Sound and complete axiomatizations of coalgebraic language equivalence

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Abstract. Coalgebras provide a uniform framework to study dynamical systems, including several types of automata. In this paper, we make use of the coalgebraic view on systems to investigate, in an uniform way, under which conditions sound and complete calculi with respect to behavioral equivalence can be extended to a coarser equivalence, which arises from a generalized powerset construction. We illustrate the framework with two examples: non-deterministic automata, for which we recover results by Rabinovich, and weighted automata, for which we present the first sound and complete calculus of weighted language equivalence.

Keywords: regular expressions, language, trace, coalgebra

1 Introduction

Coalgebras provide a uniform theory of state based systems and various kinds of infinite data structures, such as streams or infinite trees, (non-)deterministic and weighted versions of automata, labelled transition systems etc. For an endofunctor $F$ on a category $\mathcal{A}$, an $F$-coalgebra is a pair $(X,f)$, where $X$ is an object of $\mathcal{A}$ representing the state space and $f: X \to FX$ is an arrow of $\mathcal{A}$ defining the observations and transitions of the states. The strength of coalgebraic modelling lies in the fact that the type $F$ of the system determines a standard notion of equivalence called $F$-behavioral equivalence and canonical representatives of behavior, the so-called final coalgebra into which any $F$-coalgebra is mapped by a unique homomorphism that identifies all equivalent states.

The coalgebraic perspective on systems has recently been proved very relevant by the development of a number of expression calculi, sound and complete with respect to behavioral equivalence, and Kleene theorems for many types of automata, including Mealy automata \cite{11}, automata whose type is given by Kripke polynomial functors \cite{34}, automata for the so-called quantitative functors \cite{9,10} and closed stream circuits \cite{24} (which are weighted automata over a one-letter alphabet). This work generalizes Kleene’s classical theorem \cite{19} as well as Kozen’s soundness and completeness of Kleene algebra \cite{20} from automata theory to coalgebras.

It has also recently been shown \cite{33} that the classical powerset construction, which transforms a non-deterministic automaton into a deterministic one, providing language
semantics to the former, can be extended to a large class of systems, coalgebras for a given type, which includes probabilistic and weighted automata. The aforementioned paper models systems as the composite of a functor type $F$ and a monad $T$, which encodes the non-determinism or probabilities that one wants to determinize. The determinized coalgebra is actually a coalgebra in the category of Eilenberg-Moore algebras for the monad $T$. We will call the equivalence obtained by this construction, that is the $F$-behavioral equivalence in the category of $T$-algebras, coalgebraic language equivalence. For example, the construction above applied to non-deterministic automata yields a deterministic automaton in the category of join-semilattices. Coalgebraic language equivalence corresponds to ordinary language equivalence, while $FT$-behavioral equivalence is just ordinary bisimilarity [33].

In the present paper, we investigate under which conditions a sound and complete calculus with respect to behavioral equivalence can systematically be extended to a sound and complete calculus with respect to coalgebraic language equivalence. In the running examples, we will use as starting point the calculi from [34,10] mentioned above, but all the results and methodologies are formulated in general, for any given calculus sound and complete with respect to behavioral equivalence.

The contributions of this paper, which we next present in detail, can be divided into two groups: abstract category-theoretic results (Sections 3 and 4) and results for concrete calculi (Sections 5 and 6). The abstract results provide a mathematical theory and generic tools applicable in the concrete instances to reduce the work necessary in the proofs, whence leading to a pleasing simplicity of our results on concrete calculi. For instance, we explain how our category-theoretic work implies that a Kleene theorem for behavioral equivalence will always hold for coalgebraic language equivalence.

We start by systematically studying coalgebras for endofunctors on the category of (Eilenberg-Moore) algebras for a monad. In Section 3 we characterize the final coalgebra and the rational fixed points (rational fixed points can be thought of as generalizing regular languages). In Section 4 we present an abstract Kleene’s theorem and soundness and completeness theorems, and we show that it is always possible to extend a calculus for behavioral equivalence to one for coalgebraic language equivalence. This paves the way to the development of sound and complete calculi for non-deterministic systems, in Section 6 and weighted automata, in Section 5. The former calculus coincides with Rabinovich’s result for trace equivalence of labelled transition systems and trace semantics [27] and the latter is, to the best of our knowledge, the first sound and complete axiomatization of weighted language equivalence of (non-deterministic) weighted automata. This can be seen as an extension of the second author’s calculi for closed stream circuits [24] to weighted automata over alphabets of arbitrary size.

2 Preliminaries

Here we present the basic notions needed throughout the paper. We denote by Set the category of sets and maps.
2.1 Semirings and semimodules

In our applications we will consider semimodules for a semiring. A *semiring* is a tuple $(k, +, 0, 1)$ where $(k, +, 0)$ and $(k, \cdot, 1)$ are monoids, the former of which is commutative, and multiplication distributes over finite sums (i.e., $a \cdot 0 = 0 = 0 \cdot a$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$). We just write $k$ to denote a semiring. A *$k$-semimodule* is a commutative monoid $(M, +, 0)$ with an action $k \times M \to M$ denoted by juxtaposition $rm$ for $r \in k$ and $m \in M$, such that for every $r, s \in k$ and every $m, n \in M$ the following laws hold:

\[
(r + s)m = rm + sm \quad r(m + n) = rm + rn \\
0m = 0 \quad r0 = 0 \\
1m = m \quad r(sm) = (r \cdot s)m
\]

A semimodule $M$ is *finitely generated* if there is a finite set $G \subseteq M$ such that every element of $M$ can be written as a linear combination of elements from $G$. Equivalently, there exists a surjective homomorphism $k^n \to M$ for some natural number $n$. And $M$ is called *finitely presentable* if it can be presented by finitely many generators and relations. Equivalently, $M$ is a coequalizer of some parallel pair of semimodule homomorphisms $k^m \rightrightarrows k^n$, where $m$ and $n$ are natural numbers. Notice that regular epimorphisms (i.e., coequalizers) in $k$-$\text{Mod}$ are precisely the surjective homomorphisms.

A semiring $k$ is called *Noetherian* if every subsemimodule of a finitely generated $k$-semimodule is itself finitely generated. Examples of Noetherian semiring are: every finite semiring, every field and, more generally, every finitely generated commutative rings (e.g. the ring of integers). The tropical semiring $(\mathbb{R}_\infty, \min, \infty, +, 0)$ and the natural numbers with addition are not Noetherian.

**Lemma 2.1.** For every semiring $k$, finitely generated $k$-semimodules are closed under finite products.

**Proof.** Clearly the terminal semimodule $0$ is finitely generated. Given two finitely generated semimodules $M$ and $N$ with the corresponding quotients $p : k^m \to M$ and $q : k^n \to N$ we have the quotient

\[k^{m+n} = k^m \times k^n \xrightarrow{p \times q} M \times N.\]

The following proposition gives a slightly more easy criterium to verify Noetherianess of a semiring.

**Proposition 2.2.** For a semiring $k$ the following are equivalent:

1. $k$ is Noetherian,
2. every subsemimodule of a free finitely generated semimodule $k^n$ is finitely generated, and
Proof. (1) $\Rightarrow$ (2) trivially hold.

(2) $\Rightarrow$ (1). Suppose that $N$ is a subsemimodule of the finitely generated $k$-semimodule $M$ via $m : N \to M$. Take a quotient $q : k^n \to M$ and form the pullback of $m$ along $q$:

$$
\begin{array}{c}
N' \leftarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \down arrow
\end{array}
$$

Since surjective and injective homomorphism are stable under pullback, we see that $N'$ is a submodule of $k^n$ and $N$ is a quotient of $N'$. So $N'$ is finitely generated by assumption, and, hence, so is its quotient $N$. \qed

We will use the following properties of Noetherian semirings.

**Proposition 2.3.** If $k$ is a Noetherian semiring, then the following hold

1. every finitely generated semimodule is also finitely presentable.
2. finitely generated $k$-semimodules are closed under finite limits.

**Proof.** Ad (1). Let $M$ be a finitely generated $k$-module, and take a surjective homomorphism $h : k^n \to M$. Since $h$ is a regular epimorphism, it follows that $h$ is the coequalizer of its kernel pair. So we form the kernel pair $p, q : K \rightarrow k^n$ of $h$. Then $K$ is a subsemimodule of the free finitely generated module $k^{n+n}$. Hence, since $k$ is Noetherian, $K$ is a finitely generated semimodule, too. So we have a surjective homomorphism $g : k^m \to K$. This implies that $h$ is a coequalizer of the parallel pair $p \cdot g, q \cdot g : k^m \to k^n$, which shows that $M$ is finitely presentable.

Ad (2). It suffices to prove closedness under finite products and subsemimodules. Indeed, the former was established in the Lemma 2.1 and the latter is by hypothesis.

**Remark 2.4.** For a ring $k$ the item (2) of Proposition 2.3 is actually equivalent to $k$ being Noetherian.

To see this recall that the ring $k$ is Noetherian iff every of its ideals is finitely generated, see [22], VI, Proposition 1.5.

Now suppose that finitely generated $k$-modules are closed under finite limits, and let $I$ be any ideal of $k$. Form the quotient ring $k/I$, i.e., the quotient homomorphism $c : k \to k/I$ is the coequalizer of the inclusion $i : \rightarrow k$ and the $0$-morphism $I \rightarrow k$. Now notice that $I$ is a split quotient of the domain $K = \{(x, y) \mid cx = xy\}$ of the kernel pair of $c$ via $q : K \to I$ with $q(x, y) = x - y$.

The quotient $k/I$ is of course finitely generated (with one generator). Since the free $k$-module $k$ is also finitely presented, so are $K$ (by assumption) and $I$ (since finitely generated objects are closed under quotients).

We will write $k$-Mod for the category of $k$-semimodules and their homomorphisms. Let us mention a few special cases: for the Boolean semiring $k = \{0, 1\}$, $k$-Mod is the category $\text{Jsl}$ of join-semilattices and join-preserving maps$^4$; for a field $k$ $k$-Mod is the

$^4$ We consider join-semilattices with a least element $0$. So a join-semilattice is, equivalently, a commutative idempotent monoid.
category \( \text{Vec}_k \) of vector spaces over \( k \) and linear maps; for \( k \) the ring of integers we get the category of abelian groups and for \( k \) the natural numbers \( k\text{-Mod} \) is the category of commutative monoids.

### 2.2 Coalgebras

Let \( \mathcal{A} \) be a category, and let \( F : \mathcal{A} \to \mathcal{A} \) be an endofunctor. A coalgebra for \( F \) is a pair \((C, c)\) consisting of an object \( C \) and a structure morphism \( c : C \to FC \). For example, if \( \mathcal{A} = \text{Set} \), then we can understand coalgebras as systems, where the set \( C \) consists of all states of the system and where the map \( c \) provides the transitions whose type is described by the endofunctor \( F \), see e.g. [30]. Concrete examples of coalgebras for set endofunctors include various kinds of automata (deterministic, non-deterministic, Mealy, Moore), stream systems, probabilistic automata, weighted ones, labelled transition systems and many others. We mention in this paper only two leading examples more in detail; for more example see e.g. [30,9,10].

Firstly, non-deterministic automata are coalgebras for the functor \( FX = 2 \times (\mathcal{P}X)^A \), where \( A \) is the finite input alphabet. Indeed, to give a coalgebra \( c : C \to 2 \times (\mathcal{P}C)^A \) is the same as to give a set \( C \) of states an image finite transition relation \( \delta \subseteq C \times A \times C \) and a subset \( C' \subseteq C \) of final states.

Our second leading example is weighted automata [32]. Let \( k \) be a semiring. We consider the functor \( V : \text{Set} \to \text{Set} \) defined on sets \( X \) and maps \( h : X \to Y \) as follows:

\[
V X = \{ f : X \to k \mid f \text{ has finite support} \}, \quad V h(f)(y) = \sum_{x \in h^{-1}(y)} f(x).
\]

So \( V X \) consists of all formal linear combinations on elements of \( X \); in other words, \( V X \) is the free \( k \)-semimodule on \( X \). A weighted automaton with finite input alphabet \( A \) is simply a coalgebra for the functor \( X \mapsto k \times (V X)^A \). Notice that for \( k \) the Boolean semiring weighted automata are precisely the classical non-deterministic ones as \( V \) and \( \mathcal{P} \) are naturally isomorphic.

For \( F \)-coalgebras to form a category we need morphisms: a coalgebra homomorphism from a coalgebra \((C, c)\) to a coalgebra \((D, d)\) is a morphism \( h : C \to D \) preserving the transition structure, i.e., such that \( d \cdot h = F h \cdot c \).

An important concept in the theory of coalgebras is that of a final coalgebra. An \( F \)-coalgebra \((T, t)\) is said to be final if for every \( F \)-coalgebra \((C, c)\) there exists a unique coalgebra homomorphism \( c^\dagger : C \to T \). We will write \( \nu F \) for the final coalgebra \( T \) if it exists.\(^5\) The final coalgebra is uniquely determined up to isomorphism. Moreover, the structure map \( t : \nu F \to F(\nu F) \) of a final coalgebra is an isomorphism by (the dual of) Lambek’s Lemma [21]. So \( \nu F \) is a fixed point of the endofunctor \( F \). More generally, any coalgebra \((C, c)\) with \( c \) an isomorphism is said to be a fixed point of \( F \). For an endofunctor on \( \text{Set} \), the elements of the final coalgebra provide semantics for the behavior of states of systems regarded as \( F \)-coalgebras.

\(^5\) Existence of a final coalgebra can be assured by mild assumptions on \( F \), e.g., every bounded (or, equivalently, accessible) endofunctor on \( \text{Set} \) has a final coalgebra.
Let us note that finality also provides the basis for semantic equivalence. Let \((C, c)\) and \((D, d)\) be two coalgebras for an endofunctor \(F\) on \(\text{Set}\) with a final coalgebra \((\nu F, t)\). (In fact, any other concrete\(^6\) category such as \(\text{Set}\) or \(\text{Vec}_k\) is fine, too.) Then two states \(x \in C\) and \(y \in D\) are called behavioral equivalent if \(c^!(x) = d^!(y)\), and we shall write \(x \sim y\). If \(F\) preserves weak pullbacks then behavioral equivalence coincides with the well-known notion of bisimilarity. The states \(x\) and \(y\) are called bisimilar if they are in a special relation called a bisimulation \([1]\). We shall not define that concept here as it is not needed in the present paper; for details see \([30]\). Let us just remark that the coalgebraic notion of bisimulation generalizes the concepts of the same name known for concrete classes of systems, e.g., for deterministic automata or labelled transition systems (where coalgebraic bisimulation coincides with Milner’s strong bisimulation).

