Cover preserving embedding of modular lattices into partition lattices

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Abstract
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When is a finite modular lattice cover preserving embeddable into a partition lattice? We give some necessary, and slightly sharper sufficient conditions. For example, the class of cover preserving embeddable modular lattices strictly contains the class of acyclic modular lattices.

1. Introduction

Without further mention all structures considered in this paper will be finite. Any notion not defined is explained in either [1, 2, 5] or [14, 15].

According to a famous Theorem of Dilworth [2, p. 125] each lattice \( L \) is embeddable into a geometric lattice \( G \), i.e. there is an injective lattice homomorphism \( \phi : L \to G \). Denoting by \( \delta(L) \) the length of the longest chain from 0 to 1, one can always achieve \( \delta(G) \leq 2^{\delta(L) - 1} \) [4, Theorem 2.1]. This upper bound is easily seen to be sharp [6, p. 363]. Settling a long standing conjecture, Dilworth's Theorem was sharpened by Pudlak and Túma [12] who showed that each lattice \( L \) is even embeddable into a partition lattice \( P \). However, their proof yields partition lattices \( P = \text{Part}(V) \) on a ground set \( V \) whose length \( \delta(P) = |V| - 1 \) is superexponential in \( \delta(L) \).

We are interested in embeddings \( \phi : L \to G \) respectively \( \phi : L \to P \) which are cover preserving (cp). This is equivalent to the optimal bounds \( \delta(G) = \delta(L) \) (clear) respectively \( \delta(P) = \delta(L) \) (Corollary 8). It follows from the proof of Dilworth's Theorem that...
a lattice \( L \) is \( cp \) embeddable into a geometric lattice iff \( L \) is semimodular. The corresponding result for \( cp \) embeddability into partition lattices is not true. Besides semimodularity, the most obvious further necessary condition is the non-existence of covering sublattices \( M_4 \) (the length 2 lattice with 4 atoms).

If one restricts the class of semimodular lattices to the class of geometric lattices, then a neat answer to the embeddability problem can be given: According to Peele [11, Theorem 5.1] a geometric lattice \( L \) is \( cp \) embeddable into a partition lattice iff \( L \) itself is isomorphic to a direct product of partition lattices. In the present paper we restrict the class of semimodular lattices to the class of modular lattices, and derive some necessary and some sufficient conditions for cover preserving embeddability. Our methods of proof are completely different from the ones in [12] and [11]. Let us now survey the structure of the paper in more detail.

In Section 2 we give a short proof of the fact that each semimodular lattice \( L \) is \( cp \) embeddable into a geometric lattice (Theorem 4). This will be done by considering matroids on the join irreducibles \( J(L) \).

In the remainder of the paper we specialize ‘geometric lattice’ to ‘partition lattice’ and ‘semi-modular’ to ‘modular’. In Section 3 it is shown that a modular lattice \( L \) which is \( cp \) embeddable into a partition lattice, enjoys special properties (Theorem 6). First, besides covering sublattices \( M_4 \), certain configurations of covering sublattices \( M_3 \) cannot occur either. Secondly, \( L \) must be 2-distributive.

Conversely, our efforts to derive reasonable sufficient conditions for \( cp \) partition embeddability may be divided in three steps. As a first step we show in Section 4 that a modular lattice \( L \) is \( cp \) partition embeddable if there is a graphic matroid on its set \( J(L) \) of join irreducibles which satisfies two natural conditions (Theorem 7). In fact, these conditions are also necessary for \( cp \) partition embeddability, but they are somewhat hard to verify.

In [9] we associated with a modular lattice \( L \) a (not uniquely determined) partial linear space \( (J(L), A) \) with point set \( J(L) \) and line set \( \mathcal{A} \subseteq 2^{J(L)} \), such that the lattice of \( \mathcal{A} \)-closed order ideals of \( (J(L), \leq) \) is isomorphic to \( L \). These partial linear spaces, called bases of lines, generalize to arbitrary modular lattices the well-known descriptions of distributive, respectively geometric modular lattices. The concept of a base of lines was the crucial tool for many results in [9], and it will be crucial here as well. Namely, as a second step, we shall see that the existence of certain ‘regular’ bases of lines \( (J(L), A) \) allows for the construction of graphic matroids \( (J(L), -) \) satisfying the conditions mentioned above. Here the 2-distributivity of \( L \) plays a major role.

In the third step we give ‘forbidden minor’ type conditions for a 2-distributive modular lattice \( L \) which imply the existence of regular bases of lines \( (J(L), A) \). These forbidden minors are covering \( M_4 \)'s and certain configurations of covering \( M_3 \)'s which are somewhat more general than the ones considered in Section 3.

Condensing the above remarks one can say the following. The join irreducibles \( J(L) \) of a modular lattice \( L \) will be interpreted in two further ways: As edges of a graph, and as points of a partial linear space. Roughly speaking, Section 4 relates modular lattices
and graphs, Section 5 relates graphs and partial linear spaces, and Sections 6, 7 relate partial linear spaces and modular lattices.

In Section 8 the pieces are put together to obtain the main Theorem 23 and its Corollaries. For example, it turns out that acyclic modular lattices [9] are cp partition embeddable, and that cp partition embeddability is not preserved under duality. Also an illustrative example of a 71-element non-acyclic modular lattice, which is cp partition embeddable, will be given. In Section 9 we compare our results about cp partition embeddability with the ones about cp $k$-linear embeddability obtained in [9].

2. Cover preserving embeddability into geometric lattices

Theorem 4, the main result in this section, is a special case of Dilworth's Embedding Theorem. Most of the Lemmas here are interesting in their own right. In particular, Lemma 1 is a slight generalization of Lemma D in [10], and Lemma 5 exhibits an interesting property of modular lattices which will be used in Sections 4 and 9.

A map $\phi: L \to L'$ between two posets $L$ and $L'$ is an order embedding if $a \leq b$ is equivalent to $\phi(a) \leq \phi(b)$ for all $a, b \in L$ (in particular $\phi$ is injective). An order embedding $\phi$ between two lattices $L$ and $L'$ is a join (meet) embedding if $\phi(a + b) = \phi(a) + \phi(b)$ ($\phi(ab) = \phi(a)\phi(b)$) for all $a, b \in L$. It is an embedding if it is both a join and meet embedding. An order embedding is cover preserving if $a < b$ implies $\phi(a) < \phi(b)$ for all $a, b \in L$. Here $a < b$ means that $a < b$ and there is no $x$ with $a < x < b$.

Lemma 1. Let $\phi: L \to L'$ be a cover preserving order embedding between two lattices $L$ and $L'$. If $L$ is upper (lower) semimodular then $\phi$ is a join (meet) embedding.

Proof. By duality it suffices to prove the claim for upper semimodular lattices. To show $\phi(a + b) = \phi(a) + \phi(b)$ by induction on $\delta(a) + \delta(b)$ we may assume that $\delta(a) + \delta(b) \geq 2$ and that $a$ has a lower cover $a < a$. By semimodularity either $a + b = a + b$ or $a + b < a + b$. In the first case induction yields $\phi(a) + \phi(b) \leq \phi(a + b) = \phi(a) + \phi(b) = \phi(a) + \phi(b)$. In the second case induction gives $\phi(a) + \phi(b) = \phi(a) + (\phi(g) + \phi(b)) = \phi(a) + \phi(a + b)$. Now $\phi(a) \leq \phi(a + b)$ since $a \leq a + b$, and $\phi(a + b) < \phi(a + b)$ since $\phi$ is cover preserving. Hence $\phi(a) + \phi(a + b) = \phi(a + b)$. \qed

The semimodularity assumption in Lemma 1 is crucial: Let $L$ be the lattice obtained from the cube $L' := 2^3$ by removing one coatom. Then the cp order embedding $L \subseteq L'$ is not a join embedding.

For a lattice $L$ let $J(L)$ be the set of nonzero join irreducibles. Put $J(a) := \{ p \in J(L) | p \leq a \}$ for all $a \in L$ and denote by $p$ the unique lower cover of a join irreducible $p$. In the sequel 'semimodular' will always mean 'upper semimodular'. For a semimodular lattice $L$ put $\delta(L) := \delta(1)$ where $\delta$ is its height function. The rank of a matroid $(J, -)$ is denoted by $r(J, -)$. 


Lemma 2. Let \( L \) be a semimodular lattice.

(a) Let \( \phi : L \rightarrow \mathcal{L} (K, \sim) \) be a cover preserving 0-embedding (i.e. \( \phi (0) = 0 \)) into the flat lattice of a matroid \( (K, \sim) \). Then there is a simple matroid \( (J(L), -) \), isomorphic to a submatroid of \( (K, \sim) \), which satisfies

\[
(1) \quad r(J(L), -) = \delta(L)
\]

and

\[
(2) \quad (\forall a \in L) \quad J(a) = J(a).
\]

(b) If conversely \( (J(L), -) \) is a simple matroid which satisfies (1) and (2), then \( \phi : L \rightarrow \mathcal{L} (J(L), -) : a \mapsto J(a) \) is a cover preserving 0,1-embedding.

Proof. (a) Assume that \( \phi : L \rightarrow \mathcal{L} (K, \sim) \) is a cp 0-embedding. For each \( p \in J(L) \) choose a point \( p' \in \phi (p) - \phi (p) \). For distinct \( p, q \in J(L) \) we may assume that \( q \nless p \). Then the assumption \( \{ p' \} = \{ q' \} \) yields the contradiction \( \{ p' \} \subseteq \phi (p) \cap \phi (q) = \phi (p \cap q) \subseteq \phi (p) \). Therefore \( p \mapsto p' \) is injective and, moreover, the submatroid \( (J', -) \) on \( J' := \{ p' \mid p \in J(L) \} \) is simple. Put \( J'(a) := \{ p' \mid p \in J(a) \} \) for all \( a \in L \). Show that \( (J', -) \) satisfies (1) and (2). To see (1) check by induction on \( \delta (a) \) that \( r(J'(a)) = r(\phi (a)) = \delta(a) \) for all \( a \in L \). The case \( \delta (a) = 0 \) being trivial consider a covering pair \( a < b \) with \( r(J'(a)) = r(\phi (a)) = \delta(a) \). Pick a \( p \in J(b) - J(a) \). Then \( p' \notin J'(a) \). On the other hand \( a \nless p \) forces \( p' \nless \phi (a) \subseteq J'(a) \) (otherwise again \( p' \in \phi (p) \cap \phi (a) \subseteq \phi (p) \)). Hence \( r(J'(b)) = r(\phi (a)) + 1 \). Conversely, since \( \phi \) is cover preserving, \( r(J'(b)) \leq r(\phi (b)) = r(\phi (a)) + 1 \). We have just shown that \( J'(a) \) generates \( \phi (a) \). Nevertheless, the sets \( J'(a) \) are closed in the submatroid \( (J', -) \): If \( p' \notin J' - J'(a) \) then \( p \nless a \) by definition of \( J'(a) \). But this implies \( p' \notin J'(a) \), as seen above. This proves (2).

(b) Assume there is a matroid \( (J(L), -) \) which satisfies (1) and (2). By (2) the map \( \phi : L \rightarrow \mathcal{L} (J(L), -) : a \mapsto J(a) \) is well defined. It is obviously a meet 0,1-embedding. Moreover by (1) one has \( \delta(L) = \delta(\mathcal{L} (J(L), -)) \) which forces \( \phi \) to be cover preserving. By Lemma 1 it follows that \( \phi \) is also join preserving. \( \square \)

Recall that a function \( f : L \rightarrow \mathbb{N} \) on a lattice \( L \) is submodular if \( f(X \cup Y) \leq f(X) + f(Y) \) for all \( X, Y \in L \). The following Lemma is essentially Theorem 2 in [13, p. 117]; its last statement is an easy exercise.

Lemma 3 [14]. Let \( J \) be a set and \( L \subseteq 2^J \) a closure system on \( J \) with \( \emptyset \in L \). Suppose that \( f : L \rightarrow \mathbb{N} \) is a strictly monotone submodular function with \( f(\emptyset) = 0 \). Then \( \mathcal{F} := \{ A \subseteq J \mid (\forall X \in L) \ | A \cap X | \leq f(X) \} \) is the family of independent sets of a matroid \( (J, -) \) with rank function \( r(A) = \min_{X \in L} \{ f(X) + | A - X | \} \). Furthermore, each \( X \in L \) is closed in \( (J, -) \), i.e. \( L \) is a meet sublattice of \( \mathcal{L} (J, -) \).

The fact below also follows easily from the proof of Dilworth’s Embedding Theorem [2, p. 125-131]. Another proof is given in [6, Lemma 17]. The merit of the proof given here is its shortness.
Theorem 4 [2]. A lattice $L$ is cover preserving 0.1-embeddable into a geometric lattice iff $L$ is semimodular.

Proof. It is easily seen that a covering sublattice of a geometric (or semimodular) lattice is semimodular. Conversely, given a semimodular lattice $L$, identify $L$ with the closure system $\{ J(a) \mid a \in L \}$ on $J(L)$. Lemma 3 applied to the height function $\delta : L \rightarrow \mathbb{N}$ yields a matroid $(J(I), -)$ which satisfies (1) and (2). Hence $L$ is cp 0.1-embeddable into the geometric lattice $\mathcal{L}(J(L), -)$ by Lemma 2. □

The next Lemma states that for modular lattices $L$ the necessary condition (a) of Lemma 2 is also sufficient.

Lemma 5. Let $L$ be a modular lattice and $(K, \sim)$ a matroid. There is a cover preserving 0-embedding $\phi : L \rightarrow \mathcal{L}(K, \sim)$ iff the following holds: There is a simple matroid $(J(L), -)$, isomorphic to a submatroid of $(K, \sim)$, which satisfies (1) and (2). Assuming $J(L) \subseteq K$, $\phi$ can be given by $\phi(a) = J(a)$.

Proof. It remains to show the sufficiency of the above condition. So assume there is a simple matroid $(J(L), -)$ with (1) and (2) which is a submatroid of some matroid $(K, \sim)$. By Lemma 2(b) it follows that $\phi_1 : L \rightarrow \mathcal{L}(J(L), -) : a \mapsto J(a)$ is a cp 0.1-embedding. Since $(J(L), -)$ is a submatroid of $(K, \sim)$, the map $\phi_2 : L(J(L), -) \rightarrow \mathcal{L}(K, \sim) : B \mapsto \tilde{B}$ is a cp join 0-embedding (see, e.g. [15, Theorem 7.3.1]). Since $\text{Im}(\phi_1)$ is a covering lower semimodular sublattice of $\mathcal{L}(J(L), -)$, it follows by Lemma 1 that $\phi_2$ restricted to $\text{Im}(\phi_1)$ is also meet preserving. Hence $\phi := \phi_2 \circ \phi_1 : L \rightarrow \mathcal{L}(K, \sim) : a \mapsto \tilde{J(a)}$ is a cover preserving 0-embedding. □

Simple counterexamples show that Lemma 5 fails for semimodular lattices, even for geometric lattices $L$ with $r(J(L), -) = r(K, \sim)$.

3. Necessary conditions for cp embeddability of modular lattices into partition lattices

Let us start with some definitions. A lattice $L$ is called 2-distributive if

\[(\forall w, x, y, z \in L) \quad w(x + y + z) = w(x + y) + w(x + z) + w(y + z).\]

If a 2-distributive lattice is moreover modular then the dual of equation (3) holds as well. More important for us is the following fact.

(4) [7, Theorem 1.1] A modular lattice is 2-distributive iff it does not contain as an (interval) sublattice the subspace lattice of a nondegenerate projective plane.

Denote by $M_n$ the length 2 lattice with $n$ atoms ($n \geq 2$). An element $x$ of a modular lattice $L$ is an $M_n$-element if the lower covers $x_i < x$ generate a sublattice $M_n$, i.e. any
distinct \( x_i, x_j \) meet in the same element \( x_0 \). This will henceforth be the standard notation whenever we deal with an \( M_n \)-element \( x \) (analogously for \( y, z, \ldots \)). Put \( E(L) := \{ x \mid x \text{ is an } M_n \text{-element for some } n \geq 3 \} \). The elements of \( E(L) \) are the essential elements of \( L \). For distinct \( x, y \in E(L) \) write \( x \succeq y \) if a 'lower quotient' of \( x \) transposes downwards to an 'upper quotient' of \( y \), i.e. \( (x_i/x_0) \succeq (y/j) \) for some \( i, j \). Define \( x \succ y \) similarly. If we want to specify only one index we e.g. write \( x \succeq (y/j) \) or \( (x/i) \succ y \).

