Standard bases for local rings of branches and their modules of differentials

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Dedicated to Carlo Traverso on the occasion of his 60th birthday.

Abstract

This paper presents an effective method for computing Standard bases for the local ring of an algebroid branch and for its module of Kähler differentials. This allows us to determine the semigroup of values of the ring and the values of its Kähler differentials, which in the case of complex analytic branches are, respectively, important topological and analytical invariants.
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1. Introduction

The aim of this paper is to present algorithms for computing privileged bases for the local ring of an arbitrary algebroid branch and for its module of Kähler differentials. Since, in our situation, all germs are finitely determined, this will take care as well of the complex analytic case. The technique is based upon Gröbner bases and Buchberger’s algorithm extended for subalgebras of polynomial rings as in Robbiano and Sweedler (1988) and for ideals over subalgebras of polynomial rings as in Miller (1996), which we further extended and adapted in Hefez and Hernandes (2001) for complete subalgebras and submodules of the ring of formal power series.

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over a field $K$. Such bases will simply be called Standard bases, since they may be unified in the same setting.

The values of a minimal Standard basis for the local ring of a branch, with respect to the valuation determined by the branch, constitute the minimal system of generators of the semigroup of values $\Gamma$ of the branch; that is, the set of values of all elements in the local ring of the branch. In the case of plane complex analytic branches, this semigroup is classically known to be a complete discrete invariant of the equisingularity class of the branch; that is, it characterizes the topological type of the branch as an embedded germ in the plane.

The Standard basis for the module of Kähler differentials of the local ring of the branch, that we compute, will allow us to determine the set of natural numbers $\Lambda$ of values of all such differentials. For plane complex analytic branches, this set is not a topological invariant any longer, but it is an important analytic invariant of the branch.

As an application of our methods, we also present a simple way to compute, at least over the complex numbers, the Tjurina number of the generic curve in a given equisingularity class of irreducible algebroid plane curves (compare with the rather involved algorithm given in Peraire (1997)). Also, we show how to compute the Tjurina number of a complete intersection branch given in parametric form. This was so far unknown.

This work was done some years ago as part of the program of solving the analytic classification of plane branches, which we actually succeeded in realizing. The results in this paper were presented as a mini-course at the Brazilian Mathematical Colloquium in July 2001 (see Hefez and Hernandes (2001)), but they seem to have not been acknowledged by the specialists (see, for example, the recent paper Castellanos and Castellanos (2005), where parts of our result were rediscovered). For this reason, we decided to publish our work to reach a larger public.

This paper is organized as follows. In Section 2 we will survey briefly the general methods concerning Standard bases, adapted to our context. In Section 3 we specialize these methods to branches embedded in ambient spaces of arbitrary dimension and in any characteristic then show how to compute a Standard basis for the local ring of the branch, determining in this way the associated semigroup of values, when the curve is given either in cartesian form or parametrically. In Section 4 we show how to compute a Standard basis for the module of Kähler differentials of a branch and the set of values of these differentials.

2. Standard bases in $K[[X]]$

In this section we will summarize the theory of Standard bases for subalgebras and submodules of rings of formal power series. This is done in more detail in our book (Hefez and Hernandes, 2001), from which we recall some definitions and results.

Let $K[[X]] = K[[X_1, \ldots, X_n]]$ be a ring of formal power series over a field $K$, with maximal ideal $\mathcal{M}_X$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we will denote by $X^\alpha$ the monomial $\prod_{i=1}^n X_i^{\alpha_i}$ and by $T$ the set of all monomials in $K[[X]]$.

A monomial order on $K[[X]]$ is a total order $\preceq$ on $T$ such that $1 \preceq t$, and if $t_1 \preceq t_2$ then $tt_1 \preceq tt_2$ for all $t, t_1, t_2 \in T$. We will only consider on $T$ orders satisfying the finiteness property, that is, for every $t \in T$ we have

$$\# \{s \in T; s \preceq t \} < \infty.$$ 

The leading power of $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha \in K[[X]] \setminus \{0\}$ is $\text{lp}(f) = \min\{X^\alpha; \alpha \in \mathbb{N}^n$ and $a_\alpha \neq 0\}$. The height of a sum $\sum_{l \in L} f_l$, where $f_l \in K[[X]] \setminus \{0\}$ is $\text{ht}(\sum_{l \in L} f_l) = \min\{\text{lp}(f_l), l \in L\}$. Notice that the height depends upon the representation of $\sum_{l \in L} f_l$ as a
sum and not upon the element that this sum determines. In fact, we have that \( \text{ht}(\sum_{i \in L} f_i) \leq \text{lp}(\sum_{i \in L} f_i) \).

Let \( F = \{ f_1, \ldots, f_m \} \subset \mathcal{M}_X \setminus \{0\} \). We define the \( K \)-subalgebra \( K[[F]] \) of \( K[[X]] \) as
\[
K[[F]] = \{ g(f_1, \ldots, f_m); g \in K[[Y_1, \ldots, Y_m]] \}.
\]

If \( G \subset \mathcal{M}_X \setminus \{0\} \) (not empty and possibly infinite), we define the \( K \)-algebra
\[
K[[G]] = \bigcup_{F \subset G} K[[F]].
\]

From now on, we will only consider \( K \)-subalgebras of \( K[[X]] \) of the form \( K[[G]] \). All the \( K \)-subalgebras \( A \) and the \( A \)-submodules of \( K[[X]] \) we will consider will be complete with respect to the \( \mathcal{M}_X \)-adic topology.

An \( F \)-product in \( K[[F]] \) is an element of the form \( F^\alpha = \prod_{i=1}^m f_i^{\alpha_i} \), and a \( G \)-product is an \( F' \)-product for some finite subset \( F' \) of \( G \).

**Definition 2.1.** Let \( A = K[[G]] \) be a complete algebra and let \( M \) be a complete \( A \)-submodule of \( K[[X]] \). Let \( \emptyset \neq H \subset M \), where we assume \( H = \{1\} \) if \( M = A \). The pair \( (H, G) \) will be called a **Standard basis** for \( M \) if for every \( f \in A \) and every \( m \in M \), there exist \( h \in H \) and \( G \)-products \( P, Q \) such that \( \text{lp}(f) = \text{lp}(P) \) and \( \text{lp}(m) = \text{lp}(Q) \) \( \text{lp}(h) \).

