta)$ are the components of the angular velocity vector $\tilde{\omega}$.

The integrals of Eqs. (1) and (2) are given in the form

$$\dot{x} = v_{0x} = \text{const}\,, \quad \dot{y} = v_{0y} = \text{const}\,, \quad \dot{z} = -gt + v_{0z}\,, \tag{3}$$

$$\omega_\xi = \omega_{0\xi} = \text{const}\,, \;\; \omega_\eta = \omega_{0\eta} = \text{const}\,, \;\; \omega_\zeta = \omega_{0\zeta} = \text{const}\,. \tag{4}$$

Equations (3) and (4) show that during the motion the angular velocity of the die $\tilde{\boldsymbol{\omega}}$ is constant and equal to its initial value $\tilde{\boldsymbol{\omega}}_0$. The orientation of the vector $\tilde{\boldsymbol{\omega}}$ in relation to the frames $C\xi\eta\zeta$ and $Oxyz$ is not changing as the components of angular velocity vector are constant (Eq. (4)). The directions of the angular velocity vector $\tilde{\boldsymbol{\omega}}$ and the angular momentum vector $\tilde{\mathbf{K}}_{\mathbf{C}}$ coincide and are constant during the motion as shown in Figure 3(a) and depend only on initial conditions.

The initial conditions are given by the initial position of the center of die mass $C - \mathbf{q}_{10} = [x_0, y_0, z_0]^T$, its initial velocity $\mathbf{v}_0 = [v_{0x}, v_{0y}, v_{0z}]^T$, initial orientation of the axis $(C\xi\eta\zeta) - \mathbf{q}_{20} = [\psi_0, \vartheta_0, \varphi_0]^T$ (given by the Euler angles) and initial angular velocity $\omega_0 = [\omega_{0\xi}, \omega_{0\eta}, \omega_{0\zeta}]^T$.

To determine on which face the die lands, we analyze full rotation of the die, i.e., $\varphi_l \in <0, 2\pi>$ around the given rotation axes. During the die motion, the projections of the vectors $\tilde{\mathbf{n}}_i^o$ $(i = 1, \ldots, n)$ (as shown in Figure 3(b)) on the vertical axes $-\tilde{\mathbf{k}}^o$ (with the orientation towards the table)) have been calculated. This allows the calculation of the cosines of the angles $\alpha_i$: $\cos\alpha_i = -\tilde{\mathbf{k}}^o \cdot \tilde{\mathbf{n}}_i^o$. The face $j$ for which $\cos\alpha_j$ is the largest, i.e., $\cos\alpha_j = \max_i\{\cos\alpha_i\}$, $(i = 1, \ldots, n)$ is the lowest. The die lands on the face which is the lowest at the moment when the die stops on the table.

To describe a collision of the die with a table we assume that: (1) the table is modeled as flat, horizontal, elastic body (fixed to move), (2) friction force between the table and the die is omitted, (3) only one point of the die is in contact with the table during each collision. Let us consider that the die collides with a table when the vertex $A$ touches the table as shown in Figure 3(c). According to Newton's hypothesis, one gets $\chi = v'_{Az}/v_{Az}$, where $\chi$ is the coefficient of restitution, $v'_{Az}$ and $v_{Az}$ are the projections of the velocity of point $A$ on the direction $(z)$ normal to the impact surface, before and after the impact, respectively. The position of point $A$ in the body embedded frame is described by $\xi_A$, $\eta_A$, and $\zeta_A$. To describe the impacts, we consider an additional frame with an origin at point $A$ and the axes: $x'y'z'$—parallel to the fixed axes $xyz$ (Figure 3(c)). In the matrix form, the velocity vector of point $A$ is described as $\mathbf{v}_\mathrm{A} = \mathbf{v}_\mathrm{C} + \boldsymbol{\Omega}_x \mathbf{R}\,\boldsymbol{\xi}_\mathrm{A}$, where: $\mathbf{v}_\mathrm{C} = [\dot{x}\ \dot{y}\ \dot{z}]^\mathrm{T}$, $\boldsymbol{\xi}_\mathrm{A} = [\xi_A\ \eta_A\ \zeta_A]$T, $\boldsymbol{\Omega}_x = \mathbf{R}^\mathrm{T}\dot{\mathbf{R}}$, and the transformation matrix (in terms of Euler angles)