Observe that the unit element \( w \) of a covering sublattice \( M_n = \{ w_0 < w_1, \ldots, w_n < w \} \) is generally not an \( M_n \)-element; but then the quotient \( w/w_0 \) transposes downwards to a quotient \( x/x_0 \) where \( x \) is an \( M_n \)-element. Consider, e.g., the quotients \((60/39)\) respectively \((47/30)\) in Fig. 5(a).

A modular lattice is of order \( k \) if \( n \leq k + 1 \) for each \( M_n \)-element \( x \in L \). Thus the distributive lattices are the modular lattices of order 1. For a modular lattice of order 2 say that \( x \in E(L) \) is of type \((3.3s)\) if the following holds: There are (not necessarily distinct) elements \( u^1, \ldots, u^r, v^1, \ldots, v^s, w^1, \ldots, w^t \) \( y \in E(L) \) \( (r, s, t \geq 1) \) such that

\[
\begin{align*}
(x_1/x_0) & \succeq u^1 \succeq u^2 \ldots \succeq u^r \succeq y, \\
(x_2/x_0) & \succeq v^1 \succeq v^2 \ldots \succeq v^s \succeq y, \\
(x_3/x_0) & \succeq w^1 \succeq w^2 \ldots \succeq w^t \succeq y.
\end{align*}
\]

The unit element of the lattice \( L_1 \) shown in Fig. 1 is of type \((3.3s)\). The terminology 'type \((3.3s)\)' will become clear in Section 7.

For a set \( V \) denote the partition lattice on \( V \) by \( \text{Part}(V) \). It consists of all families \( x = \{ V_1, \ldots, V_r \} \) with \( V = V_1 \cup \cdots \cup V_r \) (\( V_i \neq \emptyset \)). The elements \( V_i \in x \) are the blocks of \( x \) (see [5]).

![Fig. 1](image-url)
Theorem 6. Let L be a modular lattice which is a covering sublattice of a partition lattice. Then L is 2-distributive and of order 2. Moreover L contains no $M_3$-elements of type (3.3s).

Proof. Assume that L is a covering sublattice of a partition lattice $\text{Part}(n) := \text{Part}([1, \ldots, n])$. It is easily seen that each interval $(x/y) \subseteq \text{Part}(n)$ is isomorphic to a direct product of partition lattices. In particular an interval of length two is isomorphic to $\text{Part}(3) = M_3$ or to $\text{Part}(2) \times \text{Part}(2) = M_2$. Thus $M_4$ cannot be a covering sublattice of L, i.e. L is of order 2. An interval of length 3 of $\text{Part}(n)$ is isomorphic to either $\text{Part}(4)$ or $\text{Part}(3) \times \text{Part}(2)$ or $\text{Part}(2) \times \text{Part}(2) \times \text{Part}(2)$. Since none of these lattices has more than six atoms, none of them is the subspace lattice of a nondegenerate projective plane (which has at least seven points). Therefore L is 2-distributive by (4).

Now assume that L contains a $M_3$-element $x$ of type (3.3s). The partition $x$ contains, like any $M_3$-element in L, a ‘characteristic block’ $b(x) \in x$. This is the unique block of $x$ such that for some decomposition $b(x) = X_1 \cup X_2 \cup X_3$ one has $x_0 = (x - \{b(x)\}) \cup \{X_1, X_2, X_3\}$, $x_i = (x - \{b(x)\}) \cup \{X_i, X_j \cup X_k\}$ (for $i, j, k = 1, 2, 3$). By induction on the $M_3$-elements $u^1$ we shall see that $b(y)$ must be a subset of $X_2 \cup X_3$. The analogous argument applied to the $v^1$ respectively $w^1$ then leads to the contradiction $b(y) \subseteq (X_2 \cup X_3) \cap (X_1 \cup X_3) \cap (X_1 \cup X_2) = \emptyset$.

(5) Let $u \in L$ be a $M_3$-element with $u_0 = (u - \{b(u)\}) \cup \{U_1, U_2, U_3\}$ and $u_1 = (u - \{b(u)\}) \cup \{U_1, U_2 \cup U_3\}$. If $(u_1/u_0) \succ u^1$ for another $M_3$-element $u^1$ then $b(u^1) \subseteq U_2 \cup U_3$.

Proof of (5). Since $U_2 \cup U_3$ is a block of $u_1 = u_0 + u^1$ there must be a block $b$ of $u^1$ with $b \subseteq U_2 \cup U_3$ and $b \cap U_2 \neq \emptyset \neq b \cap U_3$. If the characteristic block $b(u^1)$ were distinct from $b$ then the lower cover $u^1 := u^1u_0$ of $u^1$ would still have $b$ as a block. Thus $b$ would be contained in a block of $u_0$, which is not the case.

Applying (5) to $(x_1/x_0) \setminus u^1$ yields $b(u^1) \subseteq X_2 \cup X_3$. Applying (5) to (say) $(u_1^1/u_0) \setminus u^2$ yields $b(u^2) \subseteq U_2 \cup U_3 \subseteq b(u^1) \subseteq X_2 \cup X_3$. By induction $b(u^1) \subseteq X_2 \cup X_3$ and finally $b(y) \subseteq X_2 \cup X_3$. □

4. Modular lattices and graphs

Let us specialize Lemma 5 to the case where $(K, \sim)$ is the cycle matroid of the complete graph $G = (V, K)$ with vertex set $V$ and edge set $K := \binom{V}{2}$. It is well known [14, p. 57] and easy to see that there is a canonical lattice isomorphism $\mathcal{L}(K, \sim) \rightarrow \text{Part}(V) : B \mapsto \text{comp}(B)$. Generally, for a set of edges $B \subseteq K$ we denote by $\text{comp}(B)$ the partition of $V$ whose blocks are the vertex sets of the connected components of the subgraph $G' := (V, B)$. The above remarks together with Lemma 5 yield the following.
Theorem 7. Let $L$ be a modular lattice and $\text{Part}(V)$ a partition lattice. There is a cover preserving 0-embedding $\phi: L \rightarrow \text{Part}(V)$ if and only if the following holds: $J(L)$ is (bijective to) the edge set of a simple graph $G = (V, J(L))$ whose cycle matroid satisfies (1) and (2). In this case $\phi$ can be given by $\phi(a) := \text{comp}(J(a))$.

Let $L$ be an arbitrary lattice and $\phi: L \rightarrow \text{Part}(V)$ a (not necessarily cp) embedding with $|V|$ minimal. Since for each partition $W = \{V_1, \ldots, V_s\}$ in $\text{Part}(V)$ the interval $(1/W) \subseteq \text{Part}(V)$ is isomorphic to $\text{Part}(W)$, it follows that $\phi(0) = 0$. However, since the interval $(W/0) \subseteq \text{Part}(V)$ is isomorphic to $\text{Part}(V_1) \times \cdots \times \text{Part}(V_s)$, there is no obvious reason for $\phi(1) = 1$. That $\phi(1) = 1$ holds all the same is proven in [3]. The fact below could be shown in a similar way, but it also is an easy consequence of Theorem 7.

Corollary 8. If a modular lattice $L$ admits a cover preserving embedding in $\text{Part}(V)$, then it also admits a cover preserving 0,1-embedding in some $\text{Part}(V_0)$.

Proof. By the above remark we may assume that $L$ is cp 0-embedded in $\text{Part}(V)$. Then by Theorem 7 there is a graph $G = (V, J(L))$ whose cycle matroid satisfies (1) and (2). From $G$ one obtains a connected graph $G_0 = (V_0, J(L))$ by glueing together vertices of the connected components of $G$. The cycle matroids of $G$ and $G_0$ are identical [15] but, since $G_0$ is connected, one has achieved $r(J(L), -) = |V_0| - 1 = \delta(\text{Part}(V_0))$. Applying Theorem 7 in the other direction proves the claim.

Corollary 9. If the modular lattice $L$ is cover preserving 0,1-embeddable in $\text{Part}(V)$ then it is cover preserving 0,1-embeddable in $L(k^n)$ ($k$ an arbitrary field, $n := \delta(L)$).

Proof. By Theorem 7 there is a graphic matroid $(J(L), -)$ on $J(L)$ with (1) and (2). By [14, p. 148] each graphic matroid is $k$-linear, i.e. $(J(L), -)$ is a submatroid of $(k^n, \sim)$ (where dependence is linear dependence). Apply Lemma 5.

Observe that the embedding into $L(k^n)$ is obtainable in the following way. If $V = \{1, \ldots, n+1\}$, choose an affine base $e_1, \ldots, e_{n+1}$ of $k^n$ and associate, e.g., with the partition

$$\{\{1, 2, 3, 4\}, \{5, 6, 7\}, \{8, 9\}, \{10, 11\}, \{12\}, \ldots, \{n+1\}\} \in \text{Part}(V)$$

the linear subspace

$$\langle e_1-e_4, e_2-e_4, e_3-e_4, e_5-e_7, e_6-e_7, e_8-e_9, e_{10}-e_{11}\rangle \in L(k^n).$$

More remarks about cover preserving $k$-linear representations follow in Section 9. In particular, the lattice $L_1$ of Fig. 1 will be seen to be cp embeddable into $L(k^n)$ for any $k$, thereby proving false the converse of Corollary 9.

In the following Sections 5, 6 and 7 we shall develop verifiable sufficient conditions for the existence of a graph $G = (V, J(L))$ occurring in Theorem 7. As an easy first step,
Lemma 10 below reduces condition (2) to the behaviour of the chordless cycles of \( G \). Interestingly, at this point the modularity of \( L \) is irrelevant.

**Lemma 10.** Let \( L \) be a lattice and assume that \( J = J(L) \) is the edge set of a graph \( G = (V, J) \). Denote its cycle matroid by \( (J, -) \). Then the following conditions are equivalent.

(i) For each chordless cycle \( C \subseteq J \) of \( G \) one has \((\forall q \in C) \ q \leqslant \sum (C - \{q\})\).

(ii) For each cycle \( C \subseteq J \) of \( G \) one has \((\forall q \in C) \ q \leqslant \sum (C - \{q\})\).

(iii) \((\forall a \in L) \ J(a) = J(a)\).

**Proof.** (i) implies (ii). Assume that \( C \subseteq J \) is a cycle of \( G \) which has a chord \( p \in J - C \). Then \( C = C_1 \cup C_2 \) such that \( C_1 \cup \{p\} \) and \( C_2 \cup \{p\} \) are cycles of smaller length. Let \( q \in C \) be an arbitrary element, say \( q \in C_1 \). By assumption (i) and induction on the cycle length one has \( q \leqslant \sum (C_1 - \{q\}) + p \) and also \( p \leqslant \sum C_2 \). Hence

\[
q \leqslant \sum (C_1 - \{q\}) + \sum C_2 = \sum (C - \{q\}).
\]

(ii) implies (iii) Recall that a subset \( S \) of a matroid \((M, -)\) is closed iff \((C - \{q\}) \subseteq S\) implies \( q \in S \) for all cycles \( C \) of \((M, -)\) and \( q \in C \). In our case, it follows from \((C - \{q\}) \subseteq J(a)\) that \( \sum (C - \{q\}) \leqslant a \). By assumption (ii) \( q \leqslant \sum (C - \{q\}) \), i.e. \( q \in J(a) \).

(iii) implies (i). Assume there is a chordless cycle \( C \subseteq J \) of \( G \) with a \( q \in C \) such that \( q \notin \sum (C - \{q\}) \). Then for \( a := \sum (C - \{q\}) \) the set \( J(a) \) is not closed. \( \square \)

5. Partial linear spaces and graphs

We shall associate certain graphs \( G = (V, J) \) with certain partial linear spaces. If \( J \) happens to be the set of join irreducibles \( J(L) \) of a modular lattice \( L \), then Lemma 10 and Lemma 15 state that the cycle matroid \((J, -)\) of \( G \) satisfies (1) and (2) provided the associated partial linear space enjoys two corresponding properties (to be dealt with in Section 7).

Recall that a partial linear space is a pair \((J, A)\) consisting of a set \( J \) of points and a set \( A \subseteq 2^J \) of lines such that \(|g| \geqslant 2\) for all \( g \in A \) and \(|g \cap h| \leqslant 1\) for all distinct \( g, h \in A \). For brevity we henceforth mostly refer to \((J, A)\) as a space. Any two points \( p \neq q \) of a space \((J, A)\) lie on at most one line \( g \) which we then denote by \([p, q]\). Call a tuplet \((p_1, \ldots, p_n) \ (n \geqslant 2)\) of distinct points a path if all lines \([p_i, p_{i+1}] \ (1 \leqslant i \leqslant n - 1)\) exist and are mutually distinct. Two points \( p \) and \( q \) are connected if \( p \neq q \) or if there is a path \((p, \ldots, q)\). If \( J_i (1 \leqslant i \leqslant c) \) are the blocks of this equivalence relation then the spaces \((J_i, A_i) \ (A_i := \{g \in A \mid g \subseteq J_i\})\) are the connected components of \((J, A)\). In particular, a point \( p \in J \) is isolated if \(\{p\} \) is a connected component. Put \( c(J, A) := c \). A cycle in \((J, A)\) is a 'closed path', i.e. a tuplet \((p_1, \ldots, p_n, p_1) \ (n \geqslant 3)\) such that \((p_1, \ldots, p_n)\) and \((p_2, \ldots, p_n, p_1)\) are paths. Dealing with cycles of length \( n \) we shall always calculate modulo \( n \), i.e. \( n + 1 = 1 \).

A space \((J, A)\) is a \( m \)-space \((m \geqslant 2)\) if \(|g| = m\) for all \( g \in A \). Observe that 2-spaces are just simple graphs and that the above defined notions coincide with the corresponding
graph theoretic notions. In the next step we generalize to arbitrary spaces \((J, A)\) the matroids associated with a graph.

For this purpose define a point splitting of a space \((J, A)\) as an ordered pair \((g, p)\) \(\in A \times J\) with \(p \in g\). Let \(M(J, A) := \{(g, p) \in A \times J | p \in g\}\) be the set of all point splittings. Applying a set \(B \subseteq M(J, A)\) of point splittings to the space \((J, A)\) yields a space \((J, A)^B\) which arises from \((J, A)\) by separating for each \((g, p) \in B\) the line \(g\) at the point \(p\) from the ‘rest’. Formally, define \((J, A)^B\) as the space \((J', A')\) with \(J' := J \cup \{p^g | (g, p) \in B\}\) and \(A' := \{g' | g \in A\}\). Here \(p^g | (g, p) \in B\) is a set of cardinality \(|B|\) disjoint from \(J\), and \(g'\) is defined as \((g - \{p \mid (g, p) \in B\}) \cup \{p^g | (g, p) \in B\}\). Call a subset \(B \subseteq M(J, A)\) a base if \((J, A)^B\) is acyclic and still has the same number of connected components as \((J, A)\).

**Lemma 11.** (a) Let \((J, A)\) be a partial linear space. Then the bases of \(M(J, A)\) are the bases of a cographic matroid on \(M(J, A)\). (b) If \((J, A)\) is a graph (i.e. 2-space) then its bond matroid is isomorphic to a generating submatroid of \(M(J, A)\).

**Proof.** Let us start with the more familiar case.

(b) If \((J, A)\) is a graph then each edge \(g = \{p_1, p_2\} \in A\) induces the two point splittings \((g, p_1)\) and \((g, p_2)\) of \(M(J, A)\). Choosing one of them for each \(g \in A\) yields a subset \(M' \subseteq M(J, A)\). Show that the bases \(B\) contained in \(M'\) correspond bijectively to the complements of spanning forests of \((J, A)\); then \(M'\) is isomorphic to the bond matroid of \((J, A)\). Indeed, if \(B = \{(g_i, p_i) | i \in I\} \subseteq M'\) is a base, then \((J, A)^B\) is acyclic with the same number of connected components. Hence \(T := A - \{g_i | i \in I\}\) is (the edge set of) a spanning forest of \((J, A)\). Conversely, each spanning forest \(T\) arises in this way from a unique base \(B = \{(g, p) | g \in A - T, (g, p) \in M'\}\). Similarly \(M(J, A)\) is a cographic matroid with \(M'\) as a generating submatroid (for \(a = \{p_1, p_2\}\) the elements \((g, p_1)\) and \((g, p_2)\) are parallel in \(M(J, A)\)).