If \( M = A \) and \( \langle 1 \rangle \) is a Standard basis for \( M \), then \( G \) is a Subalgebra Analog to Gröbner Bases for Ideals (SAGBI), for the subalgebra \( A \), which we still call the Standard basis. See Robbiano and Sweedler (1988) for the polynomial theory of the SAGBI, and Hefez and Hernandes (2001) for its extension to formal power series rings.

If \( A = K[[X]] \), then the \( A \)-module \( M \) is an ideal in \( K[[X]] \). If we take \( G = \{ X_1, \ldots, X_n \} \) and if \( (H, G) \) is a Standard basis for \( M \), then \( H \) is a Gröbner basis for \( M \), which we still call the Standard basis. For more details about Gröbner bases for ideals in \( K[[X]] \) see Becker (1990), Becker (1993) and Hefez and Hernandes (2001). Observe also that an ideal always has a finite Standard basis.

If the \( A \)-module \( M \) is not finitely generated, then it never admits a finite Standard basis. In Robbiano and Sweedler (1988) there is an example, that can be adapted to our situation, of a finitely generated subalgebra that doesn’t have a finite Standard basis.

We will establish an algorithm for computing a finite Standard basis, when the \( A \)-module \( M \) admits one. So, from now on we will only consider finitely generated subalgebras and submodules in \( K[[X]] \).

In the sequel we will use the following strategy: we will present the definitions, the results and the algorithm for an \( A \)-module \( M \), where \( A \) admits a Standard basis \( G \). To obtain a Standard basis \( G \) for the subalgebra \( A \), it will be sufficient to consider \( M = A \) in our results. In the particular case where \( M \) is an ideal in \( K[[X]] \) we take \( G = \{ X_1, \ldots, X_n \} \), which is obviously a Standard basis for \( A = K[[X]] \).

The following are fundamental concepts.

Let \( A = K[[G]] \), where \( \emptyset \neq G \subset \mathcal{M}_X \setminus \{0\} \), and let \( \emptyset \neq H \subset M \). We say that \( f \in K[[X]] \) reduces to \( r \) modulo \( (H, G) \), or \( r \) is a reduction of \( f \) modulo \( (H, G) \), writing
\[
f \xrightarrow{(H,G)} r,
\]
if there exist \( b \in K \), a \( G \)-product \( P \) and \( h \in H \) such that \( r = f - bPh \), with \( r = 0 \) or \( \text{lp}(r) > \text{lp}(f) \).
When $r$ is obtained from $f$ through a chain (possibly infinite) of reductions, modulo $(H, G)$, and cannot be reduced further, we say that $r$ is a final reduction of $f$ modulo $(H, G)$, and will write

$$f \xrightarrow{(H,G)^+} r.$$ 

Notice that if $r$ is obtained from $f$ by any chain of reductions, then $f - r \in M$. Indeed, suppose that $r$ is obtained from $f$ by a chain of reductions

$$f \xrightarrow{(H,G)} r_1 \xrightarrow{(H,G)} r_2 \xrightarrow{(H,G)} \cdots \xrightarrow{(H,G)} r_m \xrightarrow{(H,G)} \cdots r,$$

then there exist $a_i \in K \setminus \{0\}$, $G$-products $P_i$ and $h_i \in H$, such that

$$f - r_m = \sum_{i=1}^{m} a_i P_i h_i \in M,$$

where, because of the definition of a reduction,

$$\text{lp}(a_1 P_1 h_1) < \text{lp}(a_2 P_2 h_2) < \cdots < \text{lp}(a_m P_m h_m) < \cdots.$$

Now, the sum $r = \sum_{i=1}^{\infty} a_i P_i h_i$ exists because of the above inequalities and the fact that the order on $T$ has the finiteness property. Finally, $f - r \in M$ because $M$ is complete.

As in the classical theory of Gröbner bases, we also have the notion of minimal Standard basis; that is, a basis where each element cannot be reduced by the other ones.

The analog, in our context, of the $S$-polynomial in the theory of Gröbner bases is the following.

**Definition 2.2.** Let $\emptyset \neq G \subset M_X \setminus \{0\}$. An $S$-process of a pair of elements $f, g \in K[[X]]$ over $G$ is an element of the form $a Pf + b Q g$, with $a, b \in K$, $P, Q$ are $G$-products and $\text{lp}(a Pf + b Q g) > \text{ht}(a Pf + b Q g)$, if $a Pf + b Q g \neq 0$.

The rest of the section will be devoted to giving several characterizations for a Standard basis of an $A$-submodule $M$ of $K[[X]]$ and formulating an algorithm for obtaining it.

Observe that an $S$-process $a F^\alpha f + b F^\beta g$, over $G$, of a pair of elements $f, g$ in $K[[X]]$, where $F = \{f_1, \ldots, f_s\} \subset G$, is determined, up to a scalar multiple, by a vector $(\alpha, \beta) \in \mathbb{N}^{2s}$, which is a solution of the following system of linear diophantine equations:

$$\sum_{i=1}^{s} \alpha_i \deg_{X_j}(\text{lp}(f_i)) + \deg_{X_j}(\text{lp}(f)) = \sum_{i=1}^{s} \beta_i \deg_{X_j}(\text{lp}(f_i)) + \deg_{X_j}(\text{lp}(g));$$

$$j = 1, \ldots, n.$$

Consider the set of minimal solutions of the above system, and the set of minimal solutions of the associated homogeneous system (see for example Hefez and Hernandes (2001, p. 4 and p. 8)). These sets are finite and may be computed by known algorithms (see Contejean and Devie (1994) and Clausen and Fortenbacher (1989)). The $S$-processes of the pair $f, g$ associated with the elements in the set of minimal solutions will be called the minimal $S$-processes of the pair $f, g$, relative to the finite set $F$.