$$\mathbf{R} = \begin{bmatrix} \cos\varphi\cos\psi - \cos\vartheta\sin\varphi\sin\psi & -\cos\psi\sin\varphi - \cos\vartheta\cos\varphi\sin\psi & \sin\vartheta\sin\psi \\ \cos\vartheta\cos\psi\sin\varphi + \cos\varphi\sin\psi & \cos\vartheta\cos\varphi\cos\psi - \sin\varphi\sin\psi & -\cos\psi\sin\vartheta \\ \sin\vartheta\sin\varphi & \cos\varphi\sin\vartheta & \cos\vartheta \end{bmatrix}. \tag{5}$$

In the analysis of the phenomena that accompany the impact besides Newton's hypothesis the laws of linear momentum and angular momentum theorems of rigid body, as well as constraint equations have been employed. Modeling the nonholonomic contact between the die and the table, we consider the case of the smooth-frictionless die,[19] i.e., the vector of the impulse base reaction $\tilde{\mathbf{S}}$ has the following form $\tilde{\mathbf{S}} = [0, 0, S_z]^T$. The above assumptions result in the following relations:

$$\begin{aligned}
&\dot{z}' + \dot{\vartheta}'(-\zeta_A\cos(\varphi+\psi)\sin\vartheta + \cos\vartheta\cos(\varphi+\psi)(\eta_A\cos\varphi + \xi_A\sin\varphi) + (\xi_A\cos\varphi - \eta_A\sin\varphi)\sin(\varphi+\psi)) \\
&\quad + \dot{\psi}'\sin\vartheta\Big(\xi_A\cos\vartheta\cos\psi\sin^2\varphi + \eta_A\cos^2\frac{\vartheta}{2}\cos\psi\sin(2\varphi) - \eta_A\sin^2\varphi\sin\psi + \xi_A\cos^2\frac{\vartheta}{2}\sin(2\varphi)\sin\psi \\
&\quad + \cos^2\varphi(-\xi_A\cos\psi + \eta_A\cos\vartheta\sin\psi) - \zeta_A\sin\vartheta\sin(\varphi+\psi)\Big) \\
&\quad = -\chi\Big(\dot{z} + \dot{\vartheta}(-\zeta_A\cos(\varphi+\psi)\sin\vartheta + \cos\vartheta\cos(\varphi+\psi)(\eta_A\cos\varphi + \xi_A\sin\varphi) + (\xi_A\cos\varphi - \eta_A\sin\varphi)\sin(\varphi+\psi)) \\
&\quad + \dot{\psi}\sin\vartheta\Big(\xi_A\cos\vartheta\cos\psi\sin^2\varphi + \eta_A\cos^2\frac{\vartheta}{2}\cos\psi\sin(2\varphi) - \eta_A\sin^2\varphi\sin\psi + \xi_A\cos^2\frac{\vartheta}{2}\sin(2\varphi)\sin\psi \\
&\quad + \cos^2\varphi(-\xi_A\cos\psi + \eta_A\cos\vartheta\sin\psi) - \zeta_A\sin\vartheta\sin(\varphi+\psi)\Big)\Big)\,,
\end{aligned} \tag{6}$$

$$\dot{x}' = \dot{x}\,, \quad \dot{y}' = \dot{y}\,, \quad m\dot{z}' - S_z = m\dot{z}\,, \tag{7}$$

$$\begin{aligned}
J(\dot{\vartheta}'\cos\psi + \dot{\varphi}'\sin\vartheta\sin\psi) &= J(\dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi) + S_z\eta_A\cos\vartheta - S_z\zeta_A\cos\varphi\sin\vartheta\,, \\
J(\dot{\vartheta}'\sin\psi - \dot{\varphi}'\cos\psi\sin\vartheta) &= J(\dot{\vartheta}\sin\psi - \dot{\varphi}\cos\psi\sin\vartheta) - S_z\xi_A\cos\vartheta + S_z\zeta_A\sin\vartheta\sin\varphi\,, \\
J(\dot{\varphi}'\cos\vartheta + \dot{\psi}') &= J(\dot{\varphi}\cos\vartheta + \dot{\psi}) + S_z\xi_A\cos\varphi\sin\vartheta - S_z\eta_A\sin\vartheta\sin\varphi\,.
\end{aligned} \tag{8}$$