The requirement that \(F\) preserve weak pullbacks is not very restrictive; many functors of interest in coalgebra theory do indeed preserve weak pullbacks. Exceptions are the above functor \(V\) and Giry’s probabilistic monad on the category of analytic spaces.

We now mention three examples of final coalgebras more in detail.

Classical deterministic automata with input alphabet \(A\) are coalgebras for the functor \(FX = 2 \times X^A\), where \(2 = \{0, 1\}\), and the final \(F\)-coalgebra is carried by the set \(\mathcal{P}(A^+)\) of all formal languages on \(A\). Moreover, for a deterministic automaton presented as an \(F\)-coalgebra \((C, c)\) the unique homomorphism \(c^!: C \to \mathcal{P}(A^+)\) assigns to every state \(s \in C\) the formal language it accepts.

In the example of non-deterministic automata the elements of the final coalgebra can be thought of as representatives of all finitely branching processes with outputs in \(2\) modulo strong bisimilarity. More concretely, consider all (rooted) finitely branching trees with edges labelled in \(A\) and nodes labelled in \(2\). Every such tree can be considered as a coalgebra in a canonical way (with the coalgebra structure assigning to a node \(x\) of a tree and an input symbol \(a \in A\) the finite set of child nodes of \(x\) reachable by \(a\)-labelled edges. A tree is said to be strongly extensional there is no non-trivial bisimulation on the coalgebra induced by the tree, and the final coalgebra consists of all finitely branching strongly extensional trees with edge labels from \(A\), cf. \([35]\).

Finally, for weighted automata considered as coalgebras for \(X \mapsto k \times (VX)^A\) it is not difficult to see that the final coalgebra is carried by the set of all behaviors of weighted automata modulo weighted bisimilarity; in fact, weighted bisimilarity \([12]\) is precisely behavioral equivalence for the above functor, see \([10]\). In this case we omit a concrete description.

### 2.3 Eilenberg-Moore-Algebras and the generalized power-set construction

The recent paper \([33]\) provides a coalgebraic version of the powerset construction applicable to many different system types expressed as coalgebras for a set endofunctor. The key idea in \(\text{loc. cit.}\) is to decompose an endofunctor giving the transition type of a class of systems into a functor \(F\) and a functor \(T\) on \(\text{Set}\) giving the behavior type and the branching behavior of systems, respectively. We already saw this in two of our examples above: non-deterministic automata are \(FT\)-coalgebras where \(FX = 2 \times X^A\)

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\(^6\) Recall that a category \(A\) is called concrete if it comes equipped with a faithful functor \(U \colon A \to \text{Set}\).
and \( T = \mathcal{P}_f \) is the finite power set functor, and weighted automata are \( FT \)-coalgebras for \( FX = k \times X^A \) and \( T = V \).

To apply the generalized power set construction to a coalgebra \( c : C \to FTC \) it is important that \( T \) is the functor part of a monad and that \( FTC \) is an Eilenberg-Moore algebra for \( T \). We now briefly recall these concepts, see e.g. \([23]\) for a detailed introduction.

A monad is a triple \((T, \eta, \mu)\), where \( \eta : \text{Id} \to T \) and \( \mu : TT \to T \) are natural transformations such that \( \mu \cdot \eta T = \text{id}_T = \mu \cdot T \eta \) and \( \mu \cdot T \mu = \mu \cdot \mu T \). Furthermore, we need the concept of an Eilenberg-Moore algebra for a monad \( T \) (or \( T \)-algebra, for short), i.e., a pair \((A, \alpha)\) consisting of an object \( A \) and a structure morphism \( \alpha : T A \to A \) such that \( \alpha \cdot \eta_A = \text{id}_A \) and \( \alpha \cdot \mu_A = \alpha \cdot T \alpha \). Furthermore, a \( T \)-algebra homomorphism from \((A, \alpha)\) to \((B, \beta)\) is a morphism \( h : A \to B \) such that \( h \cdot \alpha = \beta \cdot Th \). So Eilenberg-Moore algebras for a monad \( T \) on \( \text{Set} \) form the category denoted by \( \text{Set}^T \). Clearly, for every set \( X \), \((T X, \mu_X)\) is an Eilenberg-Moore algebra for \( T \). Moreover, this \( T \)-algebra is free on \( A \), i.e., for every \( T \)-algebra \((A, \alpha)\) and every map \( f : X \to A \) there is a unique \( T \)-algebra homomorphism \( f^\# : T X \to A \) such that \( f^\# \cdot \eta_X = f \):

\[
\begin{array}{ccc}
T X & \xrightarrow{\mu_X} & T X \\
\downarrow T f^\# & & \downarrow f^\# \\
T A & \xrightarrow{\alpha} & A
\end{array}
\]

(2.2)

Notice also that we have \( f^\# = \alpha \cdot Tf \).

Now we are ready to recall the generalized power set construction from \([33]\). Let \( F \) be an endofunctor on \( \text{Set} \) with the final coalgebra \( \nu F \) and let \( T \) be a monad. Now suppose we are given an \( FT \)-coalgebra \((C, c)\) such that \( FTC \) carries some \( T \)-algebra structure. Then we can form the \( F \)-coalgebra \( c^\# : TC \to FTC \) and consider the unique \( F \)-coalgebra homomorphism \( h \) into the final coalgebra \( \nu F \) as summarized by the following diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{c} & FTC \\
\downarrow \eta_C & & \downarrow c^\# \\
TC & \xrightarrow{h^\#} & F(\nu F)
\end{array}
\]

(2.3)

In concrete instances, the construction of the coalgebra \((TC, c^\#)\) is determination and the map \( h \cdot \eta_C : C \to \nu F \) assigns to states of the coalgebra \( C \) their language or set of traces.

For example, non-deterministic automata are \( FT \) coalgebras for \( F = 2 \times X^A \) and \( T = \mathcal{P}_f \). The construction extending the coalgebra structure \( c : C \to 2 \times (\mathcal{P}_f X)^A \) to \( c^\# : \mathcal{P}_f C \to 2 \times (\mathcal{P}_f C)^A \) is precisely the usual power set construction determining the given non-deterministic automaton. Moreover, as we saw previously, the final coalgebra for \( F \) consists of all formal languages, and the map \( h \cdot \eta_C \) from above provides the usual language semantics of a non-deterministic automata. In contrast, the
final coalgebra for $FT$ provides the bisimilarity semantics taking into account the non-deterministic branching of automata (so, for example, a non-deterministic automaton and its determinization are in general not equivalent in this semantics).

In our second leading example of weighted automata we consider $FT$-coalgebras for $FX = k \times X^A$ and $T = V$. The construction extending a coalgebra $c : C \to k \times (V X)^A$ to $c^\sharp$ can be understood as determinization of the given weighted automaton again. Moreover, the final coalgebra for $F$ is carried by the set $k^A^*$ of weighted languages (or formal power series), and so the map $h \cdot \eta_C : C \cdot k^A^*$ assigns to a state of a weighted automaton the weighted language it accepts. To summarize behavioral equivalence of $FT$-coalgebras coincides with weighted bisimilarity [12] while behavioral equivalence of $F$-coalgebras yields weighted language equivalence [32].

2.4 Lifting of functors to algebras

We have seen that the category of Eilenberg-Moore algebras for a set monad $T$ plays an important rôle for the generalized power set constructions presented in the previous section. For our work in the present paper we make use of functors $F$ the lift to the category Set$^T$ and we shall study fixed points of $F$ and its lifting. We now briefly recall the necessary background material.

Let $F : \text{Set} \to \text{Set}$ be a functor and let $(T, \eta, \mu)$ be a monad on Set. We denote by $U : \text{Set}^T \to \text{Set}$ the forgetful functor mapping a $T$-algebra to its underlying set. A lifting of $F$ to $\text{Set}^T$ is a functor $\bar{F} : \text{Set}^T \to \text{Set}^T$ such that the square below commutes:

$$
\begin{array}{ccc}
\text{Set}^T & \xrightarrow{\bar{F}} & \text{Set}^T \\
U \downarrow & & \downarrow U \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
$$

It is well-known that to have a lifting of $F$ to $\text{Set}^T$ is the same as to have a distributive law of the functor $F$ over the monad $T$, see [7,13]. Recall from loc. cit. that a distributive law of $F$ over $T$ is a natural transformation $\lambda : F T \to T F$ such that the following two laws hold:

$$
\lambda \cdot F \eta = \eta F \quad \text{and} \quad \lambda \cdot F \mu = \mu F \cdot T \lambda \cdot \lambda T.
$$

The functor in our leading examples have lifting to the respective Eilenberg-Moore categories, of course. For the case of non-deterministic automata with $FX = 2 \times X^A$ and $T = P^i$, notice that Set$^{P^i}$ is (equivalent to) the category Jsl of join-semilattices. Then, since 2 carries the join-semilattice structure with $0 \leq 1$, $F$ lifts since join-semilattices are closed under products and powers to the set $A$. More generally, every Kripke polynomial functor as presented in [34] canonically lifts to Jsl.

More generally, for the case of weighted automata we have $FX = k \times X^A$ and $T = V$, and Set$^V$ is (equivalent to) the category $k$-Mod of $k$-semimodules. Then as $k$ itself is a semimodule and semimodules are closed under product and powers to a set $A$, $F$ has the desired lifting.
Remark 2.5. It is not difficult to verify by an induction argument that for every monad $T$ every endofunctor $\text{Set}$ from the class of endofunctors defined by the following grammar has a lifting to $\text{Set}^T$:

$$F ::= B \mid \text{Id} \mid F \times F \mid F^A \mid F \cdot T \mid T \cdot G,$$

where $A$ is a finite set, $B$ ranges over $T$-algebras and $G$ over finitary endofunctors on $\text{Set}$.

3 Coalgebras on Algebras

For the results in the current paper we need to study the move from coalgebras for $F$ to ones for the lifting $\bar{F}$ more thoroughly. We develop the necessary mathematical theory we use to obtain desired general soundness and completeness in the current section. The main contributions of this section are: in subsection 3.1 the proof that the final $FT$-coalgebra is a $T$-algebra (Corollary 3.4) and the relation between the final $FT$-coalgebra and the final $\bar{F}$-coalgebra (Proposition 3.5); in subsection 3.2 the proof of preservation of locally finitely presentability under quotients (Lemma 3.18); and in subsection 3.3 the relation between the rational fixed points of $FT$ and $\bar{F}$ (Theorem 3.33).

3.1 Final coalgebras

For our soundness and completeness proof in Section 4 we need to consider coalgebras for a lifted endofunctor on categories of Eilenberg-Moore algebras. So we assume in this section that $(T, \eta, \mu)$ is a monad on $\text{Set}$ that $F : \text{Set} \rightarrow \text{Set}$ is an endofunctor with a final coalgebra $t : \nu F \rightarrow F(\nu F)$. We also assume that the final $FT$-coalgebra $t_0 : \nu(FT) \rightarrow FT(\nu(FT))$ exists that $\lambda : TF \rightarrow FT$ is a distributive law so that $F$ has a lifting $\bar{F}$ on $\text{Set}^T$. Then the final coalgebra for $F$ lifts to a final coalgebra for $\bar{F}$. This essentially follows from (the proof of) Theorem 3.2.3 in Bartels [8] (cf. also Plotkin and Turi [26]). More explicitly, one obtains the unique coalgebra homomorphism $\alpha : T(\nu F) \rightarrow \nu F$ as displayed below:

$$
\begin{array}{c}
T(\nu F) \\
\downarrow^\alpha \\
\nu F
\end{array} 
\xrightarrow{Tt_0} 
\begin{array}{c}
TF(\nu F) \\
\downarrow^\lambda_{\nu F} \\
FT(\nu F)
\end{array} 
\xrightarrow{F\alpha} 
\begin{array}{c}
\nu F \\
\downarrow^t \\
F(\nu F)
\end{array}
$$

It is then easy to prove that $(\nu F, \alpha)$ is an Eilenberg-Moore algebra for $T$ such that $t : \nu F \rightarrow \bar{F}(\nu F)$ is a $T$-algebra homomorphism, and, moreover, $(\nu F, t)$ is a final $\bar{F}$-coalgebra. So from now on we shall write $\nu F$ for both final coalgebras for $F$ and its lifting $\bar{F}$.

Example 3.1. There are many examples of the setting as described above. We only mention our two leading ones explicitly.
(1) In the case of non-deterministic automata we saw that the functor \( F X = 2 \times X^2 \) lifts to \( \text{Set}^T \) and so the final coalgebra for the lifting \( \bar{F} \) consists is carried by the set of formal languages again with the join-semilattice structure given by union of formal languages.

(2) For the case of weighted automata we saw that the functor \( F X = k \times X^A \) lifts to the category \( \text{Set}^V \) of \( k \)-semimodules. Hence, the final coalgebra for the lifting \( \bar{F} \) is carried by the set \( k^A \) with the canonical (componentwise) structure of a semimodule.

Next we would like to relate the final coalgebras for \( F \) and \( F T \).

**Lemma 3.2.** Every fixed point \((C, c)\) of \( F T \) has a canonical structure \( \gamma : T C \to C \) of a \( T \)-algebra such that \( \gamma \) is an \( \bar{F} \)-coalgebra homomorphism and \( c : C \to FTC \) a \( T \)-algebra homomorphism.