(a) For an arbitrary space \((J, A)\) consider the associated bipartite graph \(G := (J \cup A, M)\) with vertex set \(J \cup A\) and edge set \(M := M(J, A)\). Thus \(G\) just codes the point-line incidence. In particular, \(\{p_1, \ldots, p_n, p_1\}\) is a cycle in \((J, A)\) iff \(\{p_1, [p_1, p_2], \ldots, p_n, [p_n, p_1]\}\) is the vertex set of a cycle in \(G\). Similar to above the bases \(B \subseteq M\) correspond to the complements of spanning forests of \(G\). 

By Lemma 11(a) we may define the corank of a space \((J, A)\) as the uniquely determined number \(r^*(J, A)\) of point splittings required to turn \((J, A)\) into an acyclic space with the same number of components. Define the rank of \((J, A)\) as \(r(J, A) := |A| - r^*(J, A)\). By Lemma 11(b), for a graph \((J, A)\) the values \(r^*(J, A)\) and \(r(J, A)\) coincide with the rank of its bond respectively cycle matroid. (For arbitrary partial linear spaces \((J, A)\) one might well have \(r(J, A) < 0\).)

In the remainder of this section we consider spaces \((J, A)\) whose point set \(J\) bijectively corresponds to the edge set \(J'\) of some (simple) graph \(G := (V, J')\). To simplify notation identify \(J\) with \(J'\). It will always be clear from the context if an element \(p \in J\) is to be considered as a point of \((J, A)\) or as an edge of the associated graph \(G := (V, J)\).
Let \((J^1, A^1), (J^2, A^2), \cdots, (J^e, A^e)\) be spaces such that \(J^2 \cap J^1 = \{p_2\}\), 
\(J^3 \cap (J^1 \cup J^2) = \{p_3\}\), and so on until \(J^e \cap (J^1 \cup \cdots \cup J^{e-1}) = \{p_e\}\). In this situation call the space \((J, A) = \bigcup_i^e J^i, A = \bigcup_i^e A^i\) a tree of the spaces \((J^i, A^i)\). The tree is strong if the cutpoints \(p_i\) are mutually distinct. An assignment of graphs \(G^i = (V^i, J^i)\) associated to the spaces \((J^i, A^i)\) \((1 \leq i \leq e)\) is compatible if \(V_i \cap V_j = \emptyset \iff J^i \cap J^j = \emptyset\) and if the following holds for each cutpoint \(p_i\): Putting \(I := \{1 \leq i \leq e \mid p \in J^i\}\) one has \(|\bigcap_i^e V^i| = 2\) and if \(|\bigcap_i^e V^i| = \{x_p, \beta_p\}\), then \(p\) gets the (undirected) edge \((x_p, \beta_p)\) in each \(G^i\) \((i \in I)\). Given a compatible assignment, define the associated graph of \((J, A)\) by 
\[ G := (V, J) \quad (V := \bigcup_1^e V^i). \]
Note that each assignment of graphs can easily be turned into a compatible one.

**Lemma 12.** Let \((J, A)\) be a tree of the spaces \((J^i, A^i)\) \((1 \leq i \leq e)\). Assume that 
\[ G^i = (V^i, J^i) \quad (1 \leq i \leq e) \]
are a compatible assignment of associated graphs and that 
\[ G = (V, J) \]
the associated graph of \((J, A)\). Denote the cycle matroids of the 
\[ G_i \quad \text{respectively} \quad G \]
by \((J^i, -)\) respectively \((J, -)\).

(a) \(r(J, A) = \sum_i^e r(J^i, A^i)\).
(b) \(r(J, -) = \sum_i^e r(J^i, -) - (e - 1)\).
(c) If \(C \subseteq J\) is a chordless cycle of \(G\) then necessarily \(C \subseteq J^i\) for some \(G^i\).

**Proof.** (a) Because \((J, A)\) is built up from the spaces \((J^i, A^i)\) in an acyclic way, it is clear that the number \(r*(J, A)\) of point splittings required for \((J, A)\) equals \(\sum_i^e r*(J^i, A^i)\). Thus \(|A| = \sum_i^e |A^i|\) implies \(r(J, A) = \sum_i^e r(J^i, A^i)\).

(b) The claim being true for \(e = 1\) assume \(e > 1\). Let 
\[ (J^0, A^0) = (J^0 := \bigcup_i^{e-1} J^i, A^0 := \bigcup_i^{e-1} A^i) \]
be the tree of the first \(e - 1\) spaces \((J^i, A^i)\). Let \((J^0, -)\) be the cycle matroid of its associated graph \(G^0 = (V^0, J^0)\) \((V^0 := \bigcup_i^{e-1} V^i)\). By induction \(r(J^0, -) - \sum_i^{e-1} r(J^i, A^i) - (e - 2)\). From \(|V^0 \cap V^0| = 2\) follows \(r(J, -) = |V| - 1 = (|V^0| - 1) + (|V^e| - 1) - 1 = r(J^0, -) + r(J^e, -) - 1 = \sum_i^e r(J^i, -) - (e - 1)\).

(c) Let \(J^e \cap J^0 = \{p\}\) and \(V^e \cap V^0 = \{x, \beta\}\) (whence \(p = (x, \beta)\)). Assume that \(C \subseteq J\) is a chordless cycle of \(G\). If \(C \subseteq J^e\) we are done. Suppose that \(C \not\subseteq J^e\) and \(C \not\subseteq (J^0 - J^e)\). Then \(C\) 'switches' an even number of times between \(J^e\) and \((J^0 - J^e)\), i.e. there is an even number of 'switching vertices' on \(C\). Since any switching vertex clearly belongs to \(V^0 \cap V^e = \{x, \beta\}\), the vertices \(x\) and \(\beta\) lie on \(C\). But now, since \(C\) is chordless, they must be subsequent vertices. In other words, \((x, \beta) \in J^e\) is an edge of \(C\) and all other edges of \(C\) belong to \((J^0 - J^e)\). Hence \(C \subseteq J^0\) and the claim follows by induction. \(\Box\)

From now on we only consider 3-spaces \((J, A)\). The crucial notion is that of a 'middlepoint isolated space'. These will be the building stones for certain regular and quasiregular spaces.

For each line \(g \in A\) of a 3-space \((J, A)\) fix a point \(p \in g\), to be called the middlepoint of \(g\). Of course, \(p\) will be placed in the 'middle' of \(g\) in any drawing of \((J, A)\) (but cf. the remark after (9)). The other two points of \(g\) are its endpoints. A connected 3-space \((H, A)\) is middlepoint isolated (mpi) if middlepoints of lines are never middlepoints or endpoints of other lines. Thus the point set of a mpi space is the disjoint union of 'its'
middlepoints and endpoints; mpi spaces look like graphs with subdivided edges. If the mpi space is \((\{p\}, \emptyset)\) then \(p\) is considered as an endpoint.

A path \((p_1, \ldots, p_n)\) respectively cycle \((p_1, \ldots, p_n, p_1)\) of a 3-space \((J, A)\) is regular if \(p_i, p_{i+1}\) are the endpoints of the line \([p_i, p_{i+1}]\) for all \(1 \leq i \leq n - 1\) (respectively \(1 \leq i \leq n\)).

A 3-space is regular if there is an assignment of middlepoints which makes all its cycles regular. The following Lemma gives a 'analytic' description of regular spaces.

**Lemma 13.** A 3-space \((J, A)\) is regular iff each connected component is a tree of mpi spaces.

**Proof.** It is clear that a tree of mpi spaces is regular. Conversely consider a connected regular space \((J, A)\). Call two lines \(g, h \in A\) equivalent if \(g = h\) or if there is a regular path \((p_1, \ldots, p_n)\) such that \([p_1, p_2] = g\) and \([p_{n-1}, p_n] = h\). Let \(A_i \subseteq A\) \((1 \leq i \leq e)\) be the classes of this equivalence relation and put \(J^i := \bigcup A_i\). Obviously each \((J^i, A_i)\) is a mpi space. Let us sketch why \((J, A)\) is a tree of the spaces \((J^i, A_i)\). First show that the assumption \(p, q \in J^i \cap J^j\) \((p \neq q)\) yields a nonregular cycle \((p, \ldots, q, \ldots, p)\) contained in \(J^i \cup J^j\). Thus \(|J^i \cap J^j| \leq 1\) for \(i \neq j\). Call \(p \in J^i\) a cutpoint of \((J^i, A_i)\) if \(J^i \cap J^j = \{p\}\) for some \((J^j, A_j)\). Assuming that each \((J^i, A_i)\) has at least two cutpoints yields again a nonregular cycle in \((J, A)\). Thus w.l.o.g. \((J^s, A^s)\) has precisely one cutpoint, and it follows by induction that \((J, A)\) is a tree of the spaces \((J^i, A_i)\). \(\square\)

Unless stated otherwise, a graph \(G = (W, H)\) associated with a mpi space \((H, A)\) will always be of the following form. If \(p_1, \ldots, p_n\) are the (distinct) endpoints of \((H, A)\) let \(W := \{0, 1, \ldots, n\}\). An endpoint \(p_i\) gets the edge \((0, i)\) and a middlepoint \(q \in [p_i, p_j]\) gets the edge \((i, j)\). The associated graph of a connected regular space \((J, A)\), i.e. of a tree of mpi spaces \((J^i, A_i)\) \((1 \leq i \leq e)\), is the graph \(G := (V, J)\) which is induced by a compatible assignment of graphs \(G^i = (V, J^i)\) associated to the spaces \((J^i, A_i)\) \((1 \leq i \leq e)\). The associated graph \(G := (V, J)\) of a regular space \((J, A)\) is defined by \(V := \bigcup_{i=1}^{e} V_i\), where the \(G_i = (V_i, J_i)\) are the associated graphs of its connected components \((J_i, A_i)\) \((1 \leq i \leq e)\).

**Lemma 14.** Let \((J, A)\) be a regular space with associated graph \(G = (V, J)\). Denote its cycle matroid by \((J, -)\).

(a) \(r(J, -) = r(J, A) + e(J, A)\).

(b) Each chordless cycle \(C \subseteq J\) of \(G\), considered as a point set of the space \((J, A)\), is of the form (6) or (7) (see Fig. 2).

**Proof.** (a) First, let \((H, A) = (J^i, A^i)\) be a mpi space with associated graph \(G = (W, H)\) \((W := \{0, 1, \ldots, n\})\) and cycle matroid \((H, -)\). We claim that \(r(H, -) = r(H, A) + 1\). The middlepoints of \((H, A)\) being irrelevant for pointsplittings, consider \((H, A)\) as a graph on its endpoints \(p_1, \ldots, p_n\). By Lemma 11(b) the bond matroid of this graph has rank \(r^*(H, A)\) and the cycle matroid has rank \(r(H, A)\). Thus \(r(H, A) = \left|\{p_1, \ldots, p_n\}\right| - 1 = n - 1\). The claim follows by noticing that \(r(H, -) = |W| - 1 = n\).
Now let \((J, A)\) be a tree of mpi spaces \((J^i, A^i)\) \((1 \leq i \leq e)\). By the above and Lemma 12 one concludes that \(r(J, -) = \sum r(J^i, -) - (e - 1) = \sum r(J^i, A^i) + 1 = r(J, A) + 1\). Finally, if \((J^i, A^i)\) \((1 \leq i \leq c)\) are trees of mpi spaces, which are the connected components of a regular space \((J, A)\), then \(r(J, -) = \sum r(J^i, -) = \sum r(J^i, A^i) + c = r(J, A) + c\).

(b) By Lemma 12(c) it suffices to show that each chordless cycle \(C \subseteq H\) in the associated graph \(G = (W, H)\) of a mpi space \((H, A)\) is of the form (6) or (7).

First case: \(C\) contains an endpoint \(p = (0, i)\) of \((H, A)\). Then its ‘right’ neighbour edge \(q = (i, j)\) must obviously be a middlepoint of \((H, A)\) which lies on a common line with \(C\). But then \(C = \{(0, i), (i, j), (j, 0)\}\) since otherwise \((j, 0)\) would be a chord of \(C\). Thus \(C\) is of form (6).

Second case: \(C\) contains only middlepoints of \((H, A)\). Then all edges of \(C\) are of the form \(q = (i, j)\) \((i, j \neq 0)\) and it is clear that two adjacent edges \(q, q' \in C\) correspond to middlepoints of intersecting lines \(g, g' \in A\). It follows easily that \(C\) is of form (7).

It is possible to extend Lemma 14 to certain nonregular 3-spaces \((J', A')\). Assume that \((J, A)\) is a connected regular space built up from the mpi spaces \((J^k, A^k)\) \((1 \leq k \leq e)\). To each \((J^k, A^k)\) we shall add certain lines \(g = \{q, s, t\}\) joining a middlepoint \(q\) of \((J^k, A^k)\) with an endpoint \(t\) of \((J^k, A^k)\) and having \(s \notin J\) as a middlepoint. Thus one
will get nonregular cycles in the so obtained ‘quasi mpi’ space \((J^k, A^k)\) (we keep the terminology ‘mpi’ although some middlepoints of \((J^k, A^k)\) are no more isolated!). No lines \(g\) joining distinct mpi spaces will be added.

The addition of new lines to a mpi space \((H, A)\) will only be possible if \((H, A)\) itself is a strong tree of mpi spaces \((H^i, A^i)\) (1 \(\leq i \leq d > 1\)). Call such a mpi space \((H, A)\) enrichable. The definition of the associated graph of an enrichable mpi space \((H, A)\) differs from the one given before Lemma 14. Namely, take a compatible assignment of graphs \(G^i = (W^i, H^i)\) associated to the mpi spaces \((H^i, A^i)\) (the \(G^i\) being defined as before Lemma 14). But choose it in such a way that endpoints belonging to distinct components are labelled with disjoint edges. More precisely: If \(H' \cap H^j = \{p\}\) and \(W' \cap W^j = \{x, \beta\}\), label the endpoints of \((H^i, A^i)\) with edges \((x, *)\) and the endpoints of \((H^j, A^j)\) with edges \((\beta, *)\). The associated graph of \((H, A)\) is the graph \(G := (W, H)\) induced by this compatible assignment.

Let us indicate how an enrichable mpi space \((H, A)\) gives rise to a ‘quasi mpi’ space \((H', A')\). The cutpoint \(p\) between (say) \((H^1, A^1)\) and \((H^2, A^2)\) determines \((H^1, A^1)\) and \((H^2, A^2)\) uniquely since \((H, A)\) is a strong tree. One has \(H' \cap H^2 = \{p\}\), \(W' \cap W^2 = \{x, \beta\}\) and the endpoints of \((H^1, A^1)\) respectively \((H^2, A^2)\) are of the form \((u, *)\) respectively \((\beta, *)\). Fix a line \(f \in \Delta^1 \cup \Delta^2\) which is incident with \(p\). Say \(f \in \Delta^1\). The endpoints of \(f\) being \(p = (x, \beta)\) and some \(r = (a, i)\), its middlepoint is of the form \(q = (\beta, i)\). Therefore it can be joined with any endpoint \(t_1 \neq p\) of \((H^2, A^2)\) since those are of the form \(t_1 = (\beta, j)\) if \(t_1 \neq p\). More precisely, add a new point \(s_1 \in H\) with label \((i, j)\) and a new line \(g_{q, s_1} = \{q, s_1, t_1\}\) with middlepoint \(s_1\). Analogously, joining \(q\) with an endpoint \(t_2 \neq t_1, p\) of \((H^2, A^2)\) yields a line \(g_{q, s_2} = \{q, s_2, t_2\}\) etc. The point \(q\) is the basepoint of the lines \(g_{q, s_1}\) joining \((H^1, A^1)\) and \((H^2, A^2)\). Repeat the same procedure for other pairs \((H^1, A^1), (H^2, A^2)\) with cutpoint \(p'\) and basepoint \(q' \in H' \cap H^j\).