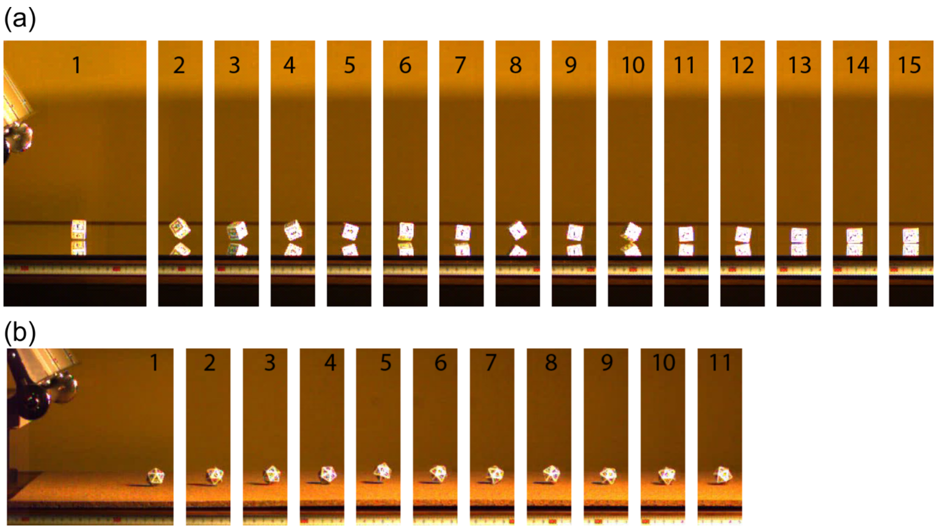


FIG. 4. Experimental observation of the die motion at the moments of successive bounces on the plane, (a) cubic die on the mirror plane, (b) icosahedron die on a cork plane.

Equations (6)–(8) allow the determination of the die velocities components after the collision ($\dot{x}'$, $\dot{y}'$, $\dot{z}'$, $\omega'_\xi$, $\omega'_\eta$, $\omega'_\zeta$) and the table reaction impulse $S_z$.

The die motion after the collisions is given by Eqs. (1)–(3) with new initial velocities: $\dot{x}'$, $\dot{y}'$, $\dot{z}'$, $\omega'_\xi$, $\omega'_\eta$, $\omega'_\zeta$ calculated from Eqs. (6)–(8) and the same initial positions $x$, $y$, $z$, $\psi$, $\vartheta$, $\varphi$ as before the impacts.

## IV. RESULTS

We perform some simple laboratory experiments that allow for monitoring of the die motion. The speed camera (Photron APX RS with the film speed at 1500 frames per second) has been used to observe the motion of a die. We observed the throw of the dice with different shapes. The dice have been released at the height of 60 cm by the special device. The examples of the die motion are shown in Figures 4(a) and 4(b). In Figure 4(a), we show the cubic die at the moments of successive bounces on the mirror plane and in Figure 4(b) the bounces of the icosahedron die on a cork plane are shown. Notice that after several bounces (11 for cube and 10 for icosahedron) the orientation of the die is not changed during the further bounces.

Our model (Eqs. (6)–(8)) allows the numerical calculations of the trajectories of dice corners (Figures 5(a) and 5(b)). In the numerical calculations, we assume typical parameters: the die mass $m = 0.016$ [kg], restitution coefficient—$\chi = 0.5$, tetrahedron die edge length—$a = 0.040793$ [m], and cube die edge length $a = 0.02$ [m]. The dice have been thrown from the height $z_0 = 0.3$ [m] with the angular velocity realistic for the throw from the hand. One can identify the position of the successive collisions (vertical lines in Figures 5(a) and 5(b)) and a die corner which collides with the table. In the considered example for the tetrahedron die during the successive bounces, the following corners collide with the table: A,B,A,C,D,A,B,A,B,D (Figure 5(a)). The cube die hits the table with C,D,D,C,B,A,D,B,B,B,B corners (Figure 5(b)). Notice that as in experiments of Figures 5(a) and 5(b) not all collisions result in the change of the die face which is the lowest one before and after the collision (For the tetrahedron die, we observe such a face change only once after the 4th collision and four changes take place for the cube die (after 3rd,4th,5th, and 7th collisions)—Figures 5(a) and 5(b)).