**Proof.** On \( FTC \) we have the \( T \)-algebra structure

\[ \bar{F}(TC, \mu_C) = (TFTC \xrightarrow{\lambda_{TC}} FTC \xrightarrow{F\mu_C} FTC) \]

Since the forgetful functor \( U : \text{Set}^T \to \text{Set} \) creates isomorphisms we have on \( C \) the unique \( T \)-algebra structure

\[ \gamma = (TC \xrightarrow{Tc} TFTC \xrightarrow{(F\mu_C \cdot \lambda_{TC})_C} FTC \xrightarrow{c^{-1}} C) \]

such that \( c \) is a \( T \)-algebra homomorphism. We only need to verify that \( \gamma \) is an \( F \)-coalgebra homomorphism, i.e., the diagram below commutes:

\[
\begin{array}{ccc}
TC & \xrightarrow{Tc} & TFTC \xrightarrow{\lambda_{TC}} FTC \\
\downarrow{\gamma} & & \downarrow{F\mu_C} \\
C & \xrightarrow{c} & FTC \xrightarrow{F\gamma} FC
\end{array}
\] (3.1)

Indeed, the coalgebras in the upper and lower row are actually \( \bar{F} \)-coalgebras, and so \( c \) is an \( \bar{F} \)-coalgebra homomorphism as desired. \(\square\)

**Remark 3.3.** It is not difficult to prove that the \( \bar{F} \)-coalgebra in the upper row of diagram (3.1) has as its structure the unique homomorphic extension of \( c : C \to FTC \) to the free algebra \( TC \); in symbols, we have the equation

\[ c^\sharp = F\mu_C \cdot \lambda_{TC} \cdot Tc. \]

Indeed, the arrows in the top row of (3.1) compose to a \( T \)-algebra homomorphism: \( Tc \) clearly is a homomorphism and so is \( F\mu_C \cdot \lambda_{TC} \) being the algebra structure of \( \bar{F}(TC, \mu_C) \). And this \( T \)-algebra homomorphism extends \( c \):

\[
\begin{align*}
F\mu_C \cdot \lambda_{TC} \cdot Tc \cdot \eta_C &= F\mu_C \cdot \lambda_{TC} \cdot \eta_{FTC} \cdot c \\
&= F\mu_C \cdot F\eta_{TC} \cdot c \\
&= c
\end{align*}
\]

\(\text{naturality of } \eta, \lambda \text{ a distributive law, since } \mu \cdot \eta T = \text{id}.\)
Corollary 3.4. The final coalgebra $\nu(FT)$ has a canonical $T$-algebra structure $\alpha_0$, and whence it is an $F$-coalgebra with the structure $F\alpha_0 \cdot t_0 : \nu(FT) \to F(\nu FT)$.

Proposition 3.5. The final $\bar{F}$-coalgebra is a quotient coalgebra of the final $FT$-coalgebra.

Proof. Consider the following $\bar{F}$-coalgebra homomorphism obtained by using the universal property of $\nu F$ (we abuse notation and write $F$ in lieu of $\bar{F}$, and we also write $Z = \nu F$ and $Z_0 = \nu FT$):

\[ \begin{array}{c}
Z_0 \xrightarrow{t_0} FTZ_0 \xrightarrow{F\alpha_0} FZ_0 \\
\downarrow{p} \quad \quad \quad \quad \quad \downarrow{Fp} \\
Z \xrightarrow{t} FZ \\
\end{array} \quad (3.2) \]

Since all horizontal morphisms are $T$-algebra homomorphisms so is $p : Z_0 \to Z$. To see that $p$ is surjective we show it has a splitting $s : Z_0 \to Z$ in $\text{Set}$. To obtain $s$ we use the universal property of $Z_0$; there is a unique $FT$-coalgebra homomorphism $s$ such that the diagram below commutes:

\[ \begin{array}{c}
Z \xrightarrow{t} FZ \xrightarrow{F\eta_Z} F\nu F \\
\downarrow{s} \quad \quad \quad \quad \quad \downarrow{FTs} \\
Z_0 \xrightarrow{t_0} FTZ_0 \\
\end{array} \quad (3.3) \]

To see that $p \cdot s = \text{id}$ holds, we verify that the following diagram commutes:

\[ \begin{array}{c}
Z \xrightarrow{t} FZ \xrightarrow{F\eta_Z} F\nu F \\
\downarrow{s} \quad \quad \quad \quad \quad \downarrow{FTs} \\
Z_0 \xrightarrow{t_0} FTZ_0 \xrightarrow{F\alpha_0} FZ_0 \\
\downarrow{p} \quad \quad \quad \quad \quad \downarrow{Fp} \\
Z \xrightarrow{t} FZ \\
\end{array} \]

Indeed, the upper left-hand and lower parts commute as indicated, but we do not claim that part $(\ast)$ commutes. This part commutes when precomposed with $F\eta_Z$; to see this remove $F$ and consider

\[ \begin{array}{c}
Z \xrightarrow{\eta_Z} TZ \xrightarrow{\alpha} Z \\
\downarrow{s} \quad \quad \quad \quad \quad \downarrow{T\alpha} \\
Z_0 \xrightarrow{\eta Z_0} TZ_0 \xrightarrow{\alpha_0} Z_0 \\
\downarrow{id} \quad \quad \quad \quad \quad \downarrow{id} \\
\end{array} \]

where the left-hand square commutes by the naturality of $\eta$ and the upper and lower triangle by the unit law of $T$-algebras. \qed
3.2 Locally finitely presentable coalgebras

For the soundness and completeness proofs in [34] locally finite coalgebras play an important rôle, and for the sound and complete calculus for linear systems presented in [24] locally finite dimensional coalgebras are important. More precisely, expressions modulo rules form a final locally finite (or locally finite dimensional, respectively) coalgebra. In loc. cit. we also introduced locally finitely presentable coalgebras as a common framework to reason about local finiteness (or local finite dimensionality). We will recall the necessary material now, and then extend the theory to be able to relate the final locally finitely presentable coalgebras for $F_T$ and $\bar{F}$.

For a general category local finiteness of coalgebras must be based on a notion of finiteness of objects of the category, and the latter is captured by locally finitely presentable categories; we recall the basics from [5]. A functor is finitary if it preserves filtered colimits, and an object $X$ of a category $\mathcal{A}$ is called finitely presentable if its hom-functor $\mathcal{A}(X, -)$ is finitary. A category $\mathcal{A}$ is called locally finitely presentable (lfp, for short) if it is cocomplete and has a set of finitely presentable objects such that every object of $\mathcal{A}$ is a filtered colimit of objects from that set. We write $\mathcal{A}_{fp}$ for the full subcategory of $\mathcal{A}$ given by all finitely presentable objects.

Our categories of interest $\mathcal{Set}$ and $k$-$\mathcal{Mod}$ are locally finitely presentable with the expected notion of finitely presentable objects: finite sets, and finitely presentable $k$-semimodules; in the special instances of $Jsl$ and vector spaces over a field $k$ the finitely presentable objects are finite join-semilattices and finite dimensional vector spaces, respectively. Other examples of lfp categories are the categories of posets, graphs or groups, in fact, every finitary variety of algebras is lfp—the corresponding notions of finitely presentable objects are: finite posets or graphs and those groups or algebras presented by finitely many generators and relations. Notice that finitary varieties are precisely the Eilenberg-Moore categories for finitary set monads, so $\mathcal{Set}^T$ is lfp for every finitary monad $T$ on $\mathcal{Set}$. In contrast, the category of complete partial orders (cpo’s) and continuous maps is not lfp; there are no non-trivial finitely presentable objects.

**Assumption 3.6.** For the rest of this section we assume that $\mathcal{A}$ is an lfp category and that $F : \mathcal{A} \to \mathcal{A}$ is a finitary functor on $\mathcal{A}$.

**Examples 3.7.** There are many examples of finitary functors on lfp categories. We mention only those two of interest in the current paper.

1. Every Kripke polynomial functor on $\mathcal{Set}$ as presented in [34] is finitary. These functors lift to finitary functors on $Jsl$ (e.g. the functor $F_X = 2 \times X^A$).
2. The functor $FX = k \times X^A$ is finitary on $\mathcal{Set}$ and it lifts to a finitary functor of $k$-$\mathcal{Mod}$.

**Remark 3.8.** (1) We shall need the following property of lfp categories, and we recall this from [5]:

Every morphism $f$ in the lfp category $\mathcal{A}$ can be factorized as a strong epi $e$ followed by a monomorphism $m: f = m \cdot e$. This factorization system has the following
diagonalization property: for every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \leftarrow{m} & D
\end{array}
\]

with \(m\) a monomorphism and \(e\) a strong epimorphism there exists a unique morphism \(d : B \to C\) such that \(m \cdot d = g\) and \(d \cdot e = f\).

(2) It follows that \(\text{Coalg}(F)\) has a factorization system, too, whenever \(F\) preserves monomorphisms. Indeed, given the coalgebra homomorphism \(f : (C, c) \to (D, d)\) we take its strong epi-mono factorization \(f = m \cdot e\) in \(A\). And by diagonalization, we obtain a unique \(F\)-coalgebra structure on the codomain of \(e\) such that \(e\) and \(m\) are coalgebra homomorphisms:

\[
\begin{array}{ccc}
C & \xrightarrow{e} & FC \\
\downarrow{e} & & \downarrow{Fe} \\
E & \xrightarrow{m} & FE \\
\downarrow{m} & & \downarrow{Fm} \\
D & \xrightarrow{d} & FD
\end{array}
\]

Notice that we do not claim that \(e\) is a strong epimorphism in \(\text{Coalg}(F)\) (and, in general, this claim is false).

In the current setting local finiteness of coalgebras is captured by the following notion introduced in [24).

**Notation 3.9.** We denote by \(\text{Coalg}_f(F)\) the category of all coalgebras \(p : P \to FP\) with a finitely presentable carrier \(P\).

**Definition 3.10.** An \(F\)-coalgebra \((S, s)\) is called locally finitely presentable if the canonical forgetful functor \(\text{Coalg}_f(F)/(S, s) \to A_0/S\) is cofinal.

**Remark 3.11.** More explicitly \((S, s)\) is locally finitely presentable iff the following two conditions are satisfied:

(1) for every \(f : X \to S\) where \(X\) is a finitely presentable object of \(A\) there exists a coalgebra \((P, p)\) from \(\text{Coalg}_f(F)\), a coalgebra homomorphism \(h : (P, p) \to (S, s)\) and a morphism \(f' : X \to P\) such that \(h \cdot f' = f\).

(2) The factorization in (1) is essentially unique in the sense that for every \(f'' : X \to P\) with \(h \cdot f'' = f\) there exists a homomorphism \(\ell : (P, p) \to (Q, q)\) in \(\text{Coalg}_f(F)\) and a coalgebra homomorphism \(h' : (Q, q) \to (S, s)\) such that \(\ell \cdot f' = \ell \cdot f''\).

**Example 3.12.** (1) For \(A = \text{Set}\) an \(F\)-coalgebra is locally finitely presentable iff every finite subset of its carrier is contained in a finite subcoalgebra. As discussed in [24].
for a $F$ preserving weak pullbacks this coincided with the notion of local finiteness considered in [34].

(2) Analogously for $A = \text{Jsl}$, an $F$-coalgebra is locally finitely presentable if every finite sublattice of its carrier is contained in a finite subcoalgebra.

(3) For $A$ the category of vector spaces over a field $k$ an $F$-coalgebra is locally finitely presentable if every finite dimensional subspace of its carrier is contained in a finite dimensional subcoalgebra, i.e., the given coalgebra is locally finite dimensional.

**Theorem 3.13** ([24]). (1) A coalgebra is locally finitely presentable iff it is a filtered colimit of a diagram of coalgebras from $\text{Coalg}_l(F)$.

(2) A locally finitely presentable coalgebra $(R, r)$ is final in the category of all locally finitely presentable coalgebras iff for every coalgebra $(P, p)$ from $\text{Coalg}_l(F)$ there exists a unique homomorphism from $(P, p)$ to $(R, r)$.

As a consequence of point (1) above, the final locally finitely presentable coalgebra exists and is constructed as the colimit of (the inclusion functor of) $\text{Coalg}_l(F)$. This colimit can be proved to be a fixed point of the functor $F$ (see [3], Theorem 3.3), and we will write $\nu_r F$ for this fixed point in analogy to the notation $\nu F$ for the final $F$-coalgebra, and we will call $\nu_r F$ the rational fixed point of $F$.

**Example 3.14.** We mention a number of examples of final locally finitely presentable coalgebras $\nu_r F$ to illustrate that they capture finite system behavior; further examples are in [34].

(1) Let $F = F\Sigma$ be a polynomial endofunctor on $\text{Set}$ associated to a signature $\Sigma$ of operation symbols with prescribed arities. Then the final coalgebra for $F$ consists of all (finite and infinite) $\Sigma$-trees and $\nu_r F$ consists of all rational $\Sigma$-trees (where recall that a $\Sigma$-tree is a rooted and ordered tree $t$ labelled in $\Sigma$ such that a node with $n$ children is labelled by an $n$-ary operation symbol, and $t$ is rational if it has, up to isomorphism, only finitely many subtrees, see [16]).

(2) For the special case $FX = 2 \times X^A$ on $\text{Set}$, where $2 = \{0, 1\}$, a coalgebra is a deterministic automaton, and the final coalgebra is carried by the set of $P(A^*)$ of all formal languages on $A$. Here $\nu_r F$ is the subcoalgebra given by all regular languages.

(3) For the functor $FX = k \times X$ on $\text{Set}$, $\nu_r F$ consists of all streams $\sigma$ that are eventually periodic, i.e., $\sigma = u\sigma$ where $u$ and $v$ are finite words on $k$. However, for the lifting of $F$ to $\text{Vec}_k$, $\nu_r F$ is the subcoalgebra of $k^\omega$ given by all rational streams, see [24] for details.

(4) Let $k$ be a field. For the functor $FX = k \times X^A$ on the category of vector spaces over $k$, the final coalgebra is carried by the set $k^{A^*}$ of formal power series (or weighted languages) on $k$, and $\nu_r F$ is the subcoalgebra of all rational formal power series [31].

Our sound and complete calculus for language equivalence of weighted automata in Section 5 is based on this example.

In all the examples above the rational fixed point $\nu_r F$ is always occurs as a subcoalgebra of $\nu F$. This is no coincidence as we will now prove.
We say that a quotient $Y$ of an object $X$ in our category $A$ is the codomain of some strong epimorphism $q : X \to Y$. Similarly, a quotient coalgebra is given by a coalgebra homomorphism $q : (X, x) \to (Y, y)$ such that $q : X \to Y$ is a strong epimorphism in $A$.

Recall from [5] that a finitely generated object is an object $X$ such that it covariant hom-functor $A(X, -)$ preserved directed unions (i.e., colimits of directed diagrams of monomorphisms). Every finitely presentable object clearly is finitely generated, but in general the converse does not hold. Finitely generated objects are closed under quotients, and an object is finitely generated iff it is a quotient of a finitely presentable object. So to say that finitely generated and finitely presentable objects coincide is equivalent to stating that finitely presentable objects be closed under quotients.