Let \(\overline{A} := \{g_{q, s_k} \mid q \text{ basepoint, } 1 \leq k \leq n_q\}\) be the collection of new lines and \(\overline{H}\) the collection of all middlepoints of lines from \(\overline{A}\). The 3-space \((H', A')\) \((H' := H \cup \overline{H}, A' := A \cup \overline{A})\) is a quasi mpi space arising from the enrichable mpi space \((H, A)\). Its associated graph is defined as \(G := (W, H')\). Ordinary mpi spaces \((H, A)\) are considered to be quasi mpi spaces as well. A 3-space \((J, A)\) is quasi-regular if its connected components \((J_i, A_i)\) (1 \(\leq i \leq c\)) are trees of quasi mpi spaces. The associated graph of \((J, A)\) is defined as \(G := (V, J)\) \((V := \bigcup_{i=1}^{c} V_i)\) where the \(G_i := (V_i, J_i)\) are the associated graphs of the spaces \((J_i, A_i)\) (1 \(\leq i \leq c\)).

**Lemma 15.** Let \((J, A)\) be a quasi-regular space with associated graph \(G = (V, J)\). Denote its cycle matroid by \((J, \overline{e})\).

(a) \(r(J, \overline{e}) = r(J, A) + c(J, A)\).

(b) Each chordless cycle \(C \subseteq J\) of \(G\), considered as a point set of the space \((J, A)\), is of the form \((6)\) or \((7)\).

**Proof.** (a) It suffices to show that \(r(H', \overline{e}) = r(H', A') + 1\) for any quasi mpi space \((H', A')\) arising from an enrichable mpi space \((H, A)\) (then (a) follows as in the proof of Lemma 14). The associated graph of the enrichable mpi space \((H, A)\) involves the same
vertex set as the associated graph of \((H, \Delta)\) considered as a plain mpi space. Also the associated graph of the quasi mpi space \((H', \Delta')\) involves no new vertices. Therefore \(r(H', -) = r(H, -) - r(H, \Delta) + 1\). Now observe that \(r(H', \Delta') - r(H, \Delta)\) since the number of additional point splittings required for \((H', \Delta')\) (with respect to \((H, \Delta)\)) obviously equals the number \(|\Delta|\) of new lines added.

(b) This claim is more interesting. By Lemma 12 it suffices to consider chordless cycles in the associated graph of a quasi mpi space. So fix an enrichable mpi space \((H, \Delta)\), which is a strong tree \((H, \Delta)\) of mpi spaces \((H^i, \Delta^i)\) \((1 \leq i \leq d)\). Denote by \((H^0, \Delta^0)\) the tree of the first \(d - 1\) components and suppose it gives rise to the quasi mpi space \((H^0, \Delta^0)\) (possibly \((H^0, \Delta^0) = (H^0, \Delta^0)\)). Let \(p = p_d\) be the cutpoint between \((H^d, \Delta^d)\) and some \((H^d, \Delta^d)\), and suppose the endpoints of these two mpi spaces are labelled as \((a, \ast)\) respectively \((b, \ast)\). Assume \(q \in H^d \cup H^s\) is the basepoint of the lines \(g_1, \ldots, g_n\) (with middlepoints \(s_1, \ldots, s_n\) which join \((H^d, \Delta^d)\) and \((H^d, \Delta^d)\)). Hence for the quasi mpi space \((H', \Delta')\) arising from \((H, \Delta)\) one has \(H' = H^0 \cup H^d \cup \{s_1, \ldots, s_n\}\) and \(\Delta' = \Delta^0 \cup \Delta^d \cup \{g_1, \ldots, g_n\}\). The inductive structure of the proof requires to show the following stronger fact.

(9) Let \(C = \{q_1, \ldots, q_m\} \subseteq H'\) be one of the following. Either a chordless cycle of \(G (q_i, q_{i+1}\) adjacent edges), or a chordless path such that \(q_1\) and \(q_m\) are middlepoints of the same mpi component \((H^i, \Delta^i)\). Then \(C,\) as a point set of the space \((H', \Delta')\), has shape (6) or (7), respectively (8). Observe that for quasiregular spaces a point \(q_i\) of diagram (7) or (8), although placed in the middle, needs not to be the designated middlepoint of that line (cf. the proof below). But all that matters in Lemma 20, will be, that the intersection points \(r_i\) in (7) form a cycle of \((H', \Delta')\).

Proof of (9). First case: The basepoint \(q\) is in \(H^d\) (see Fig. 3; there \(q = (a, 4)\)). For \(C \subseteq H^d := H^d \cup \{q, s_1, \ldots, s_n\}\) the claim follows by Lemma 14 since \((H^d, \Delta^d)\)

![Fig. 3.](image-url)
$\Delta^d := \Delta^d \cup \{g_1, \ldots, g_n\}$ is a mpi space. For $C \subseteq H^0$ the claim follows by induction since $(H^0, \Delta^0)$ is a quasi mpi space with fewer mpi components than $(H', \Delta')$. So assume $C \notin H^d$ and $C \notin H^0$. Hence $C$ switches between the parts $H^d$ and $(H^0 - H^d)$. If $C$ is a cycle, then the number of switching vertices on $C$ must be even. But the same is true if $C$ is a path since by assumption $q_1$ and $q_n$ lie in the same part.

Any switching vertex belongs to the intersection of the vertex sets of the associated graphs of $(H^0, \Delta^0)$ and $(H', \Delta')$. Because this intersection is $\{z, \beta, 4\}$, the set of switching vertices is $\{z, \beta\}$ or $\{z, 4\}$ or $\{\beta, 4\}$. As in Lemma 14, since $C$ is chordless, the switching vertices must be subsequent vertices of $C$. In other words, $C$ contains precisely one of the edges $p = (z, \beta), q = (z, 4), r = (\beta, 4)$, and the other edges of $C$ all lie in the opposite part. This unique edge cannot be $p \in H^d$ or $q \in H^d$ since this would imply $C \subseteq \{p, q\} \cup (H^0 - H^d) = H^0$. Hence the edge is $r \in (H^0 - H^d)$. It follows that $C$ is contained in the mpi space $(H^d \cup \{r\}, \Delta^d \cup \{p, q, r\})$, whence $C$ is of form (6), (7) or (8) by Lemma 14 (place $r$ between $p$ and $q$).

Second case: The basepoint $q$ is in $H^d$ (see Fig. 4; there $q = (\beta, 1)$). For $C \subseteq H^d$ the claim follows by Lemma 14 since $(H^d, \Delta^d)$ is a mpi space. For $C \subseteq H^0 := H^0 \cup \{q, s_1, \ldots, s_n\}$ the claim follows by induction since $(H^0, \Delta^0) := \Delta^0 \cup \{g_1, \ldots, g_n\}$ is a quasi mpi space with fewer mpi components than $(H', \Delta')$. So assume $C \notin H^d$ and $C \notin H^0$. Thus $C$ switches between the parts $H^0$ and $(H^d - H^0)$. As in the first case it follows that the number of switching vertices on $C$ is even. Here any switching vertex must belong to $\{z, \beta, 1\}$ which is the intersection of the vertex sets of the associated graphs of $(H^0, \Delta^0)$ and $(H^d, \Delta^d)$. Since $C$ is chordless it follows again that $C$ contains precisely one of the edges $p = (z, \beta), q = (z, 4), r = (\beta, 4)$, and that all other edges of $C$ lie in the opposite part. This unique edge cannot be $p \in H^0$ or $q \in H^0$ since this would imply $C \subseteq \{p, q\} \cup (H^d - H^0) = H^d$. Hence the edge is $r \in (H^d - H^0)$. One has $r = q_i$ for some $i$ (if $C$ is a path then $1 < i < m$). The left and right neighbour edges of $q_i := (z, 1)$ are of the form $q_{i-1} = (\ast, x)$ respectively.

![Fig. 4](image-url)
$q_{i+1} = (1, *)$ and they are both in $H^0'$. It is clear that $q_{i+1}$ must be a point $s_j$ ($j \in \{1, \ldots, n\}$). We claim that $q_{i-1}$ must be the middlepoint $u$ of some line $g \in \Delta^{d_0}$ which is incident with $p$. The only other possibility for $q_{i-1}$ is (a priori) to be the middlepoint of a line starting from such a point $u$ as a basepoint. However, one can show that this case cannot occur. We omit the somewhat messy proof. (Observe that there is nothing to prove if $(H', \Delta')$ has the property that 'no lines $g \in \Delta^1$ and $f \in \Delta^2$ $(i \neq j)$ whose middlepoints are basepoints, intersect'. This property trivially holds if, e.g., $(H, \Delta)$ is a tree of just two mpi spaces $(H^1, \Delta^1)$ and $(H^2, \Delta^2)$.) It follows that the edges $q_{i-1}$ and $q_{i+1}$ both are middlepoints of the mpi component $(H^{d_0}, \Delta^{d_0}) := (H^{d_0} \cup \{q, s_1, \ldots, s_n\}, \Delta^{d_0} \cup \{g_1, \ldots, g_m\})$ of the quasi mpi space $(H^0', \Delta^0')$. Note that this quasi mpi space has fewer mpi components than $(H', \Delta')$.

Subcase (a): $C$ is a cycle. Then $q_{i-1}$ and $q_{i+1}$ are the end-edges of the path $C - \{q_i\}$. By the above one can apply (9) and derives that $C - \{q_i\}$ is of form (8). Hence $C$ is of form (7) (place $r$ between $p$ and $q$).

Subcase (b): $C$ is a path. Then $C - \{q_i\}$ is the disjoint union of the non-empty paths $\{q_1, \ldots, q_{i-1}\}$ and $\{q_{i+1}, \ldots, q_m\}$ (possibly $q_1 = q_{i-1}$ but $q_{i+1} \neq q_m$). Recall that $q_1$ and $q_m$ lie in the same mpi component of $(H', \Delta')$ whence in the same mpi component of $(H^0', \Delta^0')$. There is a smallest index $j \in \{1, \ldots, i-1\}$ and a biggest index $k \in \{i+1, \ldots, m\}$ such that $q_j$ respectively $q_k$ are middlepoints of $(H^{d_0}, \Delta^{d_0})$. Applying (9) shows that the paths $\{q_j, \ldots, q_{i-1}\}$ and $\{q_{i+1}, \ldots, q_k\}$ are both of form (8). It remains to check that also $\{q_1, \ldots, q_j\}$ and $\{q_k, \ldots, q_m\}$ are of form (8) (then the same holds for $C$). Suppose that $q_j = (a, b)$, $q_{j+1} = (b, *)$, $q_{k-1} = (*, c)$, $q_k = (c, d)$. Because $(b, c)$ and $(b, c)$ are endpoints of the mpi space $(H^{d_0}, \Delta^{d_0})$ one can add a new line to $(H^{d_0}, \Delta^{d_0})$ with middlepoint $u := (b, c)$. The path $\{q_1, \ldots, q_j, u, q_k, \ldots, q_m\}$ is chordless since any chord would be a chord of $C$. Hence applying (9) shows that it is of form (8), and so are its parts $\{q_1, \ldots, q_j\}$ and $\{q_k, \ldots, q_m\}$. \[\square\]

6. Modular lattices and partial linear spaces

In this section we summarize, and partly reprove, some crucial facts contained in [9] (see also [8]). First arbitrary modular lattices $L$ are considered; later on $L$ will be also 2-distributive.

A line of a modular lattice $L$ is an at least 3-element subset $g \subseteq J(L)$, maximal with the property that $p + q = \sum g$ for all distinct $p, q \in g$. Let $A[L]$ be the set of all lines of $L$ and call $g, h \in A[L]$ equivalent if $\sum g = \sum h$. If $A \subseteq A[L]$ is a set of representatives for this equivalence relation then $(J(L), A)$ is a base (of lines) for $L$. Clearly each base $(J(L), A)$ is a partial linear space. A subset $I \subseteq J(L)$ is $A$-closed if $|g \cap I| \geq 2$ implies $g \subseteq I$ for all $g \in A$.

(10) [9, Theorem 2.4] Let $(J(L), A)$ be a base of lines for the modular lattice $L$. Then $a \rightarrow J(a)$ is a lattice isomorphism from $L$ onto the lattice of $A$-closed order ideals of $(J(L), \leq)$.  

Observe that the classical duality (see, e.g. [5]) between projective spaces \((P, A)\) and geometric modular lattices \(L\) is the special case of (10) where \(A = \emptyset\). In these two extreme cases the bases of lines are uniquely determined, in contrast to the general case. Note that for each interval \((a, b) \subseteq L\) one gets the induced base of lines \((J(a), A(a))(A(a) := \{g \in A \mid g \subseteq a\}\).

As a sidemark, let us mention another view of (10). Identify the modular lattice \(L\) with the closure system \(\{J(a) \mid a \in L\}\). The associated closure space \((J(L), \rightarrow)\) is given by \(A = J(\sum A)\) (do not confuse \((J(L), \rightarrow)\) with the matroid \((J(L), -)\) of Theorem 7). An implication \(X \rightarrow Y\) on the set \(J(L)\) is just an ordered pair \((X, Y)\) of subsets of \(J(L)\). A set \(I \subseteq J(L)\) is \(\Sigma\)-closed with respect to a family \(\Sigma := \{X_i \rightarrow Y_i \mid 1 \leq i \leq n\}\) of implications if \(X_i \subseteq I\) implies \(Y_i \subseteq I\) for all \(1 \leq i \leq n\). Now (10) just states that \(\Sigma(A) := \{\{p\} \rightarrow J(p) \mid p \in J(L)\} \cup \{\{p, q\} \rightarrow g \mid g \in A, p \neq q \in g\}\) is an impicational base of \((J(L), \rightarrow)\), i.e. the \(\Sigma(A)\)-closed subsets of \(J(L)\) coincide with the closed subsets of \((J(L), \rightarrow)\). It is proven in [15] that \(|\Sigma| \geq |\Sigma(A)|\) for each impicational base \(\Sigma\) of \((J(L), \rightarrow)\). Moreover, the minimal size \(s(\Sigma) := |X_1| + \cdots + |X_n| + |Y_1| + \cdots + |Y_n|\) of an impicational base \(\Sigma := \{X_i \rightarrow Y_i \mid 1 \leq i \leq n\}\) of \((J(L), \rightarrow)\) is determined.

Recall [1, p. 236] that the congruence lattice of a modular lattice \(L\) is a Boolean algebra. Denoting its height by \(s = s(L)\) and letting \(\theta_1, \ldots, \theta_s\) be the maximal congruences, \(L\) is a subdirect product of its subdirectly irreducible factors \(L_i := L/\theta_i\). Each prime quotient is separated by exactly one \(\theta_i\). In particular, for each \(p \in J(L)\) there is a unique \(\theta_i\) with \((p, p) \not\in \theta_i\). Accordingly \(J(L)\) is the disjoint union of \(s\) nonvoid 'subdirectly irreducible' components \(J_i(L)\).

(11) [9, Proposition 2.5] Let \((J(L), A)\) be a base of lines of the modular lattice \(L\) with connected components \((J(L)_1, A_1), \ldots, (J(L)_s, A_s)\). Then \(c = s\) and up to permutation \(J(L)_1 = J_1(L), \ldots, J(L)_s = J_s(L)\).

Closely related to the lines of a modular lattice \(L\) are its essential elements, which were defined in Section 3. Lemma 16 yields alternative proofs of facts (12) and (13) below which were established in [9] (however, its main application will be in the proof of Lemmas 17 and 18). For a prime quotient \((a/b) \subseteq L\) put \(J(a/b) := \{p \in J(L) \mid p \preceq a, p \not\preceq b\}\). A trivial application of modularity shows that \(J(a/b)\) is always a non-empty antichain.

**Lemma 16** [2, p. 187]. Let \(a/b\) and \(c/d\) be prime quotients of a modular lattice \(L\) which have a common upper transpose \(e/f\). Then also \((a + c)/h\) \((h := (a + c)f)\) is a common upper transpose. Furthermore, one of the following cases occurs.

(a) If \(bd = ac\) then \(\{a + c, a + d, b + c, b + d\}\) is a covering sublattice \(M_3\).