For $n$ face die there are $n$ possible final configurations (the die can land on one of its faces $F_i$ ($i = 1, 2, \ldots, n$). All

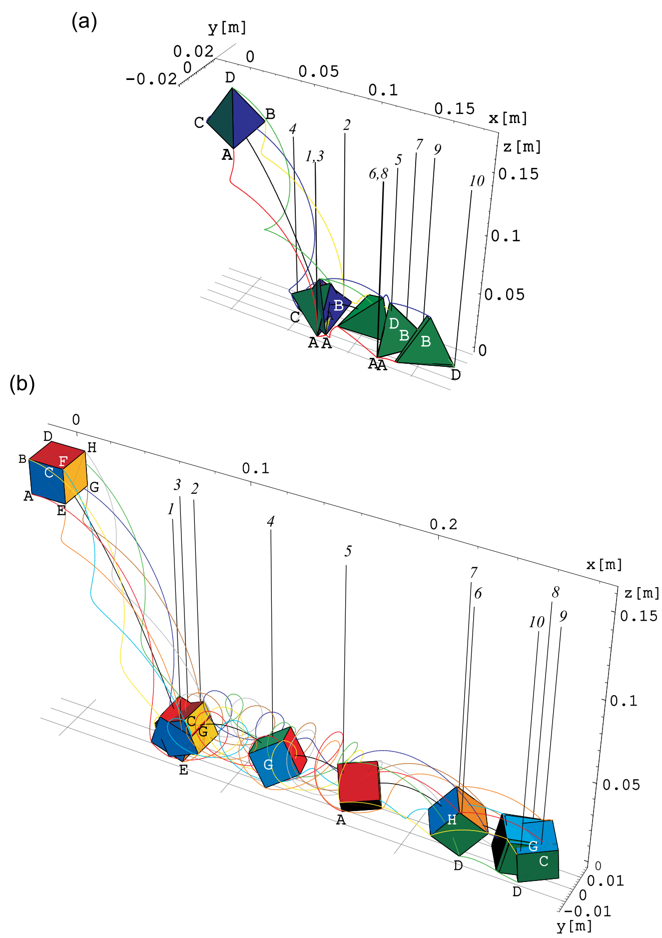


FIG. 5. Trajectories of dice vertices calculated from Eqs. (1), (2), and (6)–(8); $m = 0.016$ [kg], $\chi = 0.5$, $x_0 = y_0 = 0$, $v_{0x} = 0.6$ [m/s] $v_{0y} = 0$, $v_{0z} = -0.7$ [m/s] $\psi_0 = 0.001$ [rad], $\vartheta_0 = 0.00001$ [rad], $\varphi_0 = 0.0001$ [rad], $\omega_{\xi 0} = 0$, $\omega_{\eta 0} = 60$ [rad/s], $\omega_{\zeta 0} = 0$: (a) tetrahedron, $a = 0.040793$ [m], (b) cube, $a = 0.02$ [m].

initial conditions are mapped into one of the final configurations. The initial conditions which are mapped onto the $i$th face configuration create $i$th face basin of attraction $\beta(F_i)$. The boundaries which separate the basins of different faces consist of initial conditions mapped onto the die standing on its edge configuration which is unstable.

The structure of the boundaries between the basins of different faces is worth investigating. If in any neighborhood of the initial condition leading to one of $F_i$, there are initial conditions which are mapped to other faces (in the case of the coin in the neighborhood of any point leading to the heads there are points leading to the tail result), i.e., infinitely small inaccuracy in the initial conditions makes the result of the die throw unpredictable.