The next result will give several characterizations for a Standard basis for an $A$-module $M$. The proof will be omitted since it is similar to the standard proofs (cf. Adams and Loustaunau (1994) for ideals in polynomial rings, Robbiano and Sweedler (1988) for subalgebras.
in polynomial rings and Hefez and Hernandes (2001) in both cases and for submodules in rings of formal power series).

**Theorem 2.3.** Let $A = K[[G]]$, where $\emptyset \neq G \subset \mathcal{M}_X \setminus \{0\}$, and let $M$ be an $A$-submodule of $K[[X]]$, such that $A$ and $M$ are complete. Given $\emptyset \neq H \subset M$, the following assertions are equivalent:

(a) $(H, G)$ is a Standard basis for $M$.

(b) All final reductions, modulo $(H, G)$, of elements of $M$ are zero.

(c) $H$ is closed under $S$-processes; that is, every $S$-process of any pair of elements of $H$ over $G$ has a vanishing final reduction modulo $(H, G)$.

(d) Any non-zero $S$-process of a pair of elements of $H$ over $G$ has a representation of the form $\sum b_i P_i h_i$, where $b_i \in K$, $P_i$ is a $G$-product and $h_i \in H$, with height greater than the height of the $S$-process itself.

(e) Every minimal $S$-process of $H$ over $G$ has a vanishing final reduction modulo $(H, G)$.

As a consequence of the above theorem, one gets easily the following algorithm for computing Standard bases for submodules of $K[[X]]$.

**Theorem 2.4.** Let $A$ be a subalgebra of $K[[X]]$ with Standard basis $G$. If $M$ is a complete $A$-module generated by a finite subset $B$ of $K[[X]]$, then we always (at least theoretically) obtain a Standard basis $(H, G)$ for $M$ with the following algorithm:

**input:** $G$, $B$;

**define:** $H_0 := \emptyset$, $H_1 := B$ and $i := 0$;

**while** $H_i \neq H_{i+1}$ **do**

- $S := \{s; \ s$ is a minimal $S$-process of $H_i$ over $G\}$;
- $R := \{r; \ s \xrightarrow{(H_i, G)_+} r \text{ and } r \neq 0, \ \forall s \in S\}$;
- $H_{i+1} := H_i \cup R$;

**output:** $H = \bigcup_{i \geq 1} H_i$.

Moreover, if $M$ has a finite Standard basis, then the above procedure will produce such a basis after finitely many steps.

**Proof.** Let $f, g \in H = \bigcup_{i \geq 1} H_i$. A minimal $S$-process of the pair $f, g$ over $G$ is a minimal $S$-process of $f$ and $g$ relative to $H_i$, for some $i$.

By the algorithm, this $S$-process has a vanishing final reduction, modulo $(H_{i+1}, G)$, and consequently also modulo $(H, G)$. Hence $H$ is a Standard basis for $M$.

Suppose that $M$ has a finite Standard basis $L$. We will show that there exists an index $j$ such that $H_j$ is a Standard basis for $M$.

Let $q = \max\{\text{lp}(h); \ h \in L\}$. Since the monomial order has the finiteness property, then either there exists an index $j$ such that $H = H_j$ or the leading power of any element of $H \setminus H_j$ is greater than $q$.

Given $h \in L$, we have $\text{lp}(h) = \text{lp}(Ph)$ for some $G$-product $P$ and some $g$ in $H$, because $H$ is a Standard basis.

Since $\text{lp}(h) \leq q$, we have that $g \in H_j$. Hence, for all $h \in L$ we have that $\text{lp}(h) = \text{lp}(P'g)$ for some $G$-product $P'$ and $g \in H_j$.

In this way, given an element $m \in M$, we have that $\text{lp}(m) = \text{lp}(Ph) = \text{lp}(PP'g) = \text{lp}(Qg)$, where $P$, $P'$ and $Q = PP'$ are $G$-products, $h \in L$ and $g \in H_j$. Hence $H_j$ is a finite Standard basis for $M$. \qed
The above algorithm may be specialized to obtain a Standard basis for a finitely generated complete subalgebra $A$ of $K[[X]]$. In this case, we take for $B$ any finite set of generators of $A$, and take $H = \{1\}$. In the present case we will call an $S$-process of $H$ over $G$ simply an $S$-process of $G$ and a reduction modulo $(H, G)$ simply a reduction modulo $G$.

3. Application to the local ring of a branch

In what follows, $K$ will be an algebraically closed field. By a branch over $K$ we mean a prime ideal $C = (f_1, \ldots, f_r)$ of $K[[X_1, \ldots, X_n]]$, such that its associated local ring $\mathcal{O} = K[[x_1, \ldots, x_n]] = K[[X_1, \ldots, X_n]]/(fi, \ldots, f_r)$, has Krull-dimension one.

We will denote by $\mathcal{O}$ ($= K[[t]]$) the integral closure of $\mathcal{O}$ in its field of fractions and by $v = \text{ord}_t$ its normalized valuation, which is a monomial order on $K[[t]]$.

So, when we view the generators $x_1, \ldots, x_n$ of $\mathcal{O}$ as a set of elements $p_1(t), \ldots, p_n(t)$ in $\mathcal{O}$, called a parametrization of $C$, we have an isomorphism

$$\mathcal{O} = K[[x_1, \ldots, x_n]] \cong K[[p_1(t), \ldots, p_n(t)]] \subset K[[t]],$$

where the field of fractions of $K[[p_1(t), \ldots, p_n(t)]]$ is equal to the field of fractions of $K[[t]]$. We will frequently identify the local ring $\mathcal{O}$ with its isomorphic image $K[[p_1(t), \ldots, p_n(t)]]$ by $\varphi$.

The semigroup of values $\Gamma = v(\mathcal{O} \setminus \{0\})$, associated with $C$, may be described as follows:

$$\Gamma = \langle v_0, v_1, \ldots, v_g \rangle = \left\{ \sum_{i=0}^g \alpha_i v_i ; \ \alpha_i \in \mathbb{N} \right\} \subset \mathbb{N},$$

where $v_0 = \min (\Gamma \setminus \{0\})$ and $v_i = \min (\Gamma \setminus \{v_0, \ldots, v_{i-1}\})$, for $i > 0$. The natural numbers $v_0, v_1, \ldots, v_g$ form the minimal set of generators of $\Gamma$. The number $v_0$ is called the multiplicity of $\Gamma$ or of the branch $C$, and is denoted by $\text{mult} \ \Gamma$ or by $\text{mult} \ C$. The elements in $\mathbb{N} \setminus \Gamma$ will be called the gaps of $\Gamma$.