**Proposition 3.15.** Suppose that in $A$ finitely generated objects are finitely presentable, and that $F$ preserves monomorphisms.

Then $\nu F$ is the subcoalgebra of $\nu F$ given by the union of images of all coalgebra homomorphism $(P, p) \to (\nu F, t)$ where $(P, p)$ ranges over $\text{Coalg}_f(F)$.

Indeed, for a proof see [2], Proposition 4.6 and Remark 4.3.

**Example 3.16.** Let us list some examples of categories in which our assumption in Proposition 3.15 that finitely generated and finitely presentable objects coincide holds.

1. The categories of sets, of posets and of graphs obviously have the desired property since finitely presentable objects are just finite sets (or posets or graphs, respectively).

2. The categories $Jsl$ of join-semilattices, of vector spaces over a field and of abelian groups satisfy the property. More generally, the category $k\text{-Mod}$ satisfies this assumption whenever $k$ is a Noetherian semiring, see Proposition 2.3.

3. A locally finite variety is a finitary variety in which free algebras on finite sets are themselves finite (e.g., Boolean algebras). It is not difficult to prove that in such a category the finitely presentable objects are precisely the finite ones, and so the assumption holds.

4. In the categories of commutative monoids and commutative semigroups finitely presentable and finitely generated objects coincide, see [28,13,29]. Notice that commutative monoids are $k\text{-Mod}$ for $k$ the natural numbers, which do not form a Noetherian semiring. So the proof is different (and more involved) than what we saw in Proposition 2.3.

5. The category of presheafs on finite sets (equivalently, finitary endofunctors of Set), see [4].

We have seen that the assumption in Proposition 3.15 holds for many interesting categories. But there are finitary varieties in which it fails, e.g., in the category of groups. And Proposition 3.15 does not hold without this assumption.

**Example 3.17.** We take as $A$ the category of algebras for the signature $\Sigma$ with a unary and a binary operation symbol. Then the natural numbers $\mathbb{N}$ with the operations of addition and $n \mapsto 2 \cdot n$ is an object of $A$. Thus, the set endofunctor $FX = \mathbb{N} \times X$ lifts to $A$, and its final coalgebra consists of all streams of natural numbers. Now consider the
$F$-coalgebra $\alpha : A \to FA$, where $A$ is the free (term) algebra on one generator $x$ and $\alpha$ is given by $\alpha(x) = (1, 2 \cdot x)$. The unique $F$-coalgebra homomorphism $h : A \to \nu F$ maps $x$ to the stream $(1, 2, 4, 8, \cdots)$ of powers of 2, and we have

$$h(2 \cdot x) = h(x + x) = (2, 4, 8, \cdots).$$

Now notice that $(A, \alpha)$ lies in $\text{Coalg}_l(F)$, and so there is also a unique $F$-coalgebra homomorphism $h_0 : A \to \nu_r F$. However, one can prove that $h_0(2 \cdot x) \neq h_0(x + x)$; for lack of space we omit the details. It follows that $\nu_r F$ is not a subcoalgebra of $\nu F$.

In our work we shall make use of the following lemma.

**Lemma 3.18.** Under the assumptions in Proposition 3.15, every quotient coalgebra of a locally finitely presentable coalgebra is itself locally finitely presentable.

*Proof.* Let $q : (C, c) \to (D, d)$ be a quotient coalgebra, where $(C, c)$ is a locally finitely presentable $F$-coalgebra. So $q : C \to D$ is a strong epimorphism in $A$. By Theorem 3.13(1), $(C, c)$ is a filtered colimit of a diagram of coalgebras $(C_i, c_i)$ from $\text{Coalg}_l(F)$ with the colimit injections $i_n : (C_i, c_i) \to (C, c)$. For every $i$ factorize $q \cdot i_n = m_i \cdot e_i$ as a strong epimorphism followed by a monomorphism in $\text{Coalg}(F)$, see Remark 3.8(2). By assumption, each $(D_i, d_i)$ lies in $\text{Coalg}_l(F)$. Moreover, each connecting morphism $c_{ij} : (C_i, c_i) \to (C_j, c_j)$ induces a coalgebra homomorphism $d_{ij} : (D_i, d_i) \to (D_j, d_j)$ turning the $D_i$ into a filtered diagram (with the same diagram scheme as for the $C_i$). To conclude our proof, it suffices to show that $D$ is a colimit of this new diagram. Indeed, we shall now prove that $D = \text{colim } D_i$, i.e., $D$ has no proper subobject containing every $m_i$. It then follows that $D = \text{colim } D_i$, see [5], 1.63. So let $m : M \to D$ be a subobject containing all $m_i$, i.e., for every $i$ we have monomorphisms $n_i : D_i \to M$ such that $m \cdot n_i = m_i$. Now the outside of the following square commutes:

$$\begin{array}{ccc}
\prod_i C_i & \xrightarrow{\text{[in]}_i} & C \\
[n_i \cdot e_i], & \downarrow & \searrow s \\
M & \xrightarrow{m} & D 
\end{array}$$

Indeed, for every $i$ we have

$$q \cdot i_n = m_i \cdot e_i = m \cdot n_i \cdot e_i.$$

Moreover, notice that the copairing $[\text{in}]_i$ is a strong epimorphism since it is the copairing of all the injections of the colimit $C$. Since strong epimorphisms compose, we see that the upper edge of the above diagram is a strong epimorphism. Hence, we get, by diagonalization, the morphism $s : D \to M$ such that $m \cdot s = \text{id}$ showing $m$ to be split epimorphism, whence an isomorphism. And this completes the proof. \qed
3.3 Locally Finitely Presentable Coalgebras on Algebras

Assumption 3.19. For the rest of the section we assume that \( A = \text{Set}^T \) for a finitary monad \((T, \eta, \mu)\), and we also assume that in \( \text{Set}^T \) finitely generated algebras are closed under taking kernel pairs. In addition we require that \( F : \text{Set} \to \text{Set} \) is an endofunctor weakly preserving pullbacks and having a lifting \( \bar{F} : \text{Set}^T \to \text{Set}^T \).

Example 3.20. All categories in Example 3.16 satisfy the condition that finitely generated objects are closed under taking kernel pairs. For sets, posets, graphs and locally finite varieties this clearly holds, for semimodules of a Noetherian semiring see Proposition 2.3 and for commutative monoids and semigroups this follows from Rédei’s result that every congruence of commutative semigroups is finitely generated, see [28]. Notice that all these categories except posets, graphs and finitary endofunctors of \( \text{Set} \) are (equivalent to) \( \text{Set}^T \) for an appropriate finitary monad \( T \).

Lemma 3.21. Every finitely generated algebra is finitely presentable.

Proof. Let \( A \) be a finitely generated algebra. Then \( A \) is the quotient of some finitely presentable algebra \( B \) via \( q : B \to A \). Then \( q \) is the coequalizer of its kernel pair \( f, g : K \to B \). So since \( A \) and \( B \) are finitely generated so is \( K \). Hence, \( K \) is the quotient of the finitely presentable algebra \( L \) via \( p : L \to K \). As \( p \) is an epimorphism it follows that \( q \) is the coequalizer of \( p \cdot f \) and \( p \cdot g \). So since \( L \) and \( B \) are finitely presentable, so is \( A \).

Remark 3.22. The previous lemma ensures that the results in Proposition 3.15 and Lemma 3.18 hold in \( A = \text{Set}^T \). Also note the proof works in any lfp category where strong epimorphisms are regular and finitely generated objects are closed under kernel pairs.

Our aim for the rest of this section is to establish the same relationship between \( \nu_r(FT) \) and \( \nu_r \bar{F} \) that we saw for the corresponding final coalgebras in Proposition 3.5. In addition, we are going to improve on the finality criterium for \( \nu_r \bar{F} \) from Theorem 3.13(2).

Remark 3.23. (1) It is easy to see that every free algebra \( TX \) is projective: for every (strong) epimorphism \( q : A \to B \) in \( \text{Set}^T \) (i.e., \( q \) is a surjective homomorphism) and every \( T \)-algebra homomorphism \( f : TX \to B \) there exists a homomorphism \( g : TX \to A \) such that \( q \cdot g = f \):

\[
\begin{array}{ccc}
TX & \xrightarrow{g} & A \\
\downarrow{f} & & \downarrow{q} \\
B & & 
\end{array}
\]

Indeed, take since \( q \) is surjective we have a (not necessarily homomorphic) map \( s : B \to A \) with \( q \cdot s = \text{id} \). Then use the freeness of \( TX \) to extend the map \( s \cdot f \cdot \eta_X : X \to A \) to the homomorphism \( g : TX \to A \), which has the desired property.
(2) As we mentioned already finitely presentable algebras are precisely those algebras that are presentable by finitely many generators and relations. In category theoretic terms, an algebra $A$ is finitely presentable iff it is the (reflexive) coequalizer of a parallel pair $f, g : TX \to TY$ of homomorphisms between free finitely presentable algebras, i.e., free algebras on the finite sets $X$ and $Y$, cf. Adámek, Rosický and Vitale [6], Proposition 5.17.

(3) The monad $T$ yields a functor $T' : \text{Coalg}(FV) \to \text{Coalg}(F)$; it assigns to every $FT$-coalgebra $c : C \to FTC(\text{cf. (2.2)})$, and on morphisms $T'$ acts like $T$. It is easy to see that $T'$ is finitary; this follows essentially from the fact that the filtered colimits in $\text{Coalg}(FT)$ and $\text{Coalg}(\bar{F})$ are formed on the level of $\text{Set}$ (since the forgetful functors of $\text{Coalg}(FT)$, $\text{Coalg}(\bar{F})$ and $\text{Set}^{T}$ create filtered colimits).

**Notation 3.24.** We denote by $\mathcal{D}$ the full subcategory of $\text{Coalg}(\bar{F})$ given by coalgebras with a free finitely presentable carrier. That means that the objects of $\mathcal{D}$ are of the form $TX \to FTX$ with $X$ a finite set.

**Remark 3.25.** Observe that every coalgebra in $\mathcal{D}$ arises as an extension of an $FT$-coalgebra $c : X \to FTX$. Indeed, notice that $FTX$ is the carrier of a $T$-algebra, and so $c$ extends uniquely to the algebra homomorphism $\bar{c} : TX \to FTX$ (cf. (2.2)).

**Lemma 3.26.** The category $\mathcal{D}$ is closed in $\text{Coalg}(F)$ under finite coproducts.

**Proof.** The empty $FT$-coalgebra $0 \to FT0$ extends uniquely to an $\bar{F}$-coalgebra $T0 \to FT0$, and this is the initial object of $\mathcal{D}$.

Let $c : TX \to FTX$ and $d : TY \to FTY$ be objects of $\mathcal{D}$ with the corresponding $FT$-coalgebras $c : X \to FTX$ and $d : Y \to FTY$. Now form

$$k = (X + Y \xrightarrow{c+d} FTX + FTY \xrightarrow{\text{can}} FT(X + Y)).$$

where can $= [FT\text{inl}, FTT\text{inr}]$, and extend $k$ to the $T$-algebra homomorphism $k^\beta : T(X + Y) \to FT(X + Y)$. It is not difficult to verify that this $\bar{F}$-coalgebra is the coproduct of $(TX, c^\beta)$ and $(TY, d^\beta)$ in $\mathcal{D}$. To see this, first verify that $T\text{inl} : TX \to T(X + Y)$ and $T\text{inr} : TY \to T(Y + Y)$ are $\bar{F}$-coalgebra homomorphisms. Next we show that they serve as the coproduct injections. Indeed, suppose we have two $\bar{F}$-coalgebra homomorphisms $f : (TX, c^\beta) \to (A, a)$ and $g : (TY, d^\beta) \to (A, a)$. Let $f_0 = f \cdot \eta_X$ and $g_0 = g \cdot \eta_Y$. Now extend the morphism $h_0 = [f_0, g_0] : X + Y \to A$ to a $T$-algebra homomorphism $h : T(X + Y) \to A$. Then one readily verifies using the universal properties of free $T$-algebras that $h$ is the unique coalgebra homomorphism from $(T(X + Y), k^\beta)$ to $(A, a)$ such that $h \cdot T\text{inl} = f$ and $h \cdot T\text{inr} = g$. □

**Proposition 3.27.** Every coalgebra in $\text{Coalg}_0(\bar{F})$ is the coequalizer of a pair of morphisms in $\mathcal{D}$.

**Proof.** Let $a : A \to \bar{F}A$ be a coalgebra from $\text{Coalg}_0(\bar{F})$, i.e., $A$ is a finitely presentable $T$-algebra. From Remark 3.23(2) we recall that $A$ is the coequalizer of some pair $TX' \rightrightarrows TX$ of $T$-algebra homomorphisms with $X'$ and $X$ finite sets via some $q : TX \to A$. 

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Being a functor on Set, $F$ preserves epimorphisms. Thus, $Fq$ is a strong epimorphism in $\text{Set}^T$. Now we use that $TX$ is projective to obtain a coalgebra structure $c : TX \rightarrow FTX$ as displayed below:

\[
\begin{array}{c}
TX \xrightarrow{c} FTX \\
\downarrow \quad \downarrow Fq \\
A \xrightarrow{a} FA
\end{array}
\] (3.4)

Now since $\text{Set}^T$ is a category with pullbacks we know that every coequalizer in that category is the coequalizer of its kernel pair. So let $f, g : K \rightarrow TX$ be the kernel pair of $q$ in $\text{Set}^T$, i.e., the pullback of $q$ along itself. Notice that since $TX$ and $A$ are finitely presentable $T$-algebras, so is $K$ because finitely presentable (equivalently, finitely generated) $T$-algebras are closed under taking kernel pairs. Since the forgetful functor $\text{Set}^T \rightarrow \text{Set}$ preserves limits we have a pullback in $\text{Set}$ and since $F$ weakly preserves pullbacks $Ff, Fg$ form a weak pullback of $Fq$ in $\text{Set}$. Thus, we have a map $k : K \rightarrow FK$ such that the diagram below commutes:

\[
\begin{array}{c}
K \xrightarrow{k} FK \\
f \downarrow \quad \downarrow Ff \quad \quad \downarrow Fg \\
TX \xrightarrow{c} FTX \\
\downarrow \quad \downarrow Fq \\
A \xrightarrow{a} FA
\end{array}
\] (3.5)

Notice that we do not claim that $k$ is a $T$-algebra homomorphism. However, since $K$ is a finitely presentable $T$-algebra it is the coequalizer of some pair $TY' \rightrightarrows TY$ of $T$-algebra homomorphisms, $Y'$ and $Y$ finite, via $p : TY \rightarrow K$. Now we choose some splitting $s : K \rightarrow TY$ of $p$ in $\text{Set}$, i.e., $s$ is a map such that $p \cdot s = \text{id}$. Next we extend the map $d_0 = Fs \cdot k \cdot p \cdot \eta_Y$ to a $T$-algebra homomorphism $d : TY \rightarrow FTY$:

\[
\begin{array}{c}
TY \xrightarrow{d_0} FTY \\
\downarrow p \quad \downarrow Fp \\
K \xrightarrow{k} FK
\end{array}
\] (3.6)

(Notice that to obtain $d$ we cannot simply use projectivity of $TY$ similarly as in (3.4) since $k$ is not necessarily a $T$-algebra homomorphism.)