(b) If \(bd < ac\) then \(h = b + d\) and \(\{a + c, a, b, c, d, ac, bd\}\) is a (generally not covering) sublattice isomorphic to the cube \(2^3\).

(12) [9, Lemma 2.2] For each \(M_n\)-element \(x (n \geq 3)\) of a modular lattice \(L\) and points \(p_i \in J(x_i/x_0)\) the set \(g = g_x := \{p_1, \ldots, p_n\}\) is a line of \(L\). Conversely let \(g = \{p_1, \ldots, p_n\}\)
be a $n$-element line. Then $x - x_g := \sum g$ is a $M_n$-element and up to permutation $p_i \in J(x_i/x_0), p_i + p_j = x_0 (i \neq j)$.

Proof of (12). The first part is clear (the modularity of $L$ is not used). For the second part pick distinct $p, q, r \in g$. Assuming $p \leq q + r$ yields the contradiction $q \leq p + r \leq q + r$ and $q = q + q$. Thus $p \leq q + r$ and analogously $q \leq p + r$. It follows that $p + q/h \ (h := p + q + r)$ is a common upper transpose of $p/p$ and $q/q$. Since $pq = pq$, Lemma 16(a) guarantees that $\{p + q, p + q, p + q, h, p + q\}$ is a covering sublattice $M_3$. For any $w < p + q$ one has $w \geq wp + wp \geq p + q$, i.e. $p + q$ is essential.

(13) [9, Lemma 5.2] Let $a/b$ be a prime quotient of a modular lattice $L$. Then any distinct $p, q \in J(a/b)$ lie on a common line, i.e. $p + q$ is essential.

Proof of (13). Again $pq = pq$, but now $a/b$ is a common upper transpose. As before one concludes that $p + q$ is essential. □

Analogous to $J(a)$ and $J(a/b)$ ($a/b$ a prime quotient) define $E(a) := \{x \in E(L) | x \leq a\}$ and $E(a/b) := \{x \in E(L) | x \leq a, x \leq b\}$. The localization of a base of lines $(J(L), A)$ to the prime quotient $(a/b) \subseteq L$ is the partial linear space $(J(a/b), A(a/b))$ with $A(a/b) := \{g(a/b) | x_g \in E(a/b)\}$ and $g(a/b) := g \cap J(a/b)$. As a strengthening of (13) one can show

(14) [9, Lemma 5.2] Any localization $(J(a/b), A(a/b))$ of any base $(J(L), A)$ of a modular lattice $L$ is connected.

Let us illustrate these concepts on the modular lattice $L_2$ of Fig. 5(a). It has 71 elements, among which 26 join irreducibles and 15 essential ones. Since all essential elements are $M_3$-elements, each base of lines will be a 3-space. Consider, e.g., the base $(J(L_2), A) = (J(L_2), A(L_2))$ of Fig. 5(b). Because it is connected, $L_2$ must be subdirectly irreducible by (11). Concerning (13) and (14), consider, e.g., the prime quotient $(a/b) := (70/65)$. For 24, 39 $\in J(a/b)$ the element 64 = 24 + 39 is indeed essential. The points 24 and 39 are connected in the space $(J(a/b), A(a/b)) = \{(24, 31, 39), \{(24, 31), \{31, 39\}\})$, but do not lie on a common line of $A(a/b)$.

Let us define cycles of essential elements and compare them with cycles in a base of lines $(J(L), A)$. The interplay between these two kinds of cycles will be crucial in the sequel. For $x, y \in E(L)$ we defined $x \prec y$ and $x \nrightarrow y$ in Section 3. Now put $x \sim y$ if either $x \prec y$ or $x \nrightarrow y$. A cycle of essential elements is a tuplet $(x^1, \ldots, x^n)$ with distinct essential elements $x^1, \ldots, x^n$ such that $(\forall i) x^i \sim x^{i+1}$ but $(\forall i) x^i \nrightarrow x^{i+2}$ (dealing with cycles of essential elements we calculate modulo $n$).

As a consequence of (12) it is easily seen that a cycle $(x^1, \ldots, x^n)$ of essential elements gives rise to a cycle $(p_1, \ldots, p_n, p_1)$ in an appropriate base of lines $(J(L), A)$ such that the lines $[p_1, p_2], \ldots, [p_n, p_1]$ belong to the essential elements $x^1, \ldots, x^n$. Conversely, for a cycle $(p_1, \ldots, p_n, p_1)$ in a base of lines $(J(L), A)$ the distinct essential elements $x^1, \ldots, x^n$ belonging to the lines $[p_1, p_2], \ldots, [p_n, p_1]$ do not necessarily form a cycle of essential elements. Consider, e.g., the cycle $(21, 52, 45, 21)$ in the base
Fig. 5.
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\((J(L_2), A(L_2))\) which corresponds to the essential elements \((61, 68, 56)\). Another example is the cycle \((1, 3, 7)\) in the base \((J(L_3), A(L_3))\) of the lattice \(L_3 := L(Z_3^2)\); it corresponds to the essential elements \((8, 13, 11)\) (see Fig. 6). Unlike the second counterexample, the first counterexample is harmless in the sense that \((61, 68, 56)\) can be completed to a cycle of essential elements \((61, 68, 56, 34, 61)\). This is a general property of 2-distributive modular lattices (Lemma 18(a)). Such lattices are the subject of the remainder of this section.

The essential elements and lines of a 2-distributive modular lattice \(L\) enjoy nice properties. The key is provided by Lemma 17 below. This Lemma, as well as Lemma 18, is not stated explicitly in [9], yet the basic proof ideas can there be found. Both lemmas will be of paramount importance in Section 7.

**Lemma 17** [9]. Let \(L\) be a 2-distributive modular lattice. Let \(x, y \in E(L)\) be noncomparable and such that \((x_1/x_0)\) and \((y_1/y_0)\) have some common lower transpose \((e/f)\). Then \(x + y\) is an \(M_n\)-element \((n \geq 2)\) and one has \(p + q = x + y\) for all \(p \in J(x_1/x_0), q \in J(y_1/y_0)\) \((i, j \neq 1)\).

As a guideline to the proof consider, e.g., in Fig. 5(a) the elements \(x := 61, y := 56\) and the quotients \((x_1/x_0) := (54/40), (y_1/y_0) := (44/35), (e/f) := (36/27)\).

**Proof.** From the assumption it follows that \((e/f)\) is also a common lower transpose for each pair of quotients \((x_i/x_0), (y_j/y_0)\) \((i, j \neq 1)\).

**First case:** There are indices \(i, j \neq 1\) with \(x_i \equiv y_j = x + y, \text{say } x_2 + y_2 = x + y\). According to the dual of Lemma 16(a) the elements \(\{xy, x_2y, xy_2, h_{2, 2}, x_2y_2\}\) \((h_{2, 2} := x_2y_2 + e)\) form a covering sublattice \(M_3\). For all \(i, j \neq 1\) one has \(x_0y_0 \leq x_iy_j \leq (xy)(x_iy_j) = (x_2)(x_2y) = x_0y_0\), i.e. \(x_iy_j = x_0y_0\). But this implies \(\delta(x_i + y_j) = \delta(x_i) + \delta(y_j) = \delta(x_i) + \delta(x_2) + \delta(y_2) + \delta(x_2y_2) = \delta(x_2y_2) = x + y\), i.e. \(x_1 \equiv y_j = x + y\). Next show that \(x + y\) is an \(M_n\)-element \((n \geq 2)\). From \(\delta(xy/x_0y_0) = \delta(x/x_0) = \delta(y/y_0) = 2\) follows \(\delta(x + y/x_0 + y_0) = 2\), whence it suffices to show that \(w \perp x + y\) implies \(w \perp x_0 + y_0\).
Clearly $xw \leq x$ and $yw \leq y$. The assumption $xw = x_i$, $yw = y_j$ ($i, j \neq 1$) gives the contradiction $w \geq x_i + y_j = x + y$. Now $xw = x_1$ implies $w \geq x_1 + yw \geq x_1 + x_0 \geq x_0 + y_0$, and similarly $yw = y_1$ implies $w \geq x_0 + y_0$. Finally, fix $p \in J(x_i/x_0)$ and $q \in J(y_j/y_0)$ ($i, j \neq 1$). Since $p + q \leq x + y$ it suffices to check that $w < x + y$ implies $p \not\leq w$ or $q \not\leq w$. The cases $w = x_0 + y$ and $w = x + y_0$ being symmetric, assume that $w \neq x + y_0$. Then $p \not\leq w$, because otherwise $p \leq x_i w = x_0$, contradicting $p \in J(x_i/x_0)$.

Second case: For all $i, j \neq 1$ one has $x_i + y_j < x + y$. We show that this case is impossible. According to the dual of Lemma 16(b) for each pair $x/x_i$, $y/y_j$ ($i, j \neq 1$) the set \{ $x + y$, $x_i + y_j$, $x$, $y$, $h_{i,j}$, $x_i y_j$ \} ($h_{i,j} := x_i y_j + e$) is a sublattice isomorphic to $2^3$. In particular $(x_i + y_j) h_{i,j} = x_i y_j$. Fix a $p \in J(x_i/x_0)$ and put $u := (x_2 + y_3) (x_3 + y_2)$. Show that $p \leq x_2 + y_2 + u$ but $p \not\leq x_2 + y_2$, $p \not\leq x_2 + u$, $p \not\leq y_2 + u$; this is impossible in a 2-distributive lattice. Indeed, $x_2 + y_2 + u = x_2 + (y_2 + y_3 + x_2) (x_3 + y_2) = (y_2 + y_3 + x_2) (x_2 + x_3 + y_2) = x + y$ and $x_2 + u = (x_2 + x_3 + y_2) (x_2 + y_3) = x_2 + y_3$ and $y_2 + u = (y_2 + y_3 + x_2) (x_3 + y_2) = x_3 + y_3$. The assumption $p \leq x_i + y_j$ ($i, j \in \{ 2, 3 \}$) yields $p \leq e(x_i + y_j) = e(x_i + y_j) h_{i,j} = e x_i y_j = f$, contradicting $p \in J(x_i/x_0)$.

Lemma 18 [9]. Let $L$ be a 2-distributive modular lattice. Suppose that for some noncomparable $x, y \in E(L)$ there is an irreducible $p$ with $p \in J(x_i/x_0)$ and $p \in J(y_j/y_0)$.

(a) Then $p \in J(z_3/z_0)$ for some essential $z \leq x y$ with $(x_1/x_0) \bowtie (z/z_1)$ and $(z/z_0) \bowtie (y_1/y_0)$.

(b) If moreover $x, y \in E(a/b)$ for some prime quotient $a/b$ then necessarily $z \leq b$, so $p \in J(b)$.

As before, to illustrate (a), consider, e.g., the elements $x := 61$, $y := 56$ and $p := 15$ of Fig. 5(a).

Proof. By assumption $x_1/x_0$ and $y_1/y_0$ have the common lower transpose $p/p$ whence Lemma 17 will be applicable.

(a) As in the proof of Lemma 17, \{ $x y$, $x_2 y$, $x_2 y_2$, $h_{2,2}$, $x_2 y_2$ \} ($h_{2,2} := x_2 y_2 + p$) is a covering sublattice $M_2$. Pick a quotient $z/z_0$ minimal with the property that \( (x_1/x_2 y_2) \bowtie (z/z_0) \) and $p \leq z$ (possibly $(z/z_0) = (x_1/x_2 y_2)$). It suffices to show that $z$ is essential. Assuming the converse one has $(z/z_0) \bowtie (w/w_0)$ for some quotient $w/w_0$ ($w \neq z$). But then $p \not\leq w$ by the minimality assumption for $z/z_0$. There are two elements $z_1, z_2$ with $z_0 < z_1, z_2 < z$ and $p \not\leq z_1 z_2$ (otherwise $p \leq z_0 \leq x_0$). Let $w_1, w_2$ be such that $w_0 < w_1, w_2 < w$ and $z_1 = z_0 + w_1$, $z_2 = z_0 + w_2$. Now $p \leq z_0 + w_1 + w_2$, but $p \not\leq z_0 + w_1, z_0 + w_2, w_1 + w_2$, which is the desired contradiction.

(b) From $z_1 \leq x_0 \leq b$ and $z_2 \leq y_0 \leq b$ follows $z = z_1 + z_2 \leq b$.

(15) [10] Let $a/b$ be a prime quotient of a 2-distributive modular lattice $L$. Then $p + q$ is comparable with $p + r$ for all $p, q, r \in J(a/b)$.

Proof of (15). The claim is trivial if some of the elements $p, q, r$ are identical. Otherwise $x := p + q$ and $y := p + r$ are essential by (13). Assume they were not comparable. Since
p/p is a common lower transpose of (say) \(x_1/x_0\) and \(y_1/y_0\), Lemma 18(b) yields the contradiction \(p \leq b\). \[\square\]

(16) [9, Theorem 5.1] Each 2-distributive modular lattice \(L\) is ‘locally acyclic’, i.e. any localization \((J(a/b), A(a/b))\) of any base \((J(L), A)\) is acyclic.

Proof of (16). Assume there is a cycle \(C := (p_1, p_2, \ldots, p_n, p_1)\) in \((J(a/b), A(a/b))\). By (15) any two subsequent elements of the set \(E(C) := \{x^i := p_i \mid p_{i+1} \mid 1 \leq i \leq n\}\) are comparable. In particular, it follows easily that \(C\) has at least length \(n \geq 4\). Assume now \(n\) is the smallest length of cycles in spaces \((J(a/b), A'(a/b))\). Fix a minimal element \(x^k\) of the poset \(E(C)\). Then \(x^k \sim x^k \sim x^k + 1\) by comparability. Since \(x^{k-1} \sim (x^k/x^k)\) and \((x^{k+1}/x^k) \sim x^{k+1} (i \neq j)\) would imply \(x^k = x^k + x^k \leq x^k + x^k + 1 \leq b\), one has, e.g., \(x^{k-1} \sim (x^k/x^k) \sim x^{k+1}\). But then it follows easily that \((p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n, p_1)\) is a cycle of shorter length in the localization \((J(a/b), A'(a/b))\) where \(A' := (\forall x \sim \forall g \neq x) \cup (g \neq x)\) and \(g \neq x := (g \neq x \sim \forall p_k) \cup \{p_k\}\) (here we need \(n \geq 4\)). This contradiction shows that \((J(a/b), A(a/b))\) is acyclic for each base \(A\). \[\square\]

Note that Lemmas 17, 18 and (15), (16) badly fail in modular lattices which are not 2-distributive. See, e.g., Fig. 6(a) and (b). On the other side, it is natural to consider 2-distributive modular lattices which even are ‘globally acyclic’. These lattices will reappear in the remaining Sections 7, 8 and 9.

7. More about modular lattices and partial linear spaces

This section contains four somewhat technical lemmas. Lemma 19 gives a nice formula for the rank of an arbitrary base of a 2-distributive modular lattice. In the other three lemmas the lattices \(L\) are moreover of order 2. Lemma 20 shows that the cycles of shape (6) or (7) in a base \((J(L), \Lambda)\) behave the way they should (see Lemmas 10 and 15). Lemmas 21 and 22 give necessary respectively sufficient conditions for the existence of a regular base \((J(L), \Lambda)\).

For the modular lattice \(L_3\) in Fig. 6 one has \(r(J(L_3), A(L_3)) + c(J(L_3), A(L_3)) = (-1) + 1 \neq 3 = \delta(L_3)\). The following lemma states that this cannot happen in a 2-distributive modular lattice.

Lemma 19. For each base of lines \((J(L), \Lambda)\) of a 2-distributive modular lattice \(L\) one has \(r(J(L), \Lambda) + c(J(L), \Lambda) = \delta(L)\).