Let us indicate the die faces in different colors. For the tetrahedron die, we used blue, green, yellow, and red to indicate 1, 2, 3, and 4 faces. 12-dimensional space of all possible initial conditions has been divided into small boxes. The center of each box has been taken as the initial condition in the simulations of the die trajectory. After the identification of the face on which die lands the box is painted with an appropriate color. This procedure allows the identification of the basin of attraction of each die face (Figures 6(a)–6(d)). Of course one has to consider very small boxes and for better visualization it is necessary to consider 2-dimensional cross section of the initial condition space. In the presented calculations, we have fixed all initial conditions except the height from which the die is thrown $z_0$ and the die angular velocity around the axis $\eta$. Consideration of the influence of the number of the tetrahedron die bounces on the table $\bar{n}$ on the structure of the basin boundaries in height $z_0$-angular velocity $\omega_{0\eta}$ plane (Figures 6(a)–6(d)) leads to interesting results. For the small number of bounces, the basin boundaries are smooth (Figures 6(a)–6(c)). One can easily identify initial conditions which lead to the *a priori* known final states even when the set with known finite inaccuracy $\epsilon$. With the increase of the number of bounces $\bar{n}$ (this occurs when the energy dissipation decreases), it is possible to observe that the complexity of the basin boundaries increases (Figure 6(d)). In this case, to predict the result of the throw smaller inaccuracy $\epsilon$ is necessary. The similar mechanism of the fractalization has been previously observed for the tossed coin.[11,13] The same properties of the basin boundaries have been observed for several cases of different initial conditions as well as for the dice with different shapes (in our studies we consider tetrahedron, octahedron, dodecahedron, and icosahedron shapes). This allows us to state that if one can settle the initial conditions with appropriate accuracy, the outcome of the die throw is predictable and repeatable.

We perform the following numerical experiment.[12] For a given value of $\omega_{\eta 0}$ and fixed initial conditions $x_0 = y_0 = v_x = v_{y0} = v_{z0} = \omega_{\xi 0} = \omega_{\zeta 0} = 0$, we randomly choose $2 \times 10^6$ values of the rest of initial conditions from the following set: $z_0 \in [15a, 20a]$, $\psi_0 \in [0, 2\pi]$, $\vartheta_0 \in [0, 2\pi]$, $\varphi_0 \in [0, 2\pi]$ and integrate Eqs. (1), (2), and (6)–(8). Let $<p^*>$ be the average probability that the cubic die lands on the face which is the lowest one at the start. This probability can be related to the values of $\omega_{\eta 0}$ ($\omega_{\eta 0}$ determines the initial rotation energy of the system $E_{rot0} = \frac{1}{2} J \omega_{\eta 0}^2$). When the die lands on the soft table without bouncing this probability is equal to 0.548 (significantly different from the theoretical probability $<p> = 1/6 = 0.16(6)$). When the die is thrown from the hand or the cup, the realistic values of $\omega_{\eta 0}$ are 20–40 [rad/s] (for vigorous throw) and the typical number of bounces is about 4–5, $<p^*>$ is equal to 0.199. This value is closer but is still significantly different from the value of $<p>$. The probability $<p^*>$ is close to $<p>$ for large values of $\omega_{\eta 0}$ (e.g., $\omega_{\eta 0} = 300\,[\mathrm{rad/s}] - <p^*> = 0.121, \omega_{\eta 0} = 1000$