It is well known that $\text{GCD}(v_0, \ldots, v_g) = 1$, and consequently the semigroup of values $\Gamma$ has a conductor; that is, there exists $c \in \Gamma$ such that

$$c = \min \{v \in \Gamma ; \ \mathbb{N} + v \subset \Gamma\}.$$ 

This, in particular, implies that $\Gamma$ has finitely many gaps.

Zariski, in the series of papers (Zariski, 1965, 1968), introduced and studied extensively the fundamental notion of equisingularity of algebroid plane curves, related to the semigroup of values of the curve. In this context, two irreducible algebroid curves are said to be equisingular if they have the same semigroup of values.

The semigroup of values of any branch is related to Standard bases for $\mathcal{O}$ as follows:

**Proposition 3.1.** Let $G = \{h_0, \ldots, h_g\} \subset \mathcal{O}$. Then $G$ is a minimal Standard basis for $\mathcal{O}$ if, and only if, $\{v(h_0), \ldots, v(h_g)\}$ is a minimal set of generators of the semigroup of values $\Gamma$ of $\mathcal{O}$.

**Proof.** Suppose $G$ is a minimal Standard basis for $\mathcal{O}$. By definition of Standard basis, for every element $h \in \mathcal{O}$ there exists a $G$-product $P$ such that $\text{lp}(h) = \text{lp}(P)$. It follows that $v(h) = v(P) \in \{v(h_0), \ldots, v(h_g)\}$; that is, $\Gamma = \langle v(h_0), \ldots, v(h_g) \rangle$.

On the other hand, the minimality of $G$ means that $\text{lp}(h_i)$ doesn’t belong to the semi-group in $T$ generated by $\{\text{lp}(h_0), \ldots, \text{lp}(h_i), \ldots, \text{lp}(h_g)\}$, which is equivalent to...
$v(h_i) \notin \langle v(h_0), \ldots, v(h_j), \ldots, v(h_g) \rangle$. This shows that $\{v(h_0), \ldots, v(h_g)\}$ is a minimal set of generators of $\Gamma$.

The converse is clear from the definition of a Standard basis for $\mathcal{O}$. □

Observe that the above result guarantees that the local ring of a branch admits always a finite Standard basis, since its semigroup of values is finitely generated.

Recall that in the application of the algorithm, previously described, the $S$-processes are obtained by means of minimal solutions of a linear homogeneous diophantine equation. All diophantine equations we will have to consider are of the following particular form: $\sum_{i=1}^{s} a_i W_i = \sum_{j=1}^{s} a_j Z_j$, with $a_1, \ldots, a_s \in \mathbb{N}$. For all $j = 1, \ldots, s$, the previous equation has a minimal solution of the form $(0, \ldots, 1, \ldots, 0, 0, \ldots, 1, \ldots, 0)$, where the only non-zero entries are in positions $j$ and $s + j$. These solutions will determine identically zero $S$-processes, and hence irrelevant ones. On the other hand, if $(\alpha, \beta) \in \mathbb{N}^s \times \mathbb{N}^s$ is a minimal solution, so is $(\beta, \alpha)$. But these solutions determine, modulo a constant factor, the same $S$-process. Hence, it is sufficient to consider only one of them. From now on, when we mention the minimal $S$-processes, we will exclude the trivial ones and the redundancies described above.

The algorithm for determining a Standard basis for $\mathcal{O}$ will start by taking a representation of $\mathcal{O}$ as a subring $K[[F_0]]$ of $K[[t]](\cong \mathcal{O})$, with $F_0$ the image in $K[[t]]$ of a set of generators for $\mathcal{O}$; for example a parametrization $\{p_1(t), \ldots, p_n(t)\}$ of $C$. In the step $i$, the algorithm produces a finite set $F_i$ such that $K[[F_i]] = K[[F_0]]$. Suppose also that, in this step, one can get by some means an upper bound $c_i$ for the conductor $c$ of $\Gamma$. Since every minimal $S$-process of $F_i$ which after finitely many reductions has value greater than or equal to $c_i$ will have a zero final reduction modulo $F_i$, we may disregard it. This, in general, reduces drastically the number of final reductions of $S$-processes to be performed in order to produce the finite set $F_{i+1}$.

In this way, we get the following more efficient algorithm for producing a Standard basis for $\mathcal{O}$, with a parametrization of $C$ as an input:

**Algorithm 3.2.** Standard basis for $\mathcal{O}$:

```plaintext
input: $F_0 = \{p_1(t), \ldots, p_n(t)\}$
define: $F_{-1} := \emptyset$ and $i := 0$;
while $F_i \neq F_{i-1}$ do
  $c_i :=$ upper bound for the conductor of the semigroup generated by $v(F_i)$;
  $S := \{s; \ s$ is a minimal $S$-process of $F_i$, not computed in the
     previous step with $v(ht(s)) < c_i - 1\}$;
  $R := \{r; \ s \xrightarrow{F_i} r, \ \forall s \in S$ and $r \neq 0\}$;
  $F_{i+1} := F_i \cup R$;
output: $F = F_{i+1}$.
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Notice that the above method may be used, without restriction, in any characteristic.

The upper bounds $c_i$ mentioned in the above algorithm ought to be determined, if possible, independently from the algorithm by any means. For example, if $\rho$ is an integer such that the $v_0$ consecutive integers starting with $\rho$ are in $\Gamma$, then $c \leq \rho$.

The above algorithm is not particularly interesting when $C$ is a plane branch given by a Puiseux parametrization or by its cartesian equation. In fact, a method given by Zariski (see Zariski (1986, Théorème 3.9) or Hefez (2003, Theorem 6.12)) allows us to determine a Standard basis for $\mathcal{O}$ when $C$ is given by a Puiseux parametrization. On the other hand, when $C$ is given
by a cartesian equation, the Abhyankar–Moh approximate roots (cf. Abhyankar and Moh (1973)) will do the same.

We will now present some examples which show how the algorithm works in practice.

**Example 3.3.** Suppose char$(K) = 0$ and let $C$ be the space branch given by the following Puiseux parametrization:

$$x = t^8, \quad y = t^{10} + t^{13}, \quad z = t^{12} + at^{15}.$$

Take $F_0 = \{x, y, z\}$. Since $8, 10, 12 \in \Gamma$ and $27 = v(x^3 - z^2) \in \Gamma$ if $a \neq 0$ or $23 = v(y^2 - xz) \in \Gamma$ if $a = 0$, it is easy to verify, in any case, that the consecutive integers 54, 55, ..., 61 belong to $\Gamma$, and hence the conductor $c$ of $\Gamma$ is such that $c \leq 54$.

The minimal $S$-processes of $F_0$ are:

$$y^4 - x^5, \quad y^2 - xz, \quad z^2 - x^3, \quad y^2z - x^4, \quad z^3 - x^2y^2 \quad \text{and} \quad z^4 - x^6.$$

Reducing the above elements modulo $F_0$, we get:

If $a = 0$, then $F_1 = \{t^8, t^{10} + t^{13}, t^{12}, 2t^{23} + t^{26}\}$ is a minimal Standard basis for $\mathcal{O}$. In this case, $\Gamma = \langle 8, 10, 12, 23 \rangle$ and $c = 38$.

If $a = 2$, then $F_1 = \{t^8, t^{10} + t^{13}, t^{12} + 2t^{15}, 4t^{27} + 4t^{30}, -t^{29}\}$ is a minimal Standard basis for $\mathcal{O}$. In this case, $\Gamma = \langle 8, 10, 12, 23, 27, 29 \rangle$ and $c = 34$.

If $a \neq 0$ and $a \neq 2$, then $F_1 = \{t^8, t^{10} + t^{13}, t^{12} + at^{15}, (2-a)t^{23} + t^{26}, 2at^{27} + a^2t^{30}\}$ is a minimal Standard basis for $\mathcal{O}$. In this case, $\Gamma = \langle 8, 10, 12, 23, 27 \rangle$ and $c = 30$.

When the branch is given by a Cartesian representation, it is also possible to make the computations as we show below.

Let $C$ be a branch in $K[[X]]$, given by a Cartesian representation $f_1, \ldots, f_r$. It is well known, by elementary intersection theory, that the value $v(\bar{g})$ of an element $\bar{g} \in \mathcal{O}$ is the codimension in $K[[X]]$ of the ideal $I = \langle g, f_1, \ldots, f_r \rangle$, that may be computed by means of a Standard basis for the ideal $I$.

Let $F = \{\overline{h_1}, \ldots, \overline{h_s}\}$ be a set of nonzero elements of the maximal ideal of $\mathcal{O}$, and let $H = \{h_1, \ldots, h_s\} \subset K[[X]]$. Since we know how to compute values of elements in $\mathcal{O}$, we may determine all minimal solutions of the diophantine equation associated with the equality $v(F^\alpha) = v(F^\beta)$, where $\alpha, \beta \in \mathbb{N}^s$. Now, to produce the minimal $S$-process of $F$, associated with a minimal solution $(\alpha, \beta)$ of the diophantine equation, we must find the unique $a \in K$ such that

$$v(F^\alpha - a F^\beta) > v(F^\alpha) = v(F^\beta).$$

In the same way one can perform the reduction of an element of $\mathcal{O}$ modulo $F$. This is all we need to apply the algorithm to get a Standard basis for $\mathcal{O}$.

**Example 3.4.** Suppose that char$(K) = 0$ and let $C$ be the branch defined by

$$f_1 = 3Z^2 - 4X^2Z + XY^2 - 3X^2Y - 4X^4,$$

$$f_2 = 24Y^2Z - 18XYZ - 224X^3Z + 9Y^3 + 32X^2Y^2 - 96X^3Y - 128X^5 + 9X^4.$$

Start with $F = \{x, y, z\}$. Determining Standard bases for the ideals $\langle f_1, f_2, X \rangle$, $\langle f_1, f_2, Y \rangle$ and $\langle f_1, f_2, Z \rangle$, and computing their codimensions we find $v(x) = 6$, $v(y) = 8$ and $v(z) = 10$.