Proof. In view of (11) the above equation \(|\Lambda| - r(J(L), \Lambda) + c(J(L), \Lambda) = \delta(L)\) is equivalent to

\(|\Lambda| - r(J(L), \Lambda) = \delta(L) - s(L),

where \(s(L)\) is the number of maximal congruences \(\theta_i\), i.e. the number of subdirectly irreducible factors \(L_i := L/\theta_i\).
First case: $L$ is subdirectly reducible with corresponding connected components $(J(L)_i, A_i)$. Then by induction $|A_i| - r^*(J(L)_i, A_i) = (a_i) - 1$ for all $1 \leq i \leq s(L)$. Because the length of a subdirect product of modular lattices equals the sum of the lengths of the factors, one derives $|A| - r^*(J(L), A) = \sum(|A_i| - r^*(J(L)_i, A_i)) = \sum((a_i) - 1) = (a) - s(L)$.

Second case: $L$ is subdirectly irreducible (see Fig. 7, ignoring the labelling of the points). Fix a coatom $a < 1$ and let $L'_1, ..., L'_t (t \geq 1)$ be the subdirectly irreducible factors of the interval $a/0 \leq L$. Thus the base $(J(a), A(a))$ of $a/0$ consists of $t$ connected components $(J(a)_i, A(a)_i)$. By (14) and (16) the localization $(J(1/a), A(1/a))$ is connected and acyclic. Because $(J(L), A)$ is connected, the number $\lambda_i$ of lines $g \in L(1/a)$ which 'point' to the component $(J(a)_i, A(a)_i)$ is at least 1. Since $(J(1/a), A(1/a))$ is connected, it is clear that precisely $\lambda_i - 1$ point splittings are necessary to destroy all cycles which involve points from both $(J(1/a), A(1/a))$ and $(J(a)_i, A(a)_i)$. Since $(J(1/a), A(1/a))$ is acyclic, there are no cycles involving just points from $(J(1/a), A(1/a))$. Summarizing one obtains

$$|A| - r^*(J(L), A) = (|A(a)_1| + \cdots + |A(a)_t| + \lambda_1 + \cdots + \lambda_t)$$

$$- (r^*(J(a)_1, A(a)_1) + \cdots + r^*(J(a)_t, A(a)_t) + (\lambda_1 - 1) + \cdots + (\lambda_t - 1))$$

$$= (|A(a)_1| - r^*(J(a)_1, A(a)_1) + 1) + \cdots + (|A(a)_t| - r^*(J(a)_t, A(a)_t) + 1)$$

$$= (\lambda(L'_1) + \cdots + \lambda(L'_t) - \delta(a/0) - \delta(L)) - 1. \quad \square$$

For a base $(J, A)$ the corank $r^*(J, A) = |A| + \epsilon(J, A) - \delta(L) = |E(L)| + s(L) - \delta(L)$ is an invariant of $L$ by Lemma 19. In particular, the acyclicity of one base $(J, A)$, i.e. $r^*(J, A) = 0$ is equivalent to the acyclicity of all bases $(J, A)$. As mentioned earlier,
each cycle of essential elements induces a cycle in an approximate base. Conversely, it follows easily from Lemma 18(a) that each cycle in a base induces a cycle of essential elements. Thus the acyclicity of all bases is in turn equivalent to the non-existence of cycles of essential elements.

In the remainder of this section only 2-distributive modular lattices of order 2 are considered. So any base of such a lattice is a 3-space.

**Lemma 20.** Let \( L \) be a 2-distributive modular lattice of order 2 with a base of lines \((J(L), A)\). Then any subset \( C \subseteq J(L) \) of shape (6) or (7) satisfies \((\forall q \in C)\, q \leq \sum (C - \{q\})\).

**Proof.** If \( C \) has shape (6), then the claim is trivial by definition of a line of a modular lattice. So assume that \( C = \{q_1, \ldots, q_m\} \) is of shape (7). Unfortunately (or interestingly?) this case is more complicated. Let \( x^i \) be the \( M_{2} \)-element which belongs to the line \( r_{i-1}, q_i, r_i \) (1 ≤ i ≤ m). By symmetry it suffices to show \( q_1 \leq q_2 + \cdots + q_m \), and this follows of course from \( x^2, x^m \leq q_2 + \cdots + q_m \). Again by symmetry it is enough to prove the following.

\[
(17) \quad x^2 \leq q_2 + \cdots + q_m.
\]

**Proof of (17).** We shall see that indeed \( x^2 \leq q^i + q^j \) for suitable \( q^i, q^j \) (i, j ≠ 1). The fact below will be applied two times in the subsequent case distinction (see Fig. 8).

\[
(18) \quad \text{Let} \ e/f \ \text{be a prime quotient and assume that} \ x \neq y \ \text{are comparable} \ M_{2}\text{-elements. If} \ (x_1/x_0) \succ (e/f) \ \text{and} \ (y_1/y_0) \succ (e/f) \ \text{then either} \ (x_1/x_0) \succ (y/y_2) \ \text{or} \ (y_1/y_0) \succ (x/x_3) \ \text{for a} \ x \neq 1.
\]

**Proof of (18).** Indeed, clearly \( y \not\leq x_0 \) and \( x \not\leq y_0 \), so \( x \succ y \) or \( y \succ x \). The cases being symmetric assume \( x \succ y \). Since \( (x_\beta/x_0) \succ y(\beta \neq 1) \) yields the contradiction \( x = e + y = y \) one has \( (x_1/x_0) \succ y \). The case \( (x_1/x_0) \succ (y/y_1) \) yields the contradiction \( e \leq x_0 \).

![Fig. 8.](image)
Case (i): There is no \( x^k \) with \( (x:i/x:i) \lor x^2 \) and \( (x:i/x:i) \lor x^{k+1} \) (see Fig. 9). Then in particular \( x^3 \lor x^2 \) is impossible, so either \( x^2 \lor (r_2/r_2) \lor x^3 \) or \( x^2 \lor x^3 \). In the first case \( x^2 \) and \( x^3 \) have the common lower transpose \( r_2/r_2 \). Since \( q_2 \) and \( q_3 \) are in admissible positions, Lemma 17 yields \( x_2 < x^2 + x^3 = q_2 + q_3 \). Now assume \( x^2 \lor x^3 \), say \( (x_1/x_0) \lor (x^3/x^1) \). Let \( i \geq 3 \) be maximal with the property that \((\forall 3 \leq j \leq i) (x_j^2/x_0^2) \lor x^1\). Hence \( (x_i^2/x_0^2) \lor x^1 \) (say \( x_i^2/x_0^2 \lor (x^j/x_i^1) \)) and \( (x_i^2/x_0^2) \lor x^j+1 \). Clearly \( x^j \) cannot be \( x^i \) yet \( (r_1, r_2 \in J(x_i^2/x_0^2) \) is nonsense). The cases \( (x_i^2, x_0^2) \lor x^j+1 \) \((\neq 1)\) are impossible by the maximality of \( i \). Up to symmetry there are three possible relations between \( x^{i+1} \) and \( x^1 \).

(a) \( (x_1^i/x_0^i) \lor x^{i+1} \) or \( (x_1^i/x_0^i) \lor (r_i/r_i) \lor x^{i+1} \). Then \( r_i \in J(x_1^i/x_0^i) \), forces \( q_i \notin J(x_1^i/x_0^i) \), whence \( x^2 = q_2 + q_i \).

(b) \( x^{i+1} \lor (x_1^i/x_0^i) \), say \( (x_1^i/x_0^i) \lor (x^i+1/x_1^i) \). If \( x^{i+1} \) were comparable with \( x^2 \) then by (18) either \( (x_1^i/x_0^i) \lor (x^i+1/x_0^i) \) or \( (x_1^i/x_0^i) \lor (x^i+1/x_0^i) \) for a \( x \neq 1 \). The first case is impossible by the maximality of \( i \) and the second case contradicts our general assumption (put \( k := i + 1 \)). Therefore \( x^{i+1} \) is incomparable to \( x^2 \). The elements \( x^{i+1} \) and \( x^2 \) have the common lower transpose \( (x^1/x_1^i) \) and since \( q_{i+1} \notin J(x_1^i/x_0^i) \) Lemma 17 yields \( x^2 < x^2 + x^{i+1} = q_2 + q_{i+1} \). If one had \( x^{i+1} = x^1 \) then the connection from \( x^1 \) to \( x^2 \) must be \( (x_1^i/x_0^i) \lor (x_1^i/r_i) \lor (x_1^2/x_0^2) \) \((\neq 1)\). But this implies \( x^2 = r_1 + x^1 = x^1 \). Hence \( x^{i+1} \neq x^1 \).

(c) \( x^{i+1} \lor (x_1^2/x_0^2) \) or \( (x_1^2/x_0^2) \lor (r_i/r_i) \lor x^{i+1} \) for a \( x \neq 1 \). Here it is easy to see that in any case \( x^2 \) and \( x^{i+1} \) are incomparable, have a common lower transpose, and that \( q_2, q_{i+1} \), are in the right positions. By Lemma 17 therefore \( x^2 < x^2 + x^{i+1} = q_2 + q_{i+1} \) \((x^{i+1} \neq x^1 \) by the same reason as in (b)).
Case (ii). There is a \( x \in X \) with \((x_i/x_i) \preceq x^2 \) and \((x_i/x_i) \preceq x^{i-1} \) (see Fig. 10; possibly \( x^2 = x^3 \)). Say \((x_i/x_i) \preceq (x^2/x^2_i) \). Because of \((x^{i-1}/x^{i-1}_i) \preceq (x^i/x^i) \) (say \( x \neq 1 \)) there is a \( i \geq k \) maximal with the property that

\[
(x^{i-1}/x^{i-1}_i) \preceq (x^i/x^i), \ (x^i/x^i) \preceq (x^{i+1}/x^{i+1}_i), \ldots, (x^{i-1}/x^{i-1}_i)
\]

\( \vee (x^i/x^i) \) \( (\alpha, \beta, \ldots, \gamma \neq 1) \).

In this way one achieves three things. First \( q_i \notin J(x^i/x^i) \). Secondly \((x^i/x^i) \not\preceq x^{i+1} \) \((\delta \neq 1) \) is impossible by the maximality of \( i \). Thirdly neither \((x^i/x^i) \not\preceq x^{i+1} \) nor \((x^i/x^i) \not\preceq (r_i/r_i) \not\preceq x^{i+1} \) can occur because of \((x^i/x^i) \preceq x^{i-1} \) \((r_{i-1}, r_i \in J(x^i/x^i) \) is nonsense). In particular \( x^i \neq x^i \). Therefore, up to symmetry, there are three possible relations (a), (b), (c) between \( x^{i+1} \) and \( x^i \).

(a) \( x^{i+1} \preceq (x^i/x^i) \), say \((x^{i+1}/x^{i+1}_i) \preceq (x^i/x^i) \). Then \( q_{i+1} \notin J(x^{i+1}/x^{i+1}_i) \) whence \( x^2 \prec x^{i+1} = q_i + q_{i+1} \) \((x^{i+1} \neq x^2) \) since \( x^{i+1}_i \geq x^2 \). The remaining two cases are \((x^{i+1}/x^{i+1}_i) \not\preceq (r_i/r_i) \not\preceq x^{i+1} \) and \((x^{i+1}/x^{i+1}_i) \not\preceq x^{i+1} \) \((\alpha = 2, 3) \). By symmetry it suffices to consider the case \( \alpha = 2 \).

(b) \((x^2/x^2_i) \not\preceq (r_i/r_i) \not\preceq x^{i+1} \). Then Lemma 17 applied to \( x^i, q_i, x^{i+1}, q_{i+1} \) implies \( x^2 \prec x^{i+1} = q_i + q_{i+1} \). It remains to check that \( x^{i+1} \neq x^1 \). First observe that \( x^2 < x^{i+1} \) yields the contradiction \( x^2 = x^2 + r_i \prec x^{i+1} \), and \( x^{i+1} < x^2 \) yields \( r_i < x^2 \) whence \( x^2 = x_i + x^2 \). Thus, if \( x^{i+1} = x^1 \), then the connection between \( x^1 \) and \( x^2 \) is necessarily \( x^1 \not\preceq (r_i/r_i) \not\preceq x^2 \). But then \( x^1 \prec r_i \prec x^{i+1} \). This shows that \( x^{i+1} = x^1 \) is impossible.

(c) \((x^2/x^2_i) \not\preceq x^{i+1} \). Let \( l \geq i + 1 \) be maximal with the property that \((x^2/x^2_i) \not\preceq x^l \). Say \((x^2/x^2_i) \not\preceq (x^l/x^l) \). The cases \((x^2/x^2_i) \not\preceq x^{i+1} \) and \((x^2/x^2_i) \not\preceq (r_i/r_i) \not\preceq x^{i+1} \) \((\alpha \neq 1) \) are impossible by the maximality of \( l \). This leaves us with three subcases (c1), (c2), (c3).

[Fig. 10.]

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**Fig. 10.**
(c1) \((x_i/x_{k}) \not< x_{i+1} \) \((i \neq 1)\). Then \(x^{i}, q_{i}, x^{i+1}, q_{i+1}\) qualify for Lemma 17 and one obtains \(x^{i} < x^{i+1} = q_{i} + q_{i+1}\). Again we claim that \(x^{i+1} \neq x^{i}\). The assumptions \(x^{i+1} < x^{i}\) respectively \(x^{i} < x^{i+1}\) yield the contradictions \(x^{i} = x^{j} + x^{i} = x^{j}\) respectively \(x^{i} = x^{j} < x^{i} \leq x^{i+1}\). Thus, if \(x^{i+1} = x^{i}\), then necessarily \(x^{i+1} \not> r_{i} \not< x^{2}\). But then \(r_{i} \not< J(x^{i+1}/x^{i+1})\) implies \(r_{i} \not< J(x^{i+1}/x^{i+1})\) whence \(x^{i+1} = x^{i} + r_{i} \not< x_{i}\). This shows that \(x^{i+1} = x^{i}\) is impossible.

(c2) \((x^{i}/x^{i}) \not< x^{i+1}\), say \((x^{i}/x^{i}) \not< (x^{i+1}/x^{i+1})\). If \(x^{i+1}\) is incomparable with \(x^{i}\) one concludes as in (c1) that \(x^{2} < x^{i} < x^{i+1} = q_{i} + q_{i+1}\) by precisely the same reasoning as in (c1). If \(x^{i+1}\) and \(x^{i}\) are comparable then \((x^{i+1}/x^{i+1}) \not> (x^{i}/x^{i})\) \((i \neq 2)\) by (18) and the maximality of \(l\). Because of \(q_{i+1} \not< J(x^{i+1}/x^{i+1})\) one has \(x^{2} < x^{i+1} = q_{i} + q_{i+1}\) \((x^{i+1} \neq x^{i}\) since \(x^{2} \leq x^{i+1}\).

(c3) \((x^{i}/x^{0}) \not< x^{i+1}\) or \((x^{i}/x^{0}) \not> (r_{i}/r_{i}) \not< x^{i+1}\). Then \(q_{i} \not< J(x^{i}/x^{0})\) whence \(x^{2} < x^{i} = q_{i} + q_{i}\). However, here \(x^{2} = x^{i}\) is possible! But in this case necessarily \(x^{i} \not> (r_{i}/r_{i}) \not< x^{2}\). Since \(r_{i} \not< J(x^{i+1}/x^{i+1})\) \((\alpha \neq 1)\) yields the contradiction \(x^{i} = r_{i} + x^{i} = x^{i}\), one has \(r_{i} \not< J(x^{i+1}/x^{i+1})\). Hence \(q_{i} \not< J(x^{i}/x^{0})\) and \(x^{2} < x^{i} = q_{i} + q_{i}\). □

Being sloppy, say that a lower quotient \(x_{i}/x_{0}\) of an \(M_{3}\)-element \(x\) transposes downwards if \((x_{i}/x_{0}) \not> y\) for some \(M_{3}\)-element \(y\). An \(M_{3}\)-element \(x\) is of type \(k \(k=0, 1, 2, 3)\) if precisely \(k\) of its lower quotients transpose downwards. A brace for a pair of lower quotients \((x_{i}/x_{0}), (x_{i}/x_{0})\) of \(x\) is a sequence of \(M_{3}\)-elements \((x, y^{1}, \ldots, y^{m}, x)\) such that \((x_{i}/x_{0}) \not> y^{1} \ldots y^{m} \not< (x_{j}/x_{0})\) and \(y^{i} \not> x\) for all \(1 \leq i \leq m\) (so any brace with \(y^{i} \not> y^{i+2}\) is a cycle of \(M_{3}\)-elements). It will be convenient to refine the above type hierarchy by saying that a type \(i\) \(M_{3}\)-element is of type \((i. j)\) if exactly \(j\) pairs of lower quotients admit braces. Thus the refined hierarchy is \((0.0) < (1.0) < (2.0) < (2.1) < (3.0) < (3.1) < (3.2) < (3.3)\). Observe that an \(M_{3}\)-element of type \((3.3)\) (see Section 3) is 'strongly' of type \((3.3)\), since one even has 'downwards braces' between any two lower quotients \((x_{i}/x_{0})\) and \((x_{j}/x_{0})\). In the lattice \(L_{2}\) of Fig. 5(a) the \(M_{3}\)-element 68 is of type \((3.3)\), the \(M_{3}\)-elements 28, 34, 57 are type \((2.1)\), the \(M_{3}\)-element 64 is type \((2.0)\), the \(M_{3}\)-elements 19, 23, 26, 40, 47, 56, 61, 62 are type \((1.0)\), and 7, 22 are type \((0.0)\).