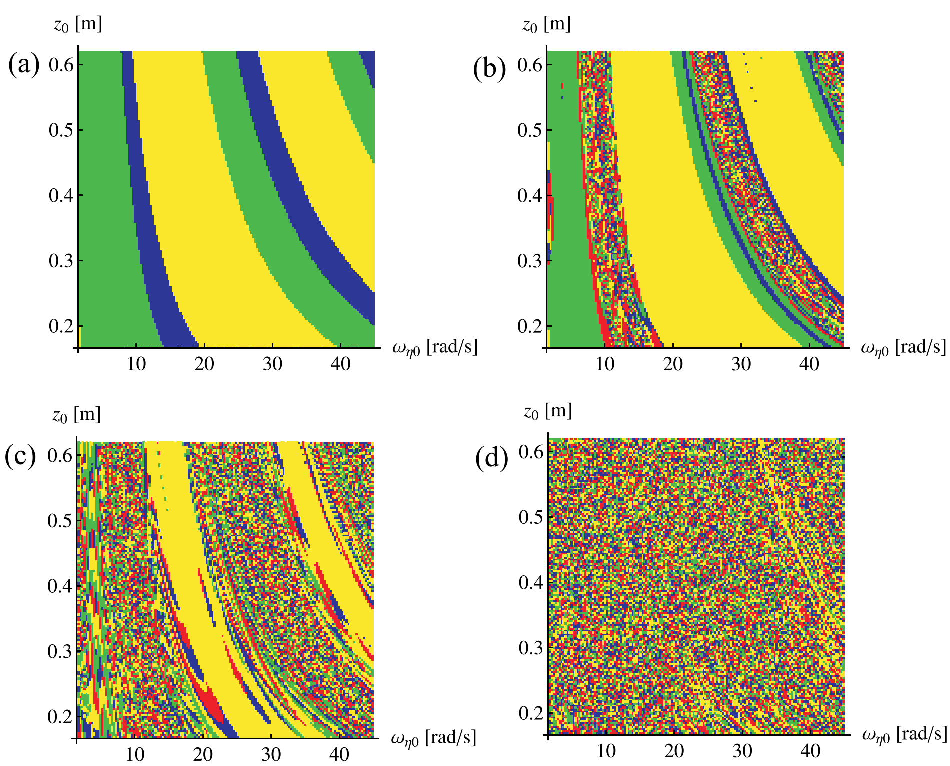


FIG. 6. Basins of attraction in $z_0 - \omega_{0\eta}$ plane for the tetrahedron die; basins of 1, 2, 3, and 4 faces are shown, respectively, in blue, green, yellow, and red; Eqs. (1), (2), and (6)–(8) have been integrated for the following initial conditions: $x_0 = y_0 = 0$, $v_{0x} = v_{0y} = v_{0z} = 0$ $\psi_0 = 0.3\,[\mathrm{rad}]$, $\vartheta_0 = 1.2\,[\mathrm{rad}]$, $\varphi_0 = 0.6\,[\mathrm{rad}]$, $v_{0x} = 0$, $v_{0y} = 0$, $v_{0z} = 0$, $\psi_0 = 0.3\,[\mathrm{rad}]$, $\omega_{0\xi} = 0$, and $\omega_{0\zeta} = 0$ (a) $\chi = 0.05$, (b) $\chi = 0.2$, (c) $\chi = 0.5$, (d) $\chi = 1$.

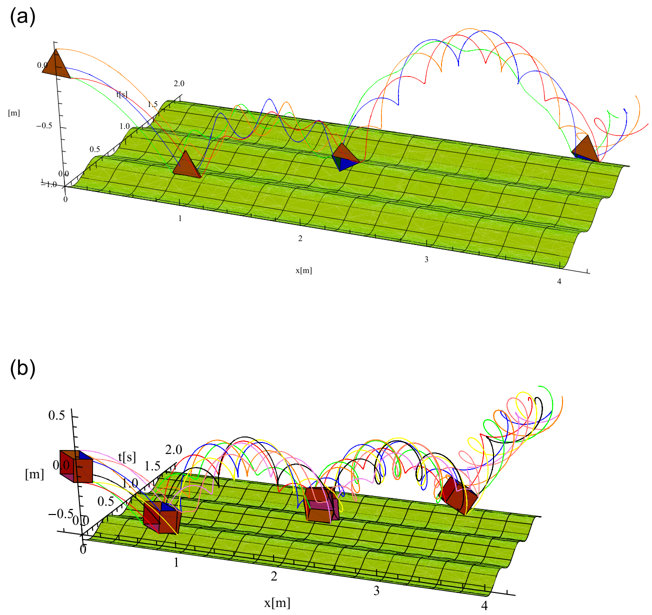


FIG. 7. Trajectory of the die bouncing on the periodically oscillating table, (a) tetrahedral, (b) cube.

$[\mathrm{rad/s}] - \langle p^* \rangle = 0.180$) when the die typically bounces on the table between 15 and 25 times before it no longer can change its orientation. This numerical experiment (results for the dice with different shapes are given in Ref. 22) gives evidence that the die face which is the lowest at the beginning is significantly more probable than other faces so in the realistic mechanical experiment the dice are not fair. It is not enough for a die which is fair by symmetry to be fair by dynamics.