We do not claim that this makes $p$ a coalgebra homomorphism (i.e., we do not claim the lower square in (3.6) commutes). However, $f \cdot p$ and $g \cdot p$ are $\tilde{F}$-coalgebra homomorphisms from $(TY, d)$ to $(TX, c)$. Indeed, to see that

\[c \cdot f \cdot p = F(f \cdot p) \cdot d\]
it suffices that this equation of $T$-algebra homomorphisms holds when both sides are precomposed with $\eta Y$. To see this we compute
\[
 c \cdot f \cdot p \cdot \eta Y = Ff \cdot k \cdot p \cdot \eta Y \quad \text{see (3.5)},
\]
\[
 = Ff \cdot Fp \cdot d_0 \quad \text{outside of (3.6)},
\]
\[
 = Ff \cdot Fp \cdot d \cdot \eta Y \quad \text{definition of $d$.}
\]

Similarly, $g \cdot p$ is a coalgebra homomorphism. Now since $p$ is an epimorphism in $\text{Set}^T$ it follows that $q$ is a coequalizer of $f \cdot p$ and $g \cdot p$. Thus $f \cdot p$ and $g \cdot p$ form the desired pair of morphisms in $\mathcal{D}$ such that $(A, a)$ is a coequalizer of them, which completes the proof. \hfill $\square$

From the previous proposition we see that $\text{Coalg}(\tilde{F})$ is the closure of $\mathcal{D}$ under coequalizers in the category $\text{Coalg}(\tilde{F})$ of all $\tilde{F}$-coalgebras; indeed, this follows since $\text{Coalg}(\tilde{F})$ is closed under coequalizers in $\text{Coalg}(\tilde{F})$.

**Corollary 3.28.** The rational fixed point of $\tilde{F}$ is the colimit of the diagram $D : \mathcal{D} \rightarrow \text{Coalg}(\tilde{F})$.

**Proof.** Indeed, since $\mathcal{D}$ is closed under finite coproducts (see Lemma 3.26) the colimit of $\mathcal{D}$ and the filtered colimit of its closure under coequalizers coincide. \hfill $\square$

**Corollary 3.29.** A locally finitely presentable $\tilde{F}$-coalgebra $(R, r)$ is final in the category of all locally finitely presentable $\tilde{F}$-coalgebras iff for every coalgebra $(TX, c^2)$ from $\mathcal{D}$ there exists a unique coalgebra homomorphism from $(TX, c^2)$ to $(R, r)$.

We are now ready to relate the rational fixed points of $FT$ and $\tilde{F}$. Recall the congruence quotient $p : \nu(FT) \rightarrow \nu F$ from Proposition 3.5 and notice that the rational fixed point $\nu \tilde{F}$ is a subcoalgebra of $\nu(FT)$, see Proposition 3.1. From our assumptions we also know that $\nu \nu \tilde{F}$ is a subcoalgebra of $\nu F$ (recall from Section 3.1 that $\nu F$ denotes the final $\nu \tilde{F}$-coalgebra). We denote the corresponding inclusion homomorphisms by $i : \nu \nu \tilde{F}(FT) \rightarrow \nu(FT)$ and $j : \nu \nu \tilde{F} \rightarrow \nu F$.

Furthermore, recall from Corollary 3.4 that $\nu(FT)$ is an $\tilde{F}$-coalgebra with the structure $Ft_0 \cdot t_0$. Similarly, we know from Lemma 3.2 that the rational fixed point $R = \nu \nu (FT)$ of $FT$ carries a $T$-algebra structure $\beta : TR \rightarrow R$ so that we have the $\tilde{F}$-coalgebra structure
\[
r = (R \xrightarrow{r_0} FR \xrightarrow{F\beta} FR_0)
\]
where $r_0$ is the $FT$-coalgebra structure of $R$.

**Lemma 3.30.** The coalgebra $(\nu \nu (FT), r)$ is a locally finitely presentable $\tilde{F}$-coalgebra.

**Proof.** We still write $R$ for $\nu \nu(FT)$, for short. By Theorem 3.13(1) the coalgebra $(R, r_0)$ is the filtered colimit of the inclusion functor $I : \text{Coalg}_0(FT) \rightarrow \text{Coalg}(FT)$. The finitary functor $T' : \text{Coalg}(FT) \rightarrow \text{Coalg}(F)$ from Remark 3.23(3) preserves this colimit, and so the coalgebra $T'(R, r_0) = (TR, r_0^2)$ is the filtered colimit of the diagram of all $\tilde{F}$-coalgebras $T'(C, c) = (TC, c^2)$. (Notice that this diagram contains the same objects as the diagram $\mathcal{D}$ from Notation 3.24 but fewer connecting morphisms—here we consider only the morphisms $Th$ for $h$ an $FT$-coalgebra homomorphism.)
Thus, since every object in this diagram has a finitely presentable algebra as its
carrier, we can apply Theorem 3.13 to conclude that $(TR_0, \rho_0^\sharp)$ is a locally finitely
presentable coalgebra. From Lemma 3.2 we see that the $T$-algebra structure $\beta : TR \to R$ is a homomorphism of $\bar{F}$ coalgebras from $(TR, r_0^\sharp)$ to $(R, r_0)$ (cf. Remark 3.3). So
finally, since the $T$-algebra structure $\beta$ is a strong epimorphism in $\text{Set}^T$ since $\beta \cdot \eta_R = \text{id}$ we conclude, using Lemma 3.18 that $(R, r_0)$ is a locally finitely presentable coalgebra as desired.

\begin{proof}
Lemma 3.31. For every locally finitely presentable $\bar{F}$-coalgebra there exists a canonical homomorphism into the coalgebra $(\nu_r(FT), r)$.

Proof. It suffices to show the statement for every coalgebra from $D$. It then follows
that every coalgebra from $\text{Coalg}(\bar{F})$ (being a coequalizer of a pair of morphisms in $D$) admits a homomorphism into $\nu_r(FT)$. So every filtered colimit of coalgebras from
$\text{Coalg}(\bar{F})$ admits a homomorphism into $\nu_r(FT)$.

Now suppose we are given $c^\sharp : TX \to FTX$ from $D$. Consider the corresponding
$FT$-coalgebra $c : X \to FTX$. Since $X$ is a finite set we obtain a unique $FT$-coalgebra
homomorphism $h$ from $(X, c)$ to the final locally finite coalgebra $\nu_r(FT)$ (once again
we write $R$ in lieu of $\nu_r(FT)$ for short):

\[
\begin{array}{ccc}
X & \xrightarrow{c} & FTX \\
R & \xrightarrow{r_0} & FTR \\
\end{array}
\]

We apply the functor $T' : \text{Coalg}(FT) \to \text{Coalg}(\bar{F})$ to obtain an $\bar{F}$-coalgebra homomorphism $Th$ from $(TX, c^\sharp)$ to $(TR, r_0^\sharp)$. Then compose with the $\bar{F}$-coalgebra homomorphisms $\beta : TR \to R$ (cf. Lemma 3.2) to obtain the desired homomorphism from $(TX, c^\sharp)$ to $(R, r)$.

Corollary 3.32. Every quotient coalgebra of $(\nu_r(FT), r)$ admits a homomorphism from
every locally finitely presentable coalgebra for $\bar{F}$.

Theorem 3.33. The rational fixed point of $\bar{F}$ is the image of $\nu_r(FT)$ under the quotient
$p : \nu(FT) \to \nu F$ from Proposition 3.5.

More precisely, there is an $\bar{F}$-coalgebra homomorphism $q : \nu_r(FT) \to \nu_r \bar{F}$ such
that the following square commutes:

\[
\begin{array}{ccc}
\nu_r(FT) & \xrightarrow{i} & \nu(FT) \\
\downarrow q & & \downarrow p \\
\nu_r \bar{F} & \xrightarrow{j} & \nu F \\
\end{array}
\]

Proof (Theorem 3.33). Let $I$ be the image in $\nu F$ of $\nu_r(FT)$ under $p$, i. e., we take the
image factorization $m \cdot e$ of $p \cdot i$. Then $I$ is a sub-$T$-algebra of $\nu F$. Now since the func-
tor $F$ preserves weak pullbacks, it preserves monos. Thus, since monomorphisms in
Set^T are precisely the injective homomorphisms, the lifting \( \hat{F} \) to \( T \)-algebras preserves monos, too. It follows that \( I \) carries the structure \( z : I \rightarrow FI \) of an \( \hat{F} \)-coalgebra making it a subcoalgebra of \( \nu F \). We will prove that \( I \) is the final locally finitely presentable \( \hat{F} \)-coalgebra.

Firstly, since the coalgebra \( (\nu_\tau(FT), r) \) is locally finitely presentable (see Lemma 3.30) so is its quotient \( (I, z) \) by Lemma 3.18. Thus, we only need to prove that for every \( \hat{F} \)-coalgebra \( c^\delta : TX \rightarrow TX \) from the category \( \mathcal{D} \) there exists a unique coalgebra homomorphism from \( (TX, c^\delta) \) to \( (I, z) \) (cf. Corollary 3.29). Since \( (I, z) \) is a subcoalgebra of the final \( \hat{F} \)-coalgebra \( \nu F \) the uniqueness of a homomorphism is clear.

For the existence of a homomorphism notice that \( (I, z) \) is a quotient coalgebra of \( (\nu_\tau(FT), r) \) via \( e : \nu_\tau(FT) \rightarrow I \) and use Corollary 3.32. \( \square \)

4 Soundness, Completeness and Kleene’s theorem in general

In this section we obtain a generalization of Kleene’s classical theorem from automata theory [19] to the setting of coalgebras for the lifting \( \hat{F} \) as presented in Section 3. We also present generic soundness and completeness results that we will then instantiate in the concrete example of weighted automata in the next section.

We still work in the setting as described in Assumption 3.19. Let us first consider our two leading examples. For the functor \( FX = 2 \times X^A \) (or, more generally, any Kripke polynomial functor \( F \) as in [34]) and the monad \( T = P_r \) consider the expression calculus obtained from (the structure of) the functor \( FT \). We have the closed syntactic expressions \( \text{Exp} \) and the least equivalence \( \equiv \) on \( \text{Exp} \) generated by the proof rules of the calculus. Then as proved in loc. cit. \( \text{Exp}/\equiv \) is isomorphic to \( \nu_\tau(FP_r) \).

Similarly, for the semiring \( k, FX = k \times X^A \) and \( T = V \) one can define an expression calculus with syntactic expressions \( \text{Exp} \), and proof rules such that \( \text{Exp}/\equiv \) is isomorphic to \( \nu_\tau(FV) \), see [910].

In each case we write \( q_0 : \text{Exp} \rightarrow \text{Exp}/\equiv \) for the canonical quotient map. This motivates the following definition.

Definition 4.1. We call a set \( \text{Exp} \) with a surjective map \( q_0 : \text{Exp} \rightarrow \nu_\tau(FT) \) an (abstract) expression calculus (for \( FT \)). The elements of \( \text{Exp} \) are referred to as expressions.

Besides the \( FT \)-bisimilarity semantics from [3410] for which the calculi given above are sound and complete, there is a different semantics that we now introduce.

Indeed, let \( q_0 : \text{Exp} \rightarrow \nu_\tau(FT) \) be an expression calculus. Recall from Lemma 3.30 that \( \nu_\tau(FT) \) carries a structure \( r \) of an \( \hat{F} \)-coalgebra. Now we see that every expression \( E \) in \( \text{Exp} \) denotes an element \( [E] \) of the final coalgebra \( \nu F \). More precisely, the semantics function \( [-] : \text{Exp} \rightarrow \nu(\hat{F}) \) is defined by

\[
[-] = (\text{Exp} \xrightarrow{q_0} \nu_\tau(FT) \xrightarrow{h} \nu(F)),
\]

(4.1)

where \( q_0 : \text{Exp} \rightarrow E = (\text{Exp}/\equiv) \) is the canonical quotient map and \( h : \nu_\tau(FT) \rightarrow \nu(F) \) is the unique \( \hat{F} \)-coalgebra homomorphism from the coalgebra \( (\nu_\tau(FT), r) \) to the final one.
In our leading examples this semantics is the usual language semantics; indeed, for non-deterministic automata \([E]\) is the formal language \(E\) denotes, and similarly, in the example of weighted automata \([E]\) is the weighted language denoted by \(E\).

Now let us fix an expression calculus \(q_0 : \text{Exp} \to \nu_\nu(FT)\). We immediately get a Kleene like theorem. First recall from Section \([22]\) that any \(F\)-coalgebra \((S, g)\) induces the unique homomorphism \(g^\dagger : S \to \nu(F)\) and for every \(s \in S\) we think of \(g^\dagger(s)\) as its behavior. Indeed, in our two leading examples \(g^\dagger\) assigns to a state the (formal or weighted) language accepted by that state. We abuse notation and write \(c^\dagger : C \to \nu(F)\) also for any \(FT\)-coalgebra \(c : C \to FTC\) meaning the language map \(h \cdot \eta_X : C \to TC \to \nu F\) obtained as in \((2.3)\).

Before we prove our generalized Kleene theorem we need a technical lemma. Recall the category \(D\) from Notation \([3.24]\).