The minimal $S$-processes of $H = \{X, Y, Z\}$ are: $Y^2 - a_1XZ$, $YZ - a_2X^3$, $Z^2 - a_3X^2Y$, $X^4 - a_4Y^3$, $XY^3 - a_5Z^3$, $Z^3 - a_6X^5$ and $Y^5 - a_7Z^4$.

Computing a Standard basis for the ideals $\langle f_1, f_2, g \rangle$ where $g$ in one of the above elements, we observe that, in order to obtain an $S$-process, we must consider $a_i = 1$, for $i = 1, \ldots, 7$. 
The first two $S$-processes give us the values 17 and 19, so $\{6, 8, 10, 17, 19\} \subseteq \Gamma$, and it is easy to verify that $c \leq 22$. But, observe that the heights of the other $S$-processes are greater than 22, the same occurring with the $S$-processes of $G = \{X, Y, Z, Y^2 - XZ, YZ - X^3\}$. This implies that $\Gamma = \{6, 8, 10, 17, 19\}$ is the semigroup of values of $C$.

4. Application to the module of Kähler differentials

Let $C = \langle f_1, \ldots, f_r \rangle$ be a branch in $K[[X_1, \ldots, X_n]]$, where $K$ is an algebraically closed field of characteristic zero.

**Definition 4.1.** The module of Kähler differentials over $\mathcal{O}$ is the $\mathcal{O}$-module

$$\mathcal{O}d\mathcal{O} = \frac{\mathcal{O}^n}{(\sum_{i=1}^{n} e_i fX_i ; \ f \in C)},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathcal{O}^n$.

We will denote by $dx_i$ the image of $e_i$ in $\mathcal{O}d\mathcal{O}$, for $i = 1, \ldots, n$. Therefore, the elements $dx_i$, $i = 1, \ldots, n$, are non-free generators of $\mathcal{O}d\mathcal{O}$ as $\mathcal{O}$-module. Indeed, they admit the following relations: $\sum_{i=1}^{n} fX_i dx_i = 0$, for all $f \in C$.

The $\mathcal{O}$-module $\mathcal{O}d\mathcal{O}$ has, in general, a non-trivial torsion submodule $T$. For example, for the plane branch given by $Y^r - X^s$, where $\min(r, s) > 1$ and $\gcd(r, s) = 1$, the non-zero differential $\omega = rx dy - sy dx$ is such that $y^{r-1}\omega = x(fy dy - fX dx) = 0$.

Let $C$ be a branch in $K[[X_1, \ldots, X_n]]$, and let $p_1(t), \ldots, p_n(t)$ be a primitive parametrization of $C$, so $\mathcal{O} \simeq K[[p_1(t), \ldots, p_n(t)]]$. Consider the $\mathcal{O}$-modules homomorphism

$$\psi : \mathcal{O}d\mathcal{O} \longrightarrow K[[t]], \quad \sum_{j=1}^{n} g_j dx_j \mapsto \sum_{j=1}^{n} \phi(g_j) \frac{dp_j(t)}{dt}.$$  

It is not difficult to see that the kernel of the homomorphism $\psi$ is the torsion submodule $T$ of $\mathcal{O}d\mathcal{O}$ (see for example Hefez and Hernandes (2001, Proposition 1, page 93)).

This implies that

$$\frac{\mathcal{O}d\mathcal{O}}{T} \cong \text{Im}(\psi) := \Omega.$$  

**Definition 4.2.** If $\omega \in \mathcal{O}d\mathcal{O} \setminus T$, then we define the value of $\omega$ as

$$v(\omega) = v(\psi(\omega)) + 1,$$

where the $v$ in the above right-hand side is the normalized valuation of $\overline{\mathcal{O}}(\simeq K[[t]])$.

**Definition 4.3.** We will say that a differential $\omega \in \mathcal{O}d\mathcal{O}$ is an exact differential if there exists $g \in \mathcal{O}$ such that $\omega = dg$. If this is not the case, we say that $\omega$ is a non-exact differential (NED).

**Remark 4.4.** Let $\Gamma$ be the semigroup of values of the curve $C$ and $c$ its conductor. If $\omega \in \mathcal{O}d\mathcal{O}$ is an exact differential, then $v(\omega) \in \Gamma$. Equivalently, if $v(\omega) \not\in \Gamma$, then $\omega$ is a NED.

On the other hand, if $v(\omega) \geq c$, then there exists an exact differential $dh$ such that $\omega - dh \in T$.
The proof of the last assertion is as follows. Since \( v(\omega) \geq c \), there exists \( h_1 \in \mathcal{O} \) such that \( v(\omega) = v(dh_1) \). Hence there exists \( a_1 \in K \) such that \( v(\omega - a_1dh_1) > v(\omega) \). In the same way we get recursively a summable family \( \{a_idh_i; \ a_i \in K, \ h_i \in \mathcal{O}, \ i \geq 1\} \) such that \( v(\omega - \sum_{i \geq 1} a_i dh_i) = \infty \), which implies that \( \psi(\omega - \sum_{i \geq 1} a_i dh_i) = 0 \). Hence \( \omega - \sum_{i \geq 1} a_i dh_i = \omega - d(\sum_{i \geq 1} a_i h_i) \in \mathcal{T} \).

**Definition 4.5.** We define \( \Lambda = v(\mathcal{O}d\mathcal{O} \setminus \mathcal{T}) := v(\Omega \setminus \{0\}) + 1 \).

Observe that for all \( h \in \mathcal{O} \) we have that \( v(dh) = v(h) \). This in particular implies that \( \Gamma \setminus \{0\} \subset \Lambda \).

The set \( \Lambda \) is what is called a \( \Gamma \)-monomodule, since it has the following property:

\[
\gamma + \lambda \in \Lambda, \quad \forall \gamma \in \Gamma, \ \forall \lambda \in \Lambda.
\]

The following lemma will show that \( \Lambda \) is finitely generated over \( \Gamma \).