Finally, let us consider two unrealistic cases of the die throw. The first one is the Hamiltonian system in which the elastic collisions with no energy dissipation are assumed. For this case, the series of die vertices which collides with the table in the successive bounces E,A,A,C,D,D,E,E,C,C, B,A,D,B,B,B,B,C,A,A,B,… is chaotic. The largest Lyapunov exponent is equal to 0.076 for the tetrahedron and 0.067 for the cube. It seems that only in this case the throw of the die can be considered as a chaotic process and the basins of different faces are fulfilling conditions of Definition 2 (up to the numerical accuracy—Figure 6(d)) and the results of the die throw are unpredictable. The second one is the case of the vibrating table. The example of a tetrahedron die bouncing on the periodically oscillating table is shown in Figures 7(a) and 7(b). In this case, the motion of the die is not terminated by the energy dissipation during the impacts and the die motion is chaotic. The largest Lyapunov exponent is equal to 0.169 for the tetrahedron and 0.183 for the cube. The analogy of the second case to the well-known chaotic dynamics of the bouncing ball[20–22] is presented. Indeed, if one considers the case in which the number of the die faces increases (theoretically to infinity) the shape of the die approaches the shape of the ball.

## V. CONCLUSIONS

To summarize, the die throw is neither random nor chaotic. From the point of view of dynamical system theory, the result of the die throw is predictable. Practically, the predictability can be realized only when the die is thrown by a special device which allows to set very precisely the initial conditions. We show that the probability, that the die lands on the face which is the lowest, is larger than on any other face, i.e., the dice are not fair by the dynamics. It is not enough for a die which is fair by symmetry to be fair by dynamics. By mechanical experiments or simulations, one cannot construct the die which is fair by continuity. If an experienced player can reproduce the initial conditions with small finite uncertainty, there is a good chance that the desired final state will be obtained.

The probabilities of landing on any face approach the same value $1/n$ only for the large values of the initial rotational energy and a great number of die bounces on table $\bar{n}$. In the limit case when $\bar{n} \to \infty$, the die throw can be considered as a chaotic process. This can be done in computer simulations but not in the real experiment when a die is thrown from the hand or the cup as due to the limitation of the initial energy the die can bounce only a few times. The dynamics of the die is also chaotic in the case of the die bouncing on the oscillating table.

Our studies give evidence that the origin of randomness in mechanical systems is connected with the discontinuity of the phase space (in the case of the die throw this discontinuity occurs during the bounces on the table). The grazing bifurcation[23–27] characteristic for the discontinuous systems leads to the sensitivity to initial conditions and chaotic behavior. As it has been already stated in Ref. 14, the mechanical gambling devices (dice, roulette, etc.) cannot show chaotic behavior as its evolution is finite due to the energy dissipation. In this case, the phenomena connected with the grazing bifurcation result in the occurrence of the transient chaos and fractalization of the boundaries between the basins of the different final configurations. This can be also concluded from the studies of other mechanical randomizers, like for e.g., roulette, pinball machine, or Buffon's needle.[22,29–31]

The simplifications used in our model like the neglection of air resistance, friction between the die and the table, and the consideration of the energy dissipation also through the restitution coefficient have a negligible effect on the predictability of die throw. The consideration of the influence of the air resistance and different models of the contact between the coin and the table results only in the small quantitative difference in the die trajectories.[11,28] Our numerical simulations are in good agreement with the experimental observations using high-speed camera.

Before the appearance of chaos theory, concepts of determinism and predictability were mostly the same. Our studies show that the throw of the die is both deterministic and predictable (when the initial conditions are set with a sufficient accuracy). This confirms that closed dissipative mechanical systems cannot show chaotic behavior. When the die tossing is taken into the realms of conservative systems (vanishing dissipation) or open dissipative systems (oscillating table) one can identify chaotic behavior but both cases are far away from the classical understanding of die games.

Although the deterministic description of the mechanical systems is possible in some cases the use of probabilistic arguments is unavoidable. Three main situations can be distinguished: (1) if the number of degrees of freedom is very large (on the order of Avogadros number) a detailed dynamical description is not useful: one does not care about the velocity of a particular molecule in gas, all that is needed is the probability distribution of the velocities, (2) even when the number of degrees of freedom is small (but larger than three) the sensitivity to initial conditions of the chaotic dynamics makes determinism irrelevant in practice, because one cannot control the initial conditions with infinite accuracy as in the case of the die tossing. Our ignorance of initial conditions is translated into a probabilistic description, (3) in quantum mechanics (for details see Refs. 32 and 33).

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