**Lemma 4.2.** Let \((C, c)\) be a locally finitely presentable \(F\)-coalgebra. For every map \(m : X \to C\) with \(X\) a finite set there exists a coalgebra \(TY\) in \(D\), a coalgebra homomorphism \(f : TY \to C\) and a map \(m' : X \to Y\) such that the following triangle commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{m'} & Y \\
\downarrow m & & \downarrow \eta_Y \\
TY & \xrightarrow{f} & \nu_\nu \tilde{F}
\end{array}
\]

**Proof.** Given \(m : X \to C\) we have, since \(C\) is a locally finitely presentable coalgebra, some coalgebra \((P, p)\) from \(\text{Coalg}_i(\tilde{F})\), a coalgebra homomorphism \(g : P \to C\) and a map \(n : X \to P\) such that \(g \cdot n = m\). By Proposition \([3.27]\) \(P\) is the coequalizer of some parallel pair in \(D\), and so we have some coalgebra \((TZ, c^\dagger)\) in \(D\) and a surjective homomorphism \(g' : TZ \to P\) in \(\text{Coalg}_i(\tilde{F})\). Choose some map \(s : P \to TZ\) with \(g' \cdot s = \text{id}\) and let \(n' = s \cdot n\). Then \(g' \cdot n' = n\).

Now let \(Y = X + Z\) and consider the \(T\)-algebra homomorphism \([g', \eta_Z]^\sharp : TY \to TZ\). This is a split epimorphism in \(\text{Set}^T\); indeed, we have \(T\text{inr} : TZ \to TY\) with

\[n', \eta_Z]^\sharp \cdot T\text{inr} = \eta_T^\sharp = \text{id}_{TZ}.
\]

Therefore we have the coalgebra structure

\[d^\sharp = (TY \xrightarrow{\eta_Y} TZ \xrightarrow{c^\dagger} FTZ \xrightarrow{FT\text{inr}} FTY)\]

such that \([n', \eta_Z]^\sharp\) is a \(\tilde{F}\)-coalgebra homomorphism from \((TY, d^\sharp)\) to \((TZ, c^\sharp)\). So \(Y\) together with \(m' = \text{inl} : X \to TY\) and \(f = g \cdot g' \cdot [n', \eta_Z]^\sharp\) are the required data; indeed the diagram below commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{\eta_Y} & TY \\
\downarrow m' & & \downarrow \eta_Y \\
TZ & \xrightarrow{g'} & P
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{m'} & C \\
\downarrow n & & \downarrow g \\
TZ & \xrightarrow{n'} & P
\end{array}
\]
This completes the proof. \(\square\)

**Theorem 4.3.** Every state of a finite coalgebra for \(FT\) can equivalently be presented by an expression and vice versa. More precisely, we have:

1. Let \(E\) be an expression in \(\text{Exp}\), then there exists a finite \(FT\)-coalgebra \((S, g)\) and a state \(s \in S\) having the behavior \([E]\), i.e., \(g^\dagger(s) = [E]\).

2. Conversely, let \((S, g)\) be a finite \(FT\)-coalgebra and let \(s \in S\) be a state. Then there exists an expression \(E\) such that the behavior of \(s\) is \([E]\); in symbols: \(g^\dagger(s) = [E]\).

**Proof.** Ad (1). Given the expression \(E\) we have \(q \cdot q_0(E) \in \nu_r(\bar{F})\). Since \(\nu_r(\bar{F})\) is a locally finitely presentable coalgebra we can apply Lemma 4.2 to obtain a \(\bar{F}\)-coalgebra \((TS, g^\dagger)\) in \(\mathcal{D}\) and a homomorphism \(f : TS \rightarrow \nu_r(\bar{F})\) such that \(q \cdot q_0(E) = f \cdot \eta_S(s)\) for some \(s \in S\). Now compose with the homomorphism \(j : \nu_r(\bar{F}) \rightarrow \nu F\) from Theorem 3.33 to obtain:

\[
[E] = h \cdot q_0(E) = j \cdot q \cdot q_0(E) = j \cdot f \cdot \eta_S(s) = g^\dagger(s),
\]

where the last equation uses the finality of \(\nu F\).

Ad (2). Given the \(FT\)-coalgebra \((S, g)\) and \(s \in S\) form the \(\bar{F}\)-coalgebra \((TS, g^\dagger)\) and take the unique \(\bar{F}\)-coalgebra homomorphism \(f\) into the final locally finitely presentable coalgebra \(\nu_r \bar{F}\). Let \(E\) be such that \(q_0 \cdot q(E) = f \cdot \eta_S(s)\). Now composing with \(j\) yields \([E] = g^\dagger(s)\) as before. \(\square\)

Next, we will show that it always possible to “add proof rules” to an existing expression calculus in order to arrive at a sound and complete calculus w. r. t. the language semantics given by \([\cdot]\).

**Definition 4.4.** Let \((E, e)\) be an \(\bar{F}\)-coalgebra and let \(f : \text{Exp} \rightarrow E\) be a map. We call \((E, e, f)\) sound if for two expressions \(E\) and \(F\) in \(\text{Exp}\), \(f(E) = f(F)\) implies that \([E] = [F]\), and \((E, e, f)\) is called complete if \([E] = [F]\) implies \(f(E) = f(F)\).

One should think of \(E\) in the above definition as a quotient coalgebra of \((\text{Exp}/\equiv) = \nu_r(FT)\) obtained by adding proof rules so as to obtain coarser equivalence \(\equiv_D\) with \(E = (\text{Exp}/\equiv_D)\). In fact, we have the following

**Theorem 4.5 (Soundness).** Every quotient coalgebra of \(\nu_r(FT)\) is sound.

**Proof.** Let \(E\) be a quotient coalgebra of \(\nu_r(FT)\) via \(q : E \rightarrow \nu_r(FT)\) and let \(j : E \rightarrow \nu F\) be the unique coalgebra homomorphism. We consider the map \(q \cdot q_0 : \text{Exp} \rightarrow E\) and verify the soundness by proving that the diagram below commutes:

\[
\begin{array}{ccc}
\text{Exp} & \xrightarrow{q_0} & \nu_r(FT) \\
& \downarrow h & \downarrow j \\
& \nu F & \\
\end{array}
\]

Indeed, the left-hand part commutes by the definition of the semantic map \([\cdot]\) (see (4.1)), and the right-hand part commutes since all its arrows are \(\bar{F}\)-coalgebra homomorphisms into the final coalgebra.

Now whenever for two expressions \(E\) and \(F\) in \(\text{Exp}\) we have \(q \cdot q_0(E) = q \cdot q_0(F)\) we clearly have \([E] = [F]\), and this is the desired soundness. \(\square\)
In particular, we see \((\nu_r(FT), r, q_0)\) is sound. Now recall that the final locally finitely presentable coalgebra \(\nu_r \bar{F}\) is the (greatest) quotient of \((\nu_r(FT), r)\) via the homomorphism \(q : \nu_r(FT) \rightarrow \nu_r \bar{f}\), see Theorem 3.33. In addition we have

**Theorem 4.6 (Completeness).** The final locally finitely presentable coalgebra \(\nu_r \bar{F}\) together with the map \(q \cdot q_0 : \text{Exp} \rightarrow \nu_r \bar{F}\) is complete.

*Proof.* Recall the four \(\bar{F}\)-coalgebra homomorphisms from the statement of Theorem 3.33. Now consider diagram 4.2 where \(E = \nu_r \bar{F}\). If for two expression \(E\) and \(F\) in \(\text{Exp}\) we have \([E] = [F]\) then \(q \cdot q_0(E) = q \cdot q_0(F)\) since \(j : \nu_r \bar{F} \rightarrow \nu F\) is injective. Therefore we obtain the desired completeness. \(\Box\)

Of course, one may wonder at this point about the relevance of the theorems in this section because we did not introduce any concrete syntax and proof rules. But we shall see in the next sections that from the above abstract results we automatically obtain soundness, completeness and Kleene theorems for concrete syntactic calculi once we’ve established that the quotient formed by concrete syntactic expressions modulo proof rules forms a (weakly) final locally finitely presentable \(\bar{F}\)-coalgebra.

### 5 Expression calculus for weighted automata

In [24] the second author has presented a sound and complete expression calculus for linear systems presented in the form of closed stream circuits. In this section we are going to use the ideas from loc. cit. and apply the results from Section 3.2 to provide a sound and complete expression calculus for the language equivalence of linear weighted automata.

In this section we work with the category \(k\)-Mod for a semiring \(k\) such that finitely generated semimodules are closed under kernel pairs. This is true whenever \(k\) is Noetherian, see Proposition 2.3, but also for \(k\) the natural numbers. We also consider the semimodule monad \(T = V\), see (2.1).

We restrict our attention to the functor \(FX = k \times X^A\), where \(A\) is a fixed finite set (of input symbols). But in fact, using the ideas from [9,10] it is not difficult to define a generic calculus for every functor from the class of set endofunctors defined by the grammar in Remark 2.5.

So our functor \(F = k \times (-)^A\) has a canonical lifting \(\bar{F}\) to \(k\)-Mod. As we saw in Example 3.14(4) the final coalgebra for \(F\) and its lifting is carried by the set \(k^A^*\) of all weighted languages.

Coalgebras for the composite \(FV\) are weighted automata, i.e., weighted automata with weights in the semiring \(k\), see [14]. The expression calculus one obtains in this particular instance from the work in [9,10] allows one to reason about the equivalence of weighted automata w.r.t. weighted bisimilarity (cf. [12]). We will now recall the syntax and proof rules of this calculus. The syntactic expressions are defined by the following grammar

\[
E :::= x | 0 | E \oplus E | \underline{r} | a.(r \bullet E) | \mu x.E^\beta.
\]
Recall from [9,10] that the variable binding operator $\mu x.$— is only applied to guarded expressions, i.e., expressions $E^g$ where each occurrence of $x$ is within the scope of a subterm $r \oplus \cdot$.

We write Exp for the set of all closed expressions defined by the above grammar. The calculus of [9,10] puts on these expressions certain rules and equations stating that $\mu$ is a unique fixed point operator, that $\oplus$ is a commutative and associative binary operation with the neutral element 0, etc; here is the list of rules:

\[
F \equiv E[F/x] \quad \Rightarrow \quad F \equiv \mu x. E \quad a.(0 \bullet E) \equiv 0 \quad 0 \oplus E \equiv 0 \\
(E \oplus F) \oplus G \equiv F \oplus (F \oplus G) \quad E \oplus F \equiv F \oplus E \\
a.(r \bullet E) \oplus a.(s \bullet E) \equiv a.((r + s) \bullet E) \quad \mu x. E \equiv E[\mu x. E/x] \\
r \oplus s \equiv r + s \\
\]

We call the two rules pertaining to $\mu$ the fixpoint rule and the uniqueness rule, respectively. In addition the rules contain $\alpha$-equivalence, i.e., renaming of bound variables does not matter and the replacement rule:

\[
E \equiv F \quad A[E/x] \equiv A[F/x],
\]

where $E$, $F$ and $A$ are expressions and $x$ is a free variable in $A$. We write $\equiv$ for the least equivalence on $\text{Exp}$ generated by the above rules.

The main result of [9,10] is that this calculus is sound and complete for bisimilarity equivalence of weighted automata. The key fact used in order to prove soundness and completeness is that the set $E = \text{Exp}/\equiv$ of closed syntactic expressions modulo the proof rules above is (isomorphic to) the locally finite final coalgebra $\nu_r(FV)$, cf. Section 3.2.

Now we will turn to a different semantics of the expressions in $\text{Exp}$, the weighted language described by them. The corresponding semantic map is $\llbracket - \rrbracket$ from (4.1).

Remark 5.1. (1) By Lemma 3.30, we see that $E = \nu_r(FV)$ has a canonical structure of a $V$-algebra, i.e., $E$ is a $k$-semimodule. It is straightforward to work out that $[0]$ is the zero vector, that the semimodule addition is

\[
[E] + [F] = [E \oplus F]
\]

and that the action of the semiring $k$ satisfies the following laws:

\[
\begin{align*}
  r[0] & = [0] \\
  r[E \oplus F] & = [rE \oplus rF] \\
  r[\mu x. E] & = [\mu x.(rE)] \\
  r[s] & = [rs] \\
  r[a.(s \bullet E)] & = [a.(rs) \bullet E]
\end{align*}
\]

From now on we will omit the square brackets indicating equivalence classes w. r. t. $\equiv$ and simply write $E$ for elements of $E$.

(2) Furthermore, since $E = \nu_r(FV)$ we have the coalgebra structure $r_0 : E \rightarrow FV(E)$ and we have the Eilenberg-Moore algebra structure $\beta : V(E) \rightarrow E$ which gives us an $\bar{F}$-coalgebra structure $r = F\beta \cdot r_0$ on $E$, cf. Lemma 3.30 and (3.7). For further
reference we note that the coalgebra structure \( r_0 : E \rightarrow FV(E) \) acts for example as follows (cf. [9,10]):
\[
\begin{align*}
    r_0(a.(r \cdot E)) &= (0, f), \\
    r_0(r) &= (r, g),
\end{align*}
\]
(5.3)
where \( f : A \rightarrow V(E) \) is the function with \( f(a) = rE \) and \( f(b) = 0 \) for \( b \neq a \) (we omit equivalence classes here, and so we do have the formal linear combination \( rE \in V(E) \)) and \( g : A \rightarrow V(E) \) is constant on 0.

(3) Recall from Section 4 that the canonical quotient map \( q_0 : \text{Exp} \rightarrow E = (\text{Exp}/\equiv) \) gives us an expression calculus in the sense of Definition 4.1 and the corresponding semantics map \( \llbracket - \rrbracket : \text{Exp} \rightarrow kA^* \) assigns to every expression the weighted language it denotes.

From the generic Kleene theorem 4.3, we obtain immediately a Kleene like theorem stating that every state of a weighted automaton can equivalently be specified by an expression of our calculus.

**Theorem 5.2.** (1) For every expression \( E \) in \( \text{Exp} \) there exists a finite weighted automaton \( S \) and a state \( s \) such that the weighted language accepted by \( s \) is \( \llbracket E \rrbracket \).
(2) For every state \( s \) of a finite weighted automaton there exists an expression that denotes the same weighted language accepted by the state \( s \).

Indeed, this is just a restatement of Theorem 4.3 noting that finite weighted automata are precisely finite \( FV \)-coalgebras.

In classical automata theory one obtains, of course an algorithmic construction of an expression for a given state of an automaton. The above theorem does not provide such a construction. However, in our theory the respective construction does occur, namely in the proof of Theorem 5.10 below.