**Lemma 4.6.** There exist \( \lambda_1, \ldots, \lambda_r \in \Lambda \) with the following property: for every element \( \lambda \in \Lambda \) there exist \( i = 1, \ldots, r \) and \( \gamma \in \Gamma \) such that \( \lambda = \gamma + \lambda_i \).

**Proof.** Consider the following sequence of integers:

\[
\lambda_1 = \min \Lambda, \quad \lambda_2 = \min \Lambda \setminus (\lambda_1 + \Gamma), \ldots, \lambda_i = \min \Lambda \setminus \bigcup_{j=1}^{i-1} (\lambda_j + \Gamma), \ldots.
\]

The proof will be complete if we show that the number of such \( \lambda_i \)'s is finite. If they were infinitely many, there would exist some \( i > 1 \) such that \( \lambda_i - \lambda_1 > c \), because the \( \lambda_j \)'s form an increasing sequence. This implies that \( \lambda_i \in \lambda_1 + \Gamma \), a contradiction. \( \square \)

**Corollary 4.7.** There exists a finite Standard basis for \( \Omega \).

**Proof.** The elements \( \psi(\omega_1), \ldots, \psi(\omega_r) \in \Omega \) such that \( v(\omega_i) = \lambda_i \) form a Standard basis for \( \Omega \). \( \square \)

We will transfer all notions such as \( S \)-process, reduction, etc., from \( \Omega \) to \( \mathcal{O}d\mathcal{O}/\mathcal{T} \) through the isomorphism \( \psi \). For example, a set \( H = \{\omega_1, \ldots, \omega_r\} \) will be called a Standard basis for \( \mathcal{O}d\mathcal{O}/\mathcal{T} \) if \( \psi(H) = \{\psi(\omega_1), \ldots, \psi(\omega_r)\} \) is so for \( \Omega \).

**Remark 4.8.** The set \( \Lambda \) plays an important role in the local theory of irreducible curves. When \( C \) is a local complete intersection, the cardinality of \( \Lambda \) is related to \( \mu \) and \( \tau \), the Milnor’s and Tjurina’s numbers of \( C \), as follows.

According to Buchweitz and Greuel (1980, Proposition 1.2.1), Milnor’s number \( \mu \) is equal to twice \( \delta \) (the codimension of \( \mathcal{O} \) in \( \mathcal{O} \)). Since \( C \) is Gorenstein, it follows that \( \delta \) is half the conductor \( c \) of the semigroup \( \Gamma \). So, \( \mu = c \).

On the other hand, Tjurina’s number \( \tau \), defined as the dimension of the complex vector space of first order deformations \( T^1 \) of \( C \), is equal to the length \( l(\mathcal{T}) \) of the torsion submodule \( \mathcal{T} \) of \( \mathcal{O}d\mathcal{O} \) (see Pinkham (1974, Lemma 10.4)).

Now, Berger in Berger (1963) proved that \( l(\mathcal{T}) = c - \#(\Lambda \setminus \Gamma) \). Then, one has that

\[
\tau = \mu - \#(\Lambda \setminus \Gamma).
\]
Now, we will refine the algorithm of Theorem 2.4. Let $B \subset K[[r]]$ and let $G$ be a Standard basis of algebras for $\mathcal{O}$. Notice that any minimal $S$-process of a pair $g, g$ in $B$, over $G$, has a zero reduction modulo $(B, G)$, so it doesn't need to be considered in the algorithm.

The algorithm starts with a set $H_0$ of generators of the module for which we want to compute a Standard basis. In the particular case of $\mathcal{O}_T \mathcal{O}/T$ we take for example $H_0 = \{d\mathcal{x}_1, \ldots, d\mathcal{x}_n\}$ as a set of generators, where $d\mathcal{x}_i$ is the image of $d\mathcal{x}_i$ in $\mathcal{O}_T \mathcal{O}/T$.

We may improve the algorithm starting instead with the following set of generators:

$$H_0 = \{dh; \ h \text{ belonging to a minimal Standard basis of } \mathcal{O}\}.$$ 

This will avoid some unnecessary computations and at the same time will allow more reductions at each step of the algorithm, possibly eliminating some steps.

Besides the above economy in the algorithm, we may use the concept of greatest gap to eliminate some irrelevant $S$-processes, as we show below.

**Definition 4.9.** The greatest gap of $A$ is $\max \mathbb{N} \setminus A$.

Observe that one always has $l \notin \Gamma$ and $l \leq c - 1$, where $c$ is the conductor of $\Gamma$.

Let $l$ be the greatest gap of $A$. In a given step $i$ of the algorithm of Theorem 2.4, consider the set $A_i = \{v(P\omega); \ \omega \in H_i \text{ and } P \text{ is a } G\text{-product}\}$, and denote by $l_i$ its greatest gap, which is obviously greater than or equal to $l$. Since every minimal $S$-process of $H_i$ over $G$ with value of its height greater or equal than $l_i$ has a zero final reduction modulo $(H_i, G)$, it can be neglected.

In this way, we get the following improvement of the algorithm for computing a Standard basis $H$ for $\mathcal{O}_T \mathcal{O}/T$, starting with a Standard basis $G$ of $\mathcal{O}$.

**Algorithm 4.10.** Standard basis for $\mathcal{O}_T \mathcal{O}/T$:

```plaintext
input: G;
define: $H_{-1} := \emptyset, \ H_0 := \{dh; \ h \in G\}$ and $i := 0$;
while $H_i \neq H_{i-1}$ do
   $A_i := \{v(P\omega); \ \omega \in H_i \text{ and } P \text{ is a } G\text{-product}\}$
   $l_i := \text{greatest gap of } A_i$
   $S := \{s; \ s \text{ is a minimal } S\text{-process of } H_i \text{ over } G \text{ with } v(\text{ht}(s)) < l_i,$
   not computed in the previous step $\}$
   $R := \{r; \ r \xrightarrow{(H_i,G)} s, \forall s \in S \text{ and } r \neq 0\}$
   $H_{i+1} := H_i \cup R$;
output: $H = H_{i+1}$.
```

Notice that the algorithm computes exclusively NED. The NED belonging to a minimal Standard basis of $\mathcal{O}_T \mathcal{O}/T$ will be called *minimal non-exact differentials*, or simply MNED.

**Remark 4.11.** The maximum number of steps in the above algorithm is equal to $v_0 - 2$, where $v_0 = \text{mult } \Gamma$.

Indeed, observe firstly that in each step $i$ of the algorithm, the NED of minimal value is a MNED, because otherwise it would be of the form $h\omega \in H_i$ with $h \in \mathcal{O}$ and $\omega$ a NED obtained in a previous step. This is not possible since $h\omega$ has a zero reduction modulo $(H_{i-1}, G)$. Secondly, remark that if $\omega_1$ and $\omega_2$ are two distinct elements of a minimal Standard basis of $\mathcal{O}_T \mathcal{O}/T$, then their values are not congruent mod $v_0$, because otherwise $v(\omega_1) = v(h\omega_2)$, with $h \in \mathcal{O}$.
such that \( v(h) \) is the appropriate multiple of \( v_0 \). This is a contradiction since \( \omega_1 \) is a MNED. Finally, observe that in the first step there are at least two minimal differentials \( dh_0 \) and \( dh_1 \) where \( v(h_0) = v_0 \) and \( v(h_1) = v_1 \).

A central problem in the theory of plane branches defined over \( \mathbb{C} \) is their classification modulo the equivalence relation we define below.

**Definition 4.12.** Let \( C_1 \) and \( C_2 \) be two plane branches, given by \( f_1 \) and \( f_2 \) in \( \mathbb{C}[X, Y] \), respectively. We will say that \( C_1 \) is *equivalent* to \( C_2 \), if there exist a unit \( u \) and an automorphism \( \Phi \) of \( \mathbb{C}[X, Y] \), such that \( \Phi(f) = u g \).

It is well known that the sets \( \Gamma \) and \( \Lambda \), and hence also the set \( \Lambda \setminus \Gamma \), are invariant under the above equivalence relation (see for example *Delorme (1978)*). Observe also that the above equivalence for plane branches is the same as Mather’s \( K \)-equivalence (see *Gibson (1979)*, for the definition).

An interesting application of Algorithm 4.10 is that it gives an answer to the problem of determining Tjurina’s number of the generic plane branch with a given semigroup \( \Gamma = \langle v_0, v_1, \ldots, v_g \rangle \). Since the branch is plane, we have that \( v_{i+1} > n_i v_i \) for all \( i = 0, \ldots, g - 1 \), where \( n_i = \frac{e_{i-1}}{e_i} \) and \( e_i = \text{GCD}(v_0, \ldots, v_i) \) (see *Zariski (1986)*, or *Hefez (2003)*). Define the *characteristic integers* \( \beta_0, \ldots, \beta_g \) of \( \Gamma \) as follows: \( \beta_0 = v_0, \beta_1 = v_1 \) and

\[
\beta_{i+1} = v_{i+1} - n_i v_i + \beta_i, \quad i = 1, \ldots, g - 1.
\]

It is also well known (see *Zariski (1986)*) that any plane branch belonging to the equisingularity class determined by \( \Gamma \) is equivalent to a branch belonging to the following family:

\[
x = t^{v_0}, \quad y = t^{v_1} + \sum_{j \in J} a_j t^j,
\]

with \( J = \{ j \in \mathbb{N}; v_1 < j \leq c - 1, e_{i-1} | j \text{ for } j < \beta_i, i = 2, \ldots, g \} \), where \( c \) is the conductor of \( \Gamma \), given by \( c = \sum_{i=1}^g (n_i - 1) v_i - v_0 + 1 \).

Now, apply Algorithm 4.10 to the curve determined by the above Puiseux expansion keeping the coefficients \( a_j \) general (i.e. without any relation among them). From the resulting Standard basis we easily obtain the finite set \( \Lambda \setminus \Gamma \) corresponding to the values of the classes of non-exact differentials in \( \mathcal{O}d\mathcal{O}/T \), for a general \( C \), whose cardinality when subtracted from \( c \) will give the Tjurina number of the generic branch in the equisingularity class determined by \( \Gamma \) (compare this with the algorithm presented in *Peraire (1997)*).

In *Hefez and Hernandes (2003)* we used the above method to compute the Tjurina number of the generic plane branch belonging to the equisingularity class determined by the semigroup of values \( (6, 9, 19) \), as well all the possible Tjurina numbers of branches is this class. This was related to Heinrich’s counterexample for a conjecture of Azevedo (see *Heinrich (1995)* or *Berger (1994)*).

The example below is taken from *Azevedo (1967, page 79)*, where using rudimental methods some of the NED were computed. The existence of a differential with value 51 wasn’t detected there, leaving the example in *Azevedo (1967)* incomplete.

**Example 4.13.** Consider

\[
C : \quad x = t^8, \quad y = t^{12} + t^{13},
\]

whose associated semigroup of values is \( \Gamma = \langle 8, 12, 25 \rangle \), with conductor \( c = 80 \).
Applying the above algorithm starting with $H_0 = \{dx, dy, dz\}$, where $z = y^2 - x^3$, whose value is 25, we find the following Standard basis for $OdO/T$:

$$3ydx - 2xdy = \omega_1; \quad \psi(\omega_1) = -2t^{20}.$$  
$$8xdz - 25zdxd = \omega_2; \quad \psi(\omega_2) = 8t^{33}.$$  
$$12ydz - 25zdy = \omega_3; \quad \psi(\omega_3) = -38t^{37} - 13t^{38}.$$  

$$25x^2zdxd - 8y^2dz \xrightarrow{20z^2dz} \omega_4; \quad \psi(\omega_4) = \frac{204}{25}t^{50} + \frac{52}{25}t^{51}.$$  

$$yzdx + 8x^3\omega_1 \xrightarrow{6z\omega_1} \omega_5; \quad \psi(\omega_5) = -4t^{46}.$$  

From Remark 4.8 we have that

$$\tau = l(T) = c - \#(\Lambda \setminus \Gamma)$$  

**Example 4.14.** Consider

$$C : \quad x = t^6, \quad y = t^8 + 2t^9, \quad z = t^{10} + t^{11}.$$  

The curve $C$ is a parametric representation of the curve in Example 3.4, so its local ring has the following minimal Standard basis:

$$F = \{x, y, z, w = xz - y^2 = -3t^{17} - 4t^{18}, u = yz - x^3 = 3t^{19} + 2t^{20}\}.$$  

The semigroup of values of $C$ is $\Gamma = \{6, 8, 10, 17, 19\}$, with $c = 22$. Applying the **Algorithm 4.10**, starting with $H_0 = \{dx, dy, dz, dw, du\}$ whose greatest gap is $l_0 = c - 1 = 21$, we have the following set of minimal $S$-processes:

$$3xdy - 4ydx, \quad 3ydy - 4x^2dx, \quad 4xdz - 5ydy, \quad 3xdz - 5zdx, \quad 3zdy - 4x^2dz, \quad 4ydz - 5zdy, \quad 3ydz - 5x^2dz, \quad 3zdz - 5xyz, \quad 4zdz - 5x^2dy.$$  

Let $\omega = 3xdy - 4ydx$; then $\psi(\omega) = 6t^{14}$. Hence, $u(\omega) = 15$ and $v(x\omega) = 21$. So, the greatest gap of $H_1$ is $l_1 = 13$, showing that the other $S$-processes reduce to zero modulo $(H_1, G)$ and also that there are no further $S$-processes to be analyzed in the following steps. Therefore the algorithm stops giving the following minimal Standard basis for $OdO/T$:

$$H = H_1 = \{dx, dy, dz, dw, du, \omega\}.$$  

Since $C$ is a complete intersection, by **Remark 4.8**, we have that

$$\tau = l(T) = c - \#(\Lambda \setminus \Gamma) = 22 - \{15, 21\} = 20.$$  

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