Lemma 21. Let \(L\) be a 2-distributive modular lattice of order 2.

(a) If \(L\) admits a mpi base of lines then all \(M_{3}\)-elements have type \(\leq (2.1)\).

(b) If \(L\) admits a regular base of lines then all \(M_{3}\)-elements have type \(\leq (3.1)\).

Proof. (a) Suppose there is an \(M_{3}\)-element \(x\) of type 3. Show that in any base \((J(L), A)\) each point of the line \(g_{x} \in A\) is also incident with other lines of \(A\) (in particular the midlepoint of \(g_{x}\) is not isolated). Consider \(p \in g_{x}\) with (say) \(p \in J(x_{i}/x_{0})\). Since by assumption \(x_{i}/x_{0}\) transposes downwards there are other points \(q\) in \(J(x_{i}/x_{0})\). Because of \((p/p) \not< (x_{i}/x_{0}) \not> (q/q)\) the points \(p\) and \(q\) are in the same subdirect component of the lattice \(x_{i}/0\), whence by (11) connected in the base \((J(x_{i}), A(x_{i}))\). If \((p, p_{1}, \ldots, q)\) is a path in \((J(x_{i}), A(x_{i}))\) then the line \([p, p_{1}] \neq g_{x}\) is incident with \(p\).

(b) We first show a general property of braces \((x_{i}/x_{0}) \not> y^{1} \cdots y^{m} \not< (x_{2}/x_{0})\).
(19) For all \( p \in J(x_1/x_0) \) and \( q \in J(x_2/x_0) \) there is a projectivity of the form \((p/p) \succ (w^1/w_1^1) \succ (r_1/r_1) \succ (w^2/w_1^2) \succ (r_2/r_2) \succ \ldots \succ (r_k/r_k) \succ (w^{k+1}/w_1^{k+1}) \succ (q/q)\) with \(M_3\)-elements \( w^i \not\supset x \) and \( r_i \in J(L)\).

Proof of (19). It is best to give a typical example (see Fig. 11). Suppose \( p \not\approx y_1 \) and \( q \approx y_7 \).

Pick arbitrary \( r_1 \in J(y_1^1/y_1^0) \) and \( r_1' \in J(y_1^2/y_1^0)\). According to (13) and (15) the \(M_3\)-element \( w^1 := p + r_1 \) is comparable with \( y^1 = r_1 + r_1' \), so \( w^1 \succ y^1 \) since \( p \not\approx y^1 \). Choosing \( r_2 \in J(y_1^2/y_1^0) \), \( r_3 \in J(y_2^1/y_3^0) \), and similarly \( r_4, \ldots, r_7 \) as indicated in Fig. 11, one gets 
\[
(p/p) \succ (w^1/w_1^1) \succ (r_1/r_1) \succ (y^1/y_1^2) \succ \ldots \succ (r_7/r_7) \succ (y_7/y_3^1) \succ (q/q).
\]

Consider now a modular lattice \( L \) with an \(M_3\)-element \( x \) of type \( \geq (3, 2) \). Say there are braces between \((x_1/x_0)\) and \((x_2/x_0)\), and between \((x_2/x_0)\) and \((x_3/x_0)\). Let \((J(L), A)\) be an arbitrary base and pick the points \( p \) and \( q \) of \( g_x \in A \) which belong to the quotients \((x_1/x_0)\) and \((x_2/x_0)\). We may assume that one of them is the middlepoint of \( g_x \) (otherwise pick \( p \) and \( q \) corresponding to the quotients \((x_2/x_0)\) and \((x_3/x_0)\) of the other brace). Thus, to create a nonregular cycle, it suffices to show that there is a path \((p, s_1, \ldots, s_n, q)\) in \((J(L), A)\) with all \([p, s_1], \ldots, [s_n, q]\) distinct from \( g_x \). Putting \( r_0 := p, r_{k+1} := q \), it is enough to check that \( r_{i-1} \) and \( r_i \) from above \((1 \leq i \leq k + 1)\) are connected by a path not using \( g_x \). Consider the line \( g_i := \{t_1, t_2, t_3\}\) of \( A \) which belongs to \( w^i \). Clearly \( g_i \neq g_x \) since \( w^i \neq x \). By (19) one has \((r_{i-1}/r_i) \succ (w^i/w_1^i) \succ (r_i/r_i)\), whence (say) \( r_{i-1}, t_1 \in J(w_1^i/w_0^i) \) and \( t_2, r_i \in J(w_2^i/w_0^i) \). Since \( r_{i-1} \) and \( t_1 \) are in the same subdirect component of the lattice \( w_1^i/0 \) they are by (11) in the same connected

Fig. 11.
component of the base \((J(w'_1), A(w'_1))\). A path in \((J(w'_1), A(w'_1))\) between \(r_{i-1}\) and \(t_i\) cannot involve \(g_x\) since \(x \not\in w'_1\). Similarly there is a path between \(t_2\) and \(r_i\) not involving \(g_x\). Concatenating these two paths with \(g_i = \{t_1, t_2, t_3\}\) yields an appropriate path between \(r_{i-1}\) and \(r_i\).

The converse of (a) and a partial converse of (b) hold as well. To establish this fact, it is necessary to have a closer look at the \(M_3\)-elements in \(E(a/b) = \{x \in E(L) | x \leq a, x \not\leq b\}\) where \(a/b\) is a prime quotient of a modular lattice \(L\) of order 2.

By (13) \(E(a/b)\) is non-empty iff \(|J(a/b)| \geq 2\). In the sequel, for \(x \in E(a/b)\), let \(x_3 := bx\) always denote the unique lower cover of \(x\) below \(b\) (similarly for \(y, z, \ldots\)). Note that \(x, y \in E(a/b), x > y\), implies already \(x \succ y\) (since \(y \not\leq x_0\)), and that \((a/b) \succ x\) for all \(x \in E(a/b)\). If \(L\) is moreover 2-distributive then, as seen in Section 6, \(E(a/b)\) contains no cycles of \(M_3\)-elements. It is not difficult to derive a more specific description of \(E(a/b)\). Consider noncomparable, \(x, y \in E(a/b)\). In order to show that \(w := x + y\) is an \(M_3\)-element we cannot apply Lemma 17 directly since \(x\) and \(y\) need not have a common lower transpose! Instead pick \(p_1 \in J(x_1/x_0)\) and \(q_1 \in J(y_1/y_0)\) \((i = 1, 2)\). Then \(p_1 \neq q_1\) by Lemma 18(b). Thus \(w := p_1 + q_1\) is an \(M_3\)-element by (13). Now \(w\) is not below \(x = p_1 + p_2\) since, again by Lemma 18(b), \(q_1\) cannot belong to \(y\) and \(x\). Thus \(w > x\) by (15). Analogously \(w > y = q_1 + q_2\), i.e. \(w = x + y\). Therefore, if \(E(a/b) \neq \emptyset\), we may denote by \(M_3(a/b)\) the unique maximal \(M_3\)-element \(w\) in \(E(a/b)\). Consider iteratively \(x := M_3(w_1/w_0), y := M_3(w_2/w_0), u := M_3(x_1/x_0), \ldots\) and so on. It follows that any non-empty \(E(a/b)\) has the structure of a rooted tree with respect to the relation \(>\). Each node has at most two sons and all leaves are of type \(\leq 1\) (there may also be other nodes of type 1).

**Lemma 22.** Let \(L\) be a 2-distributive modular lattice of order 2.

(i) If all \(M_3\)-elements have type \(\leq (2.1)\) then there is a mpi base of lines \((J(L), A)\).

(ii) Suppose that all \(M_3\)-elements have type \(\leq (3.1)\). Moreover assume:

\(20)\ For all braces \((x_2/x_0) \succ u^1 \sim \cdots \sim u^m \prec (x_\varphi/x_0)\) of \(L\) and all type 3 elements \(u^i\) one has \(u^i \not\leq x\).

Then there is a regular base of lines \((J(L), A)\).

**Proof.** The proof to be given for (ii) will settle (i) as a special case. Fix a maximal chain \(0 < a_1 < a_2 < \cdots < a_n = 1\) in \(L\). We shall construct a regular base \((J(L), A)\) via a series of regular bases \((J(a_i), A(a_i))\) \((1 \leq i \leq n)\). Given a prime quotient \((a/b) = (a_{i+1}/a_i)\), assume there is a regular base \((J(b), A(b))\) of \(b/0\). For \(E(a/b) = \emptyset\) one has \(J(a/b) = \{p\}\) and obtains a regular base \((J(a), A(a))\) of \(a/0\) (where \(A(a) := A(b)\)). For \(E(a/b) \neq \emptyset\) we are going to choose a line \(g_x\) for each \(x \in E(a/b)\) such that \((J(a), A(a)) (A(a) := A(b) \cup \{g_x | x \in E(a/b)\})\) turns out to be a regular base of \(a/0\). Treat the elements of \(E(a/b)\) in an order compatible with \(\leq\). Then, when \(x \in E(a/b)\) occurs, all nodes \(y < x\) have been treated already, but no nodes \(y > x\) have been treated yet. Suppose that \((J, A) (J \supseteq J(b), A \supseteq A(b))\) is the regular space obtained so far (before treating \(x\)). We shall carry over the following three additional properties of \((J, A)\).
(21) (i) The isolated points of \((J, A)\) are contained in \(J(b)\).
(ii) If the middlepoint of a line \(g \in A\) is incident with other lines of \(A\), then \(z_g\) is of type 3.
(iii) If \(g \in A\) is such that \(z := z_g\) is of type \((3.1)\) with braces between \((z_g/z_0)\) and \((z_+/z_0)\), then the endpoints of \(g\) belong to \(J(z_g/z_0)\) and \(J(z_+/z_0)\).

It suffices to extend \((J, A)\) to a regular space \((J', A')\) with \((21)\) which contains a line \(g_x\) belonging to \(x\). This will be done by a case distinction according to the type of \(x\).

Let us put in front the following fact which will be frequently used.

(22) Let \(i = 1\) or \(i = 2\). If \((x_i/x_0)\) does not transpose downwards, then the unique point \(p \in J(x_i/x_0)\) is not contained in \((J, A)\). If \((x_3/x_0)\) does not transpose downwards then the unique point \(r \in J(x_3/x_0)\) is an isolated point of \((J, A)\).

Proof of (22). The cases \(i = 1\) and \(i = 2\) being symmetric, assume that \((x_1/x_0)\) does not transpose downwards. Since \(p \not\in J(b)\), the assumption \(p \in J\) forces by \((21)\) (i) that \(p \in g_x\) for some line \(g_x \in A - A(b)\). By the order of treatment \(y > x\) is impossible. If \(y < x\) then \((x_3/x_0) \nless y\) since \((x_1/x_0)\) does not transpose downwards; but this yields the contradiction \(p \lneq x_1 y \lneq x_3 = x_0\). Also \(y\) cannot be incomparable to \(x\) since \(p \not\in J(\frac{y_1}{y_0})\) would contradict Lemma 18(b). Thus \(p \not\in J\). Concerning the second claim, assume to the contrary that \(r \in J\) for some line \(g_x \in A\). Again \(y > x\) is impossible by the order of treatment. If \(y < x\) then \((x_i/x_0) \nless y\) (since \(r \not\in x_0\) implies \(y \not\in x_0\)). Necessarily \(i \in \{1, 2\}\) since \((x_3/x_0)\) does not transpose downwards; but this yields the contradiction \(r \lneq x_3 \leq x_3 x_i = x_0\). Also \(y\) cannot be incomparable to \(x\) since \(r \not\in J(\frac{y_1}{y_0})\) would by Lemma 18(a) yield a \(M_3\)-element \(z\) with \((x_3/x_0) \nless z\). Thus \(r\) must be an isolated point of \((J, A)\).

First case: \(x\) has type 0. Then \(J(x_1/x_0) = \{p\}, J(x_2/x_0) = \{q\}, J(x_3/x_0) = \{r\}\). By \((22)\) the points \(p, q\) are not in \((J, A)\) and \(r\) is an isolated point of \((J, A)\). Putting \(g_x := \{p, q, r\}\) and choosing an arbitrary middlepoint for \(g_x\) yields a regular space \((J', A')\) \((J' := J \cup \{p, q\}, A' := A \cup \{g_x\})\) which still satisfies \((21)\).

Second case: \(x\) is of type 1. Thus precisely one lower quotient \((x_i/x_0)\) transposes downwards.

Subcase (a): \(i = 3\). Then \((x_3/x_0) \nless (y/y_1)\) for some \(y \in E(b)\). Say \(\{r_1, r_2, r_3\}\) is the line \(g_x \in A\) (where \(r_1 \in J(\frac{y_1}{y_0})\)). Clearly \(r_2, r_3 \in J(x_3/x_0)\), so at least one of them is an endpoint of \(g_x\), say \(r = r_2\). By \((22)\) the unique points \(p \in J(x_1/x_0)\) and \(q \in J(x_2/x_0)\) are not in \((J, A)\). Put \(g_x := \{p, q, r\}\) and choose \(p\) (or \(q\)) as its middlepoint. Then \((J', A')\) \((J' := J \cup \{p, q\}, A' := A \cup \{g_x\})\) is a regular space. What about \((21)\) (ii)? Is it not possible that \(g_x\) intersects some line \(g \in A\) in its middlepoint \(r\)? This can happen, but then \(x_0\) is necessarily of type 3 (apply \((21)\)(ii) to \((J, A)\) and the lines \(g, g_x\)). Thus \((21)\) remains valid in \((J', A')\).

Subcase (b): \(i \leq 2\), say \(i = 1\). Then \((x_1/x_0) \nless (y/y_3)\) for some \(y \in E(a/b)\) treated already. As before there is an endpoint \(p\) of \(g_x \in A\) with \(p \in J(x_1/x_0)\). By \((22)\) the unique point \(q \in J(x_2/x_0)\) is not in \((J, A)\), and the unique point \(r \in J(x_3/x_0)\) is an isolated point of \((J, A)\). Put \(g_x := \{p, q, r\}\) and choose \(q\) (or \(r\)) as its middlepoint. Then \((J', A')\) \((J' := J \cup \{q\}, A' := A \cup \{g_x\})\) is a regular space, and \((21)\) carries over by the same reason as above.
Third case: x is of type 2. Thus precisely two lower quotients \((x_i/x_0)\), \((x_j/x_0)\) transpose downwards.

Subcase (a): \(\{i, j\} = \{1, 2\}\). Then \((x_1/x_0) \succ y\) and \((x_2/x_0) \succ z\) for distinct \(y, z \in E(a/b)\) which have been treated already. Again we may pick endpoints \(p\) and \(q\) of \(g_x\) respectively \(g_z\) with \(p \in J(x_1/x_0)\) and \(q \in J(x_2/x_0)\). By (22) the unique point \(r \in J(x_3/x_0)\) is an isolated point of \((J, A)\). If \(p\) and \(q\) happen to lie in distinct components of \((J, A)\) then, for \(g_x := \{p, q, r\}\) with midpoint \(r\), the space \((J', A')\) \((J' := J \cup \{r\}, A' := A \cup \{g_x\})\) is obviously still regular. However, this construction might a priori produce a nonregular cycle in \((J', A')\) if \(p\) and \(q\) lie in the same component of \((J, A)\).