### 5.1 Axiomatization of weighted language equivalence

We are now going to add the following three additional equational laws to the calculus from the previous section:
\[
\begin{align*}
    a.(r \cdot (E \oplus F)) &\equiv_D a.(r \cdot E) \oplus a.(r \cdot F) \quad (5.4) \\
    a.(r \cdot b.(s \cdot E)) &\equiv_D a.((rs) \cdot b.(1 \cdot E)) \quad (5.5) \\
    a.(r \cdot s) &\equiv_D a.(1 \cdot rs) \quad (5.6) \\
    a.(r \cdot 0) &\equiv_D 0. \quad (5.7)
\end{align*}
\]
Notice that we write \( \equiv_D \) for the least equivalence generated by all the above rules (i.e., all the rules from the previous section and the three last ones).

**Remark 5.3.** Observe that for a field \( k \) the equational law (5.7) is provable from the other laws. Indeed, using (5.4) this follows from
\[
\begin{align*}
    a.(r \cdot 0) &\equiv_D a.(r \cdot (0 \oplus 0)) \equiv_D a.(r \cdot 0) \oplus a.(r \cdot 0)
\end{align*}
\]
since \( E_D \) is an (abelian) group w. r. t. \( \oplus \).
We denote by \( E_D = (\text{Exp}/\equiv_D) \) the closed expression modulo all these proof rules. Notice that \( E_D \) is a quotient of \( E \) via \( q : E \to E_D \), say.

**Lemma 5.4.** The quotient \( E_D \) is a \( k \)-semimodule and \( q : E \to E_D \) is a homomorphism of semimodules.

**Proof.** We only need to prove that the four additional equational laws in (5.4)–(5.6) respect the semimodule structure of \( E \), i.e., the semimodule operations are well-defined on equivalence classes.

For the addition this follows from the replacement rule (5.1). We verify well-definedness for the action of the semiring \( k \) for each of the four equational laws:

Ad (5.4) we have

\[
s(a.(r \cdot E) \oplus a.(r \cdot F)) \equiv_D s(a.(r \cdot E)) \oplus s(a.(r \cdot F)) \quad \text{see (5.2)}
\]

\[
\equiv_D a.(sr) \cdot E \oplus a.(sr) \cdot F \quad \text{see (5.2)}
\]

\[
\equiv_D a.(sr) \cdot (E \oplus F) \quad \text{see (5.4)}
\]

\[
\equiv_D s(a.(r \cdot (E \oplus F)) \quad \text{see (5.2)}.
\]

Ad (5.5) we have

\[
c(a.(r \cdot b.(s \cdot E))) \equiv_D a.((cr) \cdot b.(s \cdot E)) \quad \text{see (5.2)}
\]

\[
\equiv_D a.((crs) \cdot b.(1 \cdot E)) \quad \text{by (5.5)}
\]

\[
\equiv_D c(a.((rs) \cdot b.(1 \cdot E))) \quad \text{see (5.2)}.
\]

Ad (5.6) we have

\[
c(a.(r \cdot s)) \equiv_D a.((cr) \cdot s) \quad \text{see (5.2)}
\]

\[
\equiv_D a.(1 \cdot crs) \quad \text{by (5.6)}
\]

\[
\equiv_D a.(c \cdot rs) \quad \text{by (5.6)}
\]

\[
\equiv_D c(a.(1 \cdot rs)) \quad \text{see (5.2)}.
\]

Ad (5.7) we have

\[
c(a.(r \cdot 0)) \equiv_D a.((cr) \cdot 0) \equiv_D 0.
\]

This completes the proof. \(\square\)

**Lemma 5.5.** For the action of the semiring \( k \) in \( E_D \) we have the following provable identity:

\[
r(a.(s \cdot E)) \equiv_D a.((rs) \cdot E).
\]

**Proof.** Recall from (5.2) that \( r(a.(s \cdot E)) = a.((rs) \cdot E) \). Now the proof proceeds by induction on the complexity of expressions. Here are the different cases (we drop the subscript in \( \equiv_D \)):

1. For \( E = 0 \) we apply (5.7) and get

\[
a.((rs) \cdot 0) \equiv 0 \equiv a.(s \cdot 0) \equiv a.(s \cdot (r0)).
\]

2. For \( E = t \) we use (5.6) and (5.2) to obtain

\[
a.((rs) \cdot t) \equiv a.(1 \cdot rsr) \equiv a.(1 \cdot (sr) t) \equiv a.(s \cdot rt) \equiv a.(s \cdot (rt)).
\]
(3) For a sum \( E = A + B \) we compute
\[
a.((rs) \cdot (A + B)) \equiv a.((rs) \cdot A) \oplus a.((rs) \cdot B) \quad \text{by (5.4)}
\]
\[
\equiv a.(s \cdot rA) \oplus a.(s \cdot rB) \quad \text{by induction hypothesis}
\]
\[
\equiv a.(s \cdot (rA + rB)) \quad \text{by (5.4)}
\]
\[
\equiv a.(s \cdot r(A + B)) \quad \text{by (5.2)}.
\]

(4) For \( E = b.((t \cdot F)) \) we use (5.5) and obtain
\[
a.((rs) \cdot b.((t \cdot F))) \equiv a.((rst) \cdot b.(1 \cdot F))
\]
\[
\equiv a.(s \cdot b.((rt) \cdot F))
\]
\[
\equiv a.(s \cdot r(b.(t \cdot F))).
\]

(5) Finally, for a \( \mu \)-term \( E = \mu x.F \) one simply uses the induction hypothesis on \( F[\mu x.F/x] \) to obtain
\[
a.((rs) \cdot (\mu x.F)) \equiv a.((rs) \cdot F[\mu x.F/x])
\]
\[
\equiv a.(s \cdot (rF[\mu x.F/x]))
\]
\[
\equiv a.(s \cdot r(\mu x.F)).
\]

This completes the proof.

\[\Box\]

5.2 Soundness of the calculus

We will now show that we have obtained a sound calculus for reasoning about weighted language equivalence.

In order to achieve our goal we will show that \( E_D \) is a coalgebra for the lifting \( \bar{F} : k\text{-Mod} \to k\text{-Mod} \), and it is a quotient coalgebra of \((E, r)\) from the previous section. Then we apply the general soundness theorem from Section 4.

Lemma 5.6. The map \( Fq \cdot F\beta \cdot r_0 : E \to F(E_D) \) is well-defined w. r. t. the kernel equivalence of \( q : E \to E_D \).

Proof. It is sufficient to show that \( F\beta \cdot r_0 \) merges both sides of the four equations (5.4)–(5.7). We also use the following notation for (certain) elements of \( S^A \), where \( S \) is a semimodule: for \( s \in S \) we write \( a \mapsto s \) for the function \( f : A \to S \) with \( f(a) = s \) and \( f(b) = 0 \) for \( b \neq a \).

Ad (5.4) we compute
\[
F\beta \cdot r_0(a.(r \cdot E) \oplus a.(r \cdot F)) = F\beta(0, (a \mapsto rE) + (a \mapsto rF)) \quad \text{see (5.8)}
\]
\[
= F\beta(0, a \mapsto (rE + rF)) \quad \text{linear structure on } V(E)^A
\]
\[
= (0, a \mapsto (rE + rF)) \quad \beta \text{ is a } V\text{-algebra}
\]
\[
= (0, a \mapsto r(E \oplus F)) \quad E \text{ is a vector space}
\]
\[
= F\beta(0, a \mapsto r(E \oplus F)) \quad \beta \text{ is a } V\text{-algebra}
\]
\[
= F\beta \cdot r_0(a.(r \cdot (E \oplus F))) \quad \text{see (5.8)}.
\]

Notice that in the second line addition and action of \( k \) are formal, i.e., \( rE + rF \) lies in \( V(E) \) while in the third line the formal operations are evaluated in the semimodule \( E \).
Ad (5.5) we compute
\[ F_\beta \cdot r_0(a.(r \cdot b.(s \cdot E))) = F_\beta(0, a \mapsto r(b.(s \cdot E))) \quad \text{see (5.3)} \]
\[ = (0, a \mapsto r(b.(s \cdot E))) \quad \beta \text{ is a } V\text{-algebra} \]
\[ = (0, a \mapsto b.((rs) \cdot E)) \quad \text{see (5.2)} \]
\[ = (0, a \mapsto (rs)(b.(1 \cdot E))) \quad \text{see (5.2)} \]
\[ = F_\beta(0, a \mapsto (rs)(b.(1 \cdot E))) \quad \beta \text{ is a } V\text{-algebra} \]
\[ = F_\beta \cdot r_0(a.(rs) \cdot b.(1 \cdot E)) \quad \text{see (5.3)}. \]

Ad (5.6) we compute
\[ F_\beta \cdot r_0(a.(r \cdot 0)) = F_\beta(0, a \mapsto r.0) \quad \text{see (5.3)} \]
\[ = (0, a \mapsto r.0) \quad \beta \text{ is a } V\text{-algebra} \]
\[ = (0, a \mapsto 0) \quad \text{see (5.2)} \]
\[ = F_\beta(0, 0) \quad \beta \text{ is a } V\text{-algebra} \]
\[ = F_\beta \cdot r_0(0) \quad \text{see (5.3)}. \]

This completes the proof. \( \square \)

**Corollary 5.7.** There is a coalgebra structure \( c : E_D \to F(E_D) \) such that \( q \) is a \( \bar{F} \)-coalgebra homomorphism from the coalgebra \( (E, r) \) in Remark 5.1(2) to \( (E_D, c) \).

**Proof.** Define \( c([E]) = Fq \cdot F_\beta \cdot r_0(E) \). Then \( c \) is well-defined by Lemma 5.6 a semimodule homomorphism since \( q, \beta \) and \( r_0 \) are so, and \( c \cdot q = Fq \cdot (F_\beta \cdot r_0) \) clearly holds.

**Theorem 5.8 (Soundness).** The calculus is sound: whenever we have \( E \equiv_D F \) for two expressions, then also \( \llbracket E \rrbracket = \llbracket F \rrbracket \).

Indeed, this is just an application of Theorem 4.5 to the quotient coalgebra \( q : (E, r) \to (E_D, c) \) for \( \bar{F} \).

### 5.3 Completeness

We are ready to prove the completeness of our calculus w. r. t. weighted language equivalence of expressions. The key ingredient for our completeness proof is the fact that \( E_D \) is the final locally finite dimensional coalgebra for \( \bar{F} : k\text{-Mod} \to k\text{-Mod} \).

**Lemma 5.9.** The map \( c : E_D \to F(E_D) \) is a semimodule isomorphism.
Proof. We first define the map $d : F(E_D) \to E_D$ by

$$d(r, \langle [E^a] \rangle_{a \in A}) = [r \oplus \sum_{a \in A} a \cdot (1 \bullet E)].$$

By the replacement rule (5.1) $d$ is well-defined. We first prove that $d$ preserves sums:

$$d((r, \langle [E^a] \rangle_{a \in A}) + (s, \langle [F^a] \rangle_{a \in A}))$$

$$= d(r + s, \langle [E^a \oplus F^a] \rangle_{a \in A})$$

$$= (r + s) \oplus \sum_{a \in A} a \cdot (1 \bullet (E^a \oplus F^a))$$

$$= \left( r \oplus \sum_{a \in A} a \cdot (1 \bullet E^a) \right) \oplus \left( s \oplus \sum_{a \in A} a \cdot (1 \bullet F^a) \right)$$

by (5.4)

$$= d(r, \langle [E^a] \rangle_{a \in A}) + d(s, \langle [F^a] \rangle_{a \in A})$$

definition of $d$.

We now prove that $c$ and $d$ are mutually inverse. To see that $c \cdot d = \text{id}$ we compute:

$$c \cdot d(r, \langle [E^a] \rangle_{a \in A}) = c \left( r \oplus \sum_{a \in A} 1 \bullet E \right)$$

definition of $d$

$$= c ([r] + \sum_{a \in A} 1 \bullet E^a)$$

linearity of $c$

$$= (r, \langle [0] \rangle) + \sum_{a \in A} (0, a \to [E^a])$$

see 5.6, 5.7 and 5.3

$$= (r, \langle [E^a] \rangle_{a \in A})$$

linear structure on $k \times E_D^a$.

Finally, we verify that $d \cdot c = \text{id}$, and we show this by induction on the complexity of expressions $E$ (for easier readability we omit the square bracket indicating equivalence classes):

For $E = 0$ we have

$$d \cdot c(0) = d(0, \langle 0 \rangle_{a \in A}) = 0 \oplus \sum_{a \in A} a \cdot (1 \bullet 0) \equiv_D 0 \oplus 0 \equiv_D 0,$$

by the definitions of $c$ and $d$ and using (5.7).

For $E = r$ we obtain

$$d \cdot c(r) = d(r, \langle 0 \rangle_{a \in A}) = r \oplus \sum_{a \in A} a \cdot (1 \bullet 0) = r,$$

where the last step uses the vector space structure on $E_D$ and (5.7).

Next, for $E = A \oplus B$ we simply use that $c$ and $d$ preserve sums and the induction hypothesis to obtain

$$d \cdot c(A \oplus B) = d(c(A) + c(B)) = d(c(A)) \oplus d(c(A)) = A \oplus B.$$

For $E = a \cdot (r \bullet E)$ we compute

$$d \cdot c(a \cdot (r \bullet E)) = d(0, a \to rE)$$

see (5.3)

$$= 0 \oplus a \cdot (1 \bullet (rE))$$

definition of $d$ and linear structure of $E_D$

$$= a \cdot (1 \bullet (rE))$$

by Lemma 5.5

$$= r(a \cdot (1 \bullet E))$$

see 5.2.
Finally, for a $\mu$-expression $E = \mu x.A$ we simply use the fixpoint rule and the induction hypothesis to obtain
\[
d \cdot c(\mu x.A) = d \cdot c(\mu x.A/x) = A[\mu x.A/x] = \mu x.A.
\]
This completes the proof. 

**Theorem 5.10.** For every $F$-coalgebra $(VS, g)$ with $S$ finite there exists a unique coalgebra homomorphism from $(VS, g)$ to $(ED, c)$.

**Proof.** Since the coalgebra $(ED, c)$ is a quotient of the coalgebra $(E, r)$ we obtain the existence of a homomorphism from Corollary 3.32. It remains to verify its uniqueness.

So let $m : (VS, g) \to (ED, c)$ by any $F$-coalgebra homomorphism. Let us assume that $S = \{s_1, \ldots, s_n\}$. It suffices to prove that the $m(s_i)$ are uniquely determined.

In order to prove this we will first define closed expressions $\langle \langle s_i \rangle \rangle$ and then show that these are provably equivalent to $m(s_i)$.

The expressions $\langle \langle s_i \rangle \rangle$ are defined by an $n$-step process. Let
\[
g(s_i) = \left( r_i, \left( \sum_{j=1}^{n} r_i^a s_j \right)_{a \in A} \right), \quad i = 1, \ldots, n. \tag{5.8}
\]

Our expressions will involve the scalars $r_i$, the coefficients $r_i^a$ and $n$ variables $x_1, \ldots, x_n$. For every $i = 1, \ldots, n$ let
\[
A_i^0 = \mu x_i. \left( r_i \oplus \sum_{a \in A} (a.(r_i^a \bullet x_1) \oplus \cdots \oplus a.(r_i^{a_n} \bullet x_n)) \right).
\]

Now define for $k = 0, \ldots, n - 1$
\[
A_i^{k+1} = \begin{cases} A_i^k \{ A_i^{k+1}/x_{k+1} \} & \text{if } i = k + 1 \\ A_i^k & \text{if } i = k + 1, \end{cases}
\]

where $\{ A/x \}$ denotes syntactic replacement (i.e., substitution without renaming of bound variables). It is easy to see that the set of free variables of $A_i^k$ is $\{ x_{k+1}, \ldots, x_n \} \setminus \{ x_i \}$, and moreover, every occurrence of those variables is free.

We also see that for every $i$,
\begin{align*}
A_i^n &= A_i^0 \{ A_1^1/x_1 \} \{ A_2^2/x_2 \} \cdots \{ A_i^{i-2}/x_{i-1} \} \{ A_{i+1}^i/x_{i+1} \} \cdots \{ A_{n-1}^{n-1}/x_n \} \\
&= A_i^{n-1} \{ A_{i+1}^i/x_{i+1} \} \cdots \{ A_n^{n-1}/x_n \}.
\end{align*}

Observe that $A_i^n$ is a closed term. Moreover, the variable $x_i$ from $A_i^0$ is never syntactically replaced and it is bound by the outermost $\mu x_i$. All other occurrences of $x_i$ in $A_i^n$ are not bound by this $\mu$-operator (but by $\mu$-operators further inside the term). We define
\[
\langle \langle s_i \rangle \rangle = A_i^n.
\]

From now on we shall abuse notation and we will denote equivalence classes $[A]$ of expressions in $ED$ simply by expressions $A$ representing them.
It is our goal to prove that \( m(s_i) \equiv_D \llbracket s_i \rrbracket \). Let us write \( m_i \) for (some representative of) \( m(s_i) \), for short. Using the fact that \( m \) is a coalgebra homomorphism, Lemma 5.9 and equation (5.8) and we see that

\[
m_i = c^{-1} \cdot Fm \cdot g(s_i)
= c^{-1} \cdot Fm \left( r_1, \left( \sum_{j=1}^{n} r^a_{ij} s_j \right)_{a \in A} \right)
= c^{-1} \left( r_1, \left( \sum_{j=1}^{n} r^a_{ij} m_j \right)_{a \in A} \right)
= r_1 \oplus \sum_{a \in A} a \cdot \left( 1 \cdot \sum_{j=1}^{n} r^a_{ij} m_j \right).
\]

(5.9)

For the proof of \( m_i \equiv_D \llbracket s_i \rrbracket \), we show the case \( n = 3 \) in detail; the general case is completely analogous and is left to the reader.

We start by proving that \( m_1 \equiv_D A^0_1 \llbracket m_2/x_2 \rrbracket [m_3/x_3] \) by an application of the uniqueness rule; indeed, from (5.9) we get

\[
m_1 \equiv_D r_1 \oplus \sum_{a \in A} a \cdot (1 \cdot (r^a_{11} m_1 + r^a_{12} m_2 + r^a_{13} m_3))
= (r_1 \oplus \sum_{a \in A} a \cdot (1 \cdot (r^a_{11} m_1 + r^a_{12} m_2 + r^a_{13} m_3)))[m_2/x_2][m_3/x_3][m_1/x_1].
\]

Next, we prove that \( m_2 \equiv_D A^1_2 \llbracket m_3/x_3 \rrbracket \). Notice that

\[
A^0_1 \llbracket m_2/x_2 \rrbracket [m_3/x_3] = A^0_1 \llbracket m_3/x_3 \rrbracket [m_2/x_2]
\]

since \( m_2 \) and \( m_3 \) are closed. Then, applying (5.9), we have

\[
m_2 \equiv_D r_2 \oplus \sum_{a \in A} a \cdot (1 \cdot (r^a_{21} m_1 + r^a_{22} m_2 + r^a_{23} m_3))
\equiv_D r_2 \oplus \sum_{a \in A} a \cdot (1 \cdot (r^a_{21} A^0_1 \llbracket m_2/x_2 \rrbracket [m_3/x_3] + r^a_{22} m_2 + r^a_{23} m_3))
= (r_2 \oplus \sum_{a \in A} a \cdot (1 \cdot (r^a_{21} A^0_1 \llbracket m_3/x_3 \rrbracket + r^a_{22} m_2 + r^a_{23} m_3)))[m_2/x_2],
\]

and so we can apply the uniqueness rule to obtain the desired equation.

Now we are able to prove that

\[
m_1 \equiv_D A^0_1 \llbracket A^1_2/x_2 \rrbracket [m_3/x_3].
\]

Notice first that we have \( A^0_1 \llbracket A^1_2/x_2 \rrbracket = A^0_1 \llbracket A^1_2/x_2 \rrbracket \) since \( x_1 \) (which is bound in \( A^0_1 \)) is not free in \( A^1_2 \). Now we obtain

\[
A^0_1 \llbracket A^1_2/x_2 \rrbracket [m_3/x_3] \equiv_D A^0_1 [m_3/x_3] [A^1_2 [m_3/x_3]/x_2]
\equiv_D A^0_1 [m_3/x_3] [m_2/x_2]
\equiv_D m_1.
\]
Finally, we show that \( m_3 \equiv_D A^2_3 \) by another application of the uniqueness rule; we have

\[
m_3 \equiv_D r_3 \oplus \sum_{a \in A} a \cdot (1 \bullet (r_{31}^a m_1 + r_{32}^a m_2 + r_{33}^a m_3)) \equiv_D r_3 \oplus \sum_{a \in A} a \cdot (1 \bullet (r_{31}^a A_1^0 \{A_2^1/x_2\} [m_3/x_3] + r_{32}^a A_2^1 \{m_3/x_3\} + r_{33}^a m_3)) \]

\[
= (r_3 \oplus \sum_{a \in A} a \cdot (1 \bullet (r_{31}^a A_1^0 \{A_2^1/x_2\} + r_{32}^a A_2^1 + r_{33}^a x_3))) [m_3/x_3].
\]

So we have proved

\[
m_3 \equiv_D A^2_3 = A^3 = \langle \langle s_3 \rangle \rangle.
\]

This implies that

\[
m_2 \equiv_D A_1^2 \{m_3/x_3\} \equiv_D A_2^2 [A_3^2/x_3] = A_2^1 \{A_3^2/x_3\} = A^3 = \langle \langle s_2 \rangle \rangle,
\]

where the third step holds since the bound variables \( x_1 \) and \( x_2 \) of \( A_1^2 \) are also bound in \( A_3^2 \). Similarly, we have

\[
m_1 \equiv_D A_1^0 \{A_2^1/x_2\} [m_3/x_3] \equiv_D A_1^0 \{A_2^1/x_2\} [A_3^2/x_3] = A^3 = \langle \langle s_1 \rangle \rangle.
\]

This completes the proof. \( \square \)

**Corollary 5.11.** The coalgebra \( (E_D, c) \) is the final locally finitely presentable coalgebra for \( \bar{F} \).

**Proof.** Indeed, to see that \((E_D, c)\) is a locally finitely presentable coalgebra we use that the coalgebra \((E, r)\) from Remark 5.1 is locally finitely presentable (see Lemma 3.30). Since \( E_D \) is a quotient coalgebra of \( E \) by Corollary 5.7, we see that \( E_D \) is locally finite dimensional, too (apply Lemma 3.18). The finality of \((E_D, c)\) now follows from Corollary 3.29. \( \square \)

**Theorem 5.12 (Completeness).** Whenever we have \( \llbracket E \rrbracket = \llbracket F \rrbracket \) for two expressions, then they are provably equivalent, in symbols: \( E \equiv_D F \).

Indeed, this is just an application of Theorem 4.6 to \( E_D = \nu_r \bar{F} \) with the map \( q \cdot q_0 : \text{Exp} \to E_D \).

### 6 Sound and complete calculi for nondeterministic systems

In this section we present some details of an interesting special case of the work in the previous section—the case of non-deterministic automata. The calculus becomes somewhat simpler in this case but all results are just consequence of the more general results of Section 5.

Here \( k \) is the Boolean semiring, and so the category \( k\text{-Mod} \) is the category \( Jsl \), of join-semilattices and join-preserving maps, which is isomorphic to the category of Eilenberg-Moore algebras for the finite powerset monad \( P_t \).
Once again, we will restrict our attention to the functor $F X = 2 \times X^A$, where $A$ is a fixed set and 2 the two element join-semilattice. However, all the results we will present can easily be extended to define a generic calculus for a larger class of functors (Remark 2.5, for $T = \mathcal{P}_1$).

A coalgebra $(X, \alpha)$ for the functor $F X = 2 \times X^A$, in $\mathcal{J} \mathcal{S} \mathcal{L}$, corresponds to a coalgebra of the composite type $F \mathcal{P}_1$ in $\mathcal{S} \mathcal{E} \mathcal{T}$, that is, a function $X \rightarrow 2 \times \mathcal{P}_1(X)^A$. Hence, coalgebras for this functor are simply non-deterministic automata.

In [34], one considers the language $\text{Exp}$ of closed and guarded expressions defined by the following grammar

$$E ::= x \mid 0 \mid E \oplus E \mid 1 \mid a.E \mid \mu x.E.$$  

Notice that this is just a simplification of the syntax of the calculus from Section 5. Indeed, $a.E$ corresponds to $a.(1 \cdot E)$, and we do not need the expression $a.(0 \cdot E)$ as this is provably equivalent to 0. These syntactic expressions describe precisely the rational behaviors of non-deterministic automata. Our set of axioms from Section 5 now states that (1) $\mu$ is a unique fixed point operator, (2) $\oplus$ is a associative, commutative and idempotent binary operation with the neutral element 0 and that (3) the $\alpha$-equivalence (i.e., renaming of bound variables does not matter) and the replacement (also called congruence) rules are valid. In fact, those are exactly the axioms and rules considered in [34], where they were proven sound and complete with respect to bisimilarity. Again, $E$ denotes the set of expressions modulo the axioms for bisimilarity.

To obtain a sound and complete axiomatization for language equivalence we only need to add the following two axioms the above axiomatization:

$$a.(E_1 \oplus E_2) \equiv a.E_1 \oplus a.E_2 \quad \text{and} \quad a.0 = 0. \quad (6.1)$$

Indeed, these new axioms corresponds to (5.4) and (5.7), and the other two added axioms of the previous section already trivially hold in the current special case. In this way we recover the result of Rabinovich [27] for labelled transition systems (which are just non-deterministic automata where every state is considered final). Also note that the result of [34] coincides precisely with Milner’s results [25] for labeled transition systems and bisimilarity, which constituted the base of Rabinovich’s work.

From Section 5 we get: (1) a Kleene Theorem: every state of a non-deterministic automaton is language equivalent to an expression in the calculus and vice-versa; (2) soundness of the calculus from Theorem 5.8; (3) completeness of the calculus from Theorem 5.12, which uses that the coalgebra $E_D$ of expressions from $\text{Exp}$ modulo all the axioms is final among all locally finite coalgebra for $\bar{F}$ (cf. Corollary 5.11).

7 Conclusions and Future Work

In this paper, we have presented a general methodology to extend sound and complete calculi with respect to behavioral equivalence to sound and complete calculi with respect to coalgebraic language equivalence. We illustrated our general framework by applying it to two concrete instances, non-deterministic automata and weighted automata.

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8 That $\oplus$ is idempotent can be derived using the semiring action: $E \oplus E = 1E \oplus 1E = (1 + 1)E = 1E = E$. 
For the former, we recovered Rabinovich’s results [27], whereas for the latter we presented, to the best of our knowledge, the first sound and complete axiomatization of weighted language equivalence.

A key fact to be established in our soundness and completeness proofs is that expressions modulo proof rules form the final locally finitely presentable coalgebra. The development of the mathematical theory of these coalgebras was started in [24], and we continue this in the current paper.

Even though we did not present the details, our method is generic. For non-deterministic systems it applies to all coalgebras for $FP_i$ and for linear systems we can deal with coalgebras of type $FV$, where $F$ is from the class of functors as described in Remark 2.5.

One very interesting direction for future work concerns the question whether the calculi for coalgebraic language equivalence we have developed are decidable.

We presented the main results of the theory for the base category $Set$. In the future we plan to extend this to more general base categories in order to deal with systems whose state spaces have extra structure, e.g., they form posets, graphs or presheaves.

Unfortunately, our main result on final locally finitely presentable coalgebras (Theorem 3.33) uses the assumption that finitely generated objects be closed under kernel pairs (see Proposition 3.15). This assumption is somewhat restrictive, and we intend to study whether this can be relaxed. This would allow to consider other monads $T$, i.e., other branching types like, for instance, various kinds of probabilistic systems.

As we saw in our work, the generalized power set construction lets us move from systems of type $FT$ to systems of type $F$ (in the category of $T$-algebras). So $F$ easily can encode outputs of systems using products with constant functors. On the other hand, coalgebraic trace semantics [17] deals with functors of the form $TF$, and one works with coalgebras for (the lifting of $F$) to the Kleisli category of $T$. This easily deals with system inputs by using products with constant functors (e.g. $TF = \mathcal{P}(A \times -)$ for labelled transition systems). It would be desirable to find a framework that accommodates both these approaches and deals with input and output at the same time.

References