Namely, a nonregular cycle occurs if there is a nonregular path \((p, \ldots, q)\) in \((J, A)\). We shall show that this is impossible. Assume to the contrary that \((p_1, \ldots, p_n)\) \((p_1 := p, p_n := q)\) is nonregular. Then at least one \(p_i\) \((1 \leq i \leq n)\) is the midpoint of either \([p_{i-1}, p_i]\) or \([p_i, p_{i+1}]\), say the latter. By (21)(ii) the \(M_3\)-element \(x^i\) corresponding to \(g := [p_i, p_{i+1}]\) is of type 3. First case: \(x^i \neq x^j\) for all other \(M_3\)-elements \(x^j\) \((1 \leq j \leq n, j \neq i)\) induced by the cycle \((p_1, \ldots, p_n, p_i)\) of \((J', A')\). Then, since always either \(x^i \sim x^{i+1}\) or \(x^i \succ u \succ x^{i+1}\) for some \(M_3\)-element \(u\) (see Lemma 18(a)), there is a brace \((x_i/x_0) \succ u_1 \cdots \succ u_n \succ (x_i/x_0)\). By definition, a brace cannot occur if \(x^i\) has type (3.0). But also type (3.1) is impossible: Since \(p_i \in J(x_i/x_0), p_{i+1} \in J(x_{i+1}/x_0)\), and \(p_i\) is the midpoint of the line \(g = [p_i, p_{i+1}] \in A\) this assumption would contradict property (21)(iii). Second case: There is a \(x^i \equiv x^j\). Choose \(x^i\) maximal, i.e. \(x^j \prec x^i\) for all \(1 \leq k \leq n\). Then, as in the first case, there is a brace \((x_i/x_0) \succ \cdots \succ x^i \succ (x_i/x_0)\). But this contradicts (20) since \(x^i \succ x^j\). It follows that \((J', A')\) is always a regular space with (21), even if \(p\) and \(q\) lie in the same component of \((J, A)\).

Subcase (b): \(\{i, j\} \neq \{1, 2\}\), say \(\{i, j\} = \{1, 3\}\). Then \((x_1/x_0) \succ y\) for some \(y \in E(a/b)\) and \((x_3/x_0) \succ z\) for some \(z \in E(b)\). As before choose endpoints \(p\) and \(r\) of \(g_z\) respectively \(g_x\) with \(p \in J(x_1/x_0)\) and \(r \in J(x_3/x_0)\). By (22) the unique point \(q \in J(x_2/x_0)\) is not in \((J, A)\). Put \(g_x := \{p, q, r\}\) with midpoint \(q\). As in (a) it follows that \((J', A')\) \((J' := J \cup \{q\}, A' := A \cup \{g_x\})\) is a regular space with (21).

Fourth case: x is of type 3. By assumption x is of type (3.0) or (3.1). Choose arbitrary lower quotients \((x_i/x_0)\), \((x_j/x_0)\) if the type is (3.0), and choose them as the 'hooks' of the braces if the type is (3.1).

Subcase (a): \(\{i, j\} = \{1, 2\}\). Then \((x_1/x_0) \succ y\) and \((x_2/x_0) \succ z\) for distinct \(y, z \in E(a/b)\), and \((x_3/x_0) \succ w\) for some \(w \in E(b)\). Choose endpoints \(p, q, r\) of \(g_y, g_x, g_w\) respectively such that \(p \in J(x_1/x_0), q \in J(x_2/x_0), r \in J(x_3/x_0)\). Now the component of \((J, A)\) which contains \(r\) is distinct from the component of \(p\) and the component of \(q\); otherwise there were braces between \((x_3/x_0)\) and \((x_1/x_0)\) \((i = 1 \text{ or } 2)\), contradicting type(x)(3.1). If \(x\) has type (3.0) then also \(p\) and \(q\) lie in distinct components. But if \(x\) has type (3.1) then \(p\) and \(q\) might lie in the same component. However, each path \((p, \ldots, q)\) in \((J, A)\) is regular. Otherwise, as in the third case, this would contradict assumption (20). Putting \(g_x := \{p, q, r\}\), with midpoint \(r\), it follows that \((J', A')\) \((J' := J \cup \{q\}, A' := A \cup \{g_x\})\) is a regular space. Besides (21)(ii) also (21)(ii) remains true: The space \((J', A')\) has one more line than \((J, A)\) whose midpoint is not isolated, namely \(g_x\) with \(x\) of type 3. Finally, by the choice of \(x_i, x_j < x\), (21)(iii) is satisfied for \(g_x\) if \(x\) has type (3.1).
Subcase (b): \( \{i,j\} \neq \{1,2\} \), say \( \{i,j\} = \{1,3\} \). This case is completely analogous to (a) (here the point belonging to \((x_2/x_0)\) is the middlepoint of \(g_k\)).

We have shown that assumption (ii) guarantees a regular base \((J(L), A)\), and it is clear from the proof that \((J(L), A)\) is a mpi space if all \(M_3\)-elements have type \(\leq (2.1)\). \(\Box\)

Observe that even lattices with \(M_3\)-elements only of type \(\leq (2.1)\) can have non-regular bases, so a careful choice of the base is really necessary! The regular base \((J(L), A)\) constructed in Lemma 22 is generally not unique. There are many ways to choose a maximal chain \(0 < a_1 < a_2 < \cdots < a_n = 1\) and one can also choose the middlepoints at ease in the first and second case. Note that a suitable choice of these two parameters can be decisive for getting (all the same) regular bases, when condition (20) is violated.

8. Sufficient conditions for cp embeddability of modular lattices into partition lattices

Distributive lattices \(L\) are cp embeddable into partition lattices by a simple direct argument: Any \(n = \delta(L)\) independent atoms of \(\text{Part}(n+1)\) generate a covering Boolean sublattice which itself contains an isomorphic copy of \(L\) as a covering sublattice. For modular lattices the problem is more challenging. By Theorem 6 a modular lattice \(L\) which is cp embeddable into a partition lattice is necessarily 2-distributive and of order 2. Our main result is the following partial converse.

**Theorem 23.** Let \(L\) be a modular 2-distributive lattice of order 2. If \(L\) admits a quasiregular base of lines, then \(L\) is cover preserving embeddable into a partition lattice.

**Proof.** Let \((J(L), A)\) be a quasiregular base of lines of \(L\). By Theorem 7 it suffices to construct a graphic matroid \((J(L), -)\) on the join irreducibles \(J(L)\) which satisfies (1) and (2). Let \(G = (V, J(L))\) be the graph associated to \((J(L), A)\) (see Section 5). Its cycle matroid \((J(L), -)\) satisfies (1) by Lemma 15(a) and Lemma 19. Also it satisfies (2) by Lemma 10, Lemma 15(b) and Lemma 20. \(\Box\)

Let us embed concretely the lattice \(L_2\) of Fig. 5(a). The base of lines \((J(L_2), A)\) drawn in Fig. 5(b) happens to be quasiregular. This is indicated in Fig. 12: \((J, A)\) consists of one connected component. It is a tree of the quasi mpi spaces \((J^1, A^1)\), \((J^2, A^2)\), \((J^3, A^3)\). Only \((H', A') := (J^2, A^2)\) is a proper quasi mpi space. The underlying enricicable mpi space \((H, A)\) is a strong tree of two mpi spaces \((H^1, A^1)\) and \((H^2, A^2)\). They are joined by just one line \(g_{q,1} := \{53, 55, 24\} \ (q = 53)\). Moreover, Fig. 12 shows how the edges of the associated graph \(G := (V, J(L))\) \((V := \{a, b, c, d, e, f, g, h, i, j, k, l\})\) correspond to the points of \((J(L_2), A)\). The graph \(G\) itself is depicted in Fig. 13(a). Its cycle matroid satisfies (1) and (2). The blocks of the partition assigned to \(a \in L_2\) are the vertex sets of the connected components of the subgraph \(G' := (V, J(a))\) (see Theorem 7). For example, the subgraph \(G' := (V, J(43))\) is shown in Fig. 13(b). Table 1 lists all 71 partitions obtained in that way.
Observe that the $M_3$-element $68 \in L_2$ is of type (3.3). Moreover, all its braces are downwards braces, and the braces $(61/54) \vee 61 \sim 34 \sim 56 \sim 64 \wedge (64/54)$ respectively $(62/54) \vee 62 \sim 40 \sim 64 \wedge (64/54)$ have a smallest element 34 respectively 40. Thus the $M_3$-element 68 is 'almost' of type (3.3s). This indicates that the exclusion of type (3.3s) $M_3$-elements in Theorem 6 might be close to a sufficient embeddability condition. A bolder conjecture would be, that the exclusion of type (3.3s) $M_3$-elements even implies the existence of a quasiregular base! Unfortunately, it seems difficult to relate the existence of quasiregular bases to the 'geometry' of $M_3$-elements. At least something can be said about the more restrictive regular bases.

Namely, by Lemma 21(b), a 2-distributive modular lattice with a regular base can only have $M_3$-elements of type $\leqslant (3.1)$. It seems that in many cases this condition is also sufficient for regular bases (cf. the remarks at the end of Section 7). However, to
make a precise statement, one has to add condition (20) of Lemma 22(h). Together with Theorem 23 one then derives the following.

**Corollary 24.** Let \( L \) be a 2-distributive modular lattice of order 2. If all \( M_3 \)-elements have type \( \leq (3.1) \) and if condition (20) is satisfied, then \( L \) is cover preserving embeddable into a partition lattice. Condition (20) is in particular satisfied if no \( M_3 \)-element \( x \) of type 3 is below an \( M_3 \)-element \( y \) of type (2.1) or (3.1).

Corollary 24 comprises two interesting special cases.

**Corollary 25.** Let \( L \) be a 2-distributive modular lattice of order 2. If \( L \) has no \( M_3 \)-elements of type 3, then \( L \) is cover preserving embeddable into a partition lattice. This condition is in particular satisfied if each \( M_3 \)-element of \( L \) has at least one join irreducible lower cover.

**Proof.** The first claim is clear from Corollary 24. Note that by Lemma 22(a) there even is a mpi base of lines. If each \( M_3 \)-element \( x \) has an irreducible lower cover \( x_1 < x \) then \((x_1/x_0)\) cannot transpose downwards. So all \( M_3 \)-elements have type \( \leq (2.1) \).

Observe that canceling the doubly irreducible \( 53 \in L_2 \) yields a lattice all of whose \( M_3 \)-elements have type \( \leq (2.1) \) (but the \( M_3 \)-elements 34, 64, 68 have no irreducible lower covers).

| Table 1 |
|---|---|---|
| 1 = (a, b, c, d, e, f, g, h, i, j, k, l) | 25 = (abcd, e, f, g, h, i, j, k, l) | 49 = (acde, f, g, h, i, j, k, l) |
| 2 = (ad, b, c, e, f, g, h, i, j, k, l) | 26 = (abcdef, e, f, g, h, i, j, k, l) | 50 = (acdef, g, h, i, j, k, l) |
| 3 = (a, b, c, d, e, f, g, h, i, j, k, l) | 27 = (abcd, e, f, g, h, i, j, k, l) | 51 = (abcdf, e, f, g, h, i, j, k, l) |
| 4 = (a, b, c, d, e, f, g, h, i, j, k, l) | 28 = (abcd, e, f, g, h, i, j, k, l) | 52 = (abcdf, e, f, g, h, i, j, k, l) |
| 5 = (ac, b, d, e, f, g, h, i, j, k, l) | 29 = (abcd, e, f, g, h, i, j, k, l) | 53 = (abcdf, e, f, g, h, i, j, k, l) |
| 6 = (ab, d, e, f, g, h, i, j, k, l) | 30 = (abcdef, e, f, g, h, i, j, k, l) | 54 = (abcdf, e, f, g, h, i, j, k, l) |
| 7 = (acd, b, c, e, f, g, h, i, j, k, l) | 31 = (abcdef, e, f, g, h, i, j, k, l) | 55 = (abcdf, e, f, g, h, i, j, k, l) |
| 8 = (a, b, c, d, e, f, g, h, i, j, k, l) | 32 = (abed, e, f, g, h, i, j, k, l) | 56 = (abcdf, e, f, g, h, i, j, k, l) |
| 9 = (ac, b, c, d, e, f, g, h, i, j, k, l) | 33 = (abed, e, f, g, h, i, j, k, l) | 57 = (abcdf, e, f, g, h, i, j, k, l) |
| 10 = (abcd, e, f, g, h, i, j, k, l) | 34 = (abed, e, f, g, h, i, j, k, l) | 58 = (abcdf, e, f, g, h, i, j, k, l) |
| 11 = (acd, b, c, d, e, f, g, h, i, j, k, l) | 35 = (abed, e, f, g, h, i, j, k, l) | 59 = (abcdf, e, f, g, h, i, j, k, l) |
| 12 = (acd, b, c, d, e, f, g, h, i, j, k, l) | 36 = (abed, e, f, g, h, i, j, k, l) | 60 = (abcdf, e, f, g, h, i, j, k, l) |
| 13 = (ac, b, c, d, e, f, g, h, i, j, k, l) | 37 = (abed, e, f, g, h, i, j, k, l) | 61 = (abcdf, e, f, g, h, i, j, k, l) |
| 14 = (acid, b, c, e, f, g, h, i, j, k, l) | 38 = (abed, e, f, g, h, i, j, k, l) | 62 = (abcdf, e, f, g, h, i, j, k, l) |
| 15 = (ace, b, c, e, f, g, h, i, j, k, l) | 39 = (abed, e, f, g, h, i, j, k, l) | 63 = (abcdf, e, f, g, h, i, j, k, l) |
| 16 = (acd, b, c, d, e, f, g, h, i, j, k, l) | 40 = (abed, e, f, g, h, i, j, k, l) | 64 = (abcdf, e, f, g, h, i, j, k, l) |
| 17 = (abd, c, e, f, g, h, i, j, k, l) | 41 = (abed, e, f, g, h, i, j, k, l) | 65 = (abcdf, e, f, g, h, i, j, k, l) |
| 18 = (ad, bc, c, d, e, f, g, h, i, j, k, l) | 42 = (abed, e, f, g, h, i, j, k, l) | 66 = (abcdf, e, f, g, h, i, j, k, l) |
| 19 = (abcd, c, d, e, f, g, h, i, j, k, l) | 43 = (abed, e, f, g, h, i, j, k, l) | 67 = (abcdf, e, f, g, h, i, j, k, l) |
| 20 = (abc, d, c, e, f, g, h, i, j, k, l) | 44 = (abed, e, f, g, h, i, j, k, l) | 68 = (abcdf, e, f, g, h, i, j, k, l) |
| 21 = (abcd, c, d, e, f, g, h, i, j, k, l) | 45 = (abed, e, f, g, h, i, j, k, l) | 69 = (abcdf, e, f, g, h, i, j, k, l) |
| 22 = (acd, b, c, d, e, f, g, h, i, j, k, l) | 46 = (abed, e, f, g, h, i, j, k, l) | 70 = (abcdf, e, f, g, h, i, j, k, l) |
| 23 = (acd, b, c, d, e, f, g, h, i, j, k, l) | 47 = (abed, e, f, g, h, i, j, k, l) | 71 = (abcdf, e, f, g, h, i, j, k, l) |
| 24 = (abd, c, e, f, g, h, i, j, k, l) | 48 = (abed, e, f, g, h, i, j, k, l) | 72 = (abcdf, e, f, g, h, i, j, k, l) |

Cover preserving embedding of modular lattices
Corollary 26. Let $L$ be a 2-distributive modular lattice of order 2. Assume that $L$ is acyclic, i.e. admits an acyclic base of lines. Then $L$ is cover preserving embeddable into a partition lattice.

Proof. An acyclic base $(J(L), A)$ is regular, so the claim follows directly from Theorem 23. □

By the remarks following Lemma 19, in a 2-distributive modular lattice $L$ the acyclicity of one base is equivalent to the acyclicity of all bases, and also to the non-existence of cycles of essential elements. If $L$ is moreover of order 2, this amounts to say that all $M_3$-elements are of type 0, 1, (2.0), or (3.0). In this form Corollary 26 is an immediate special case of Corollary 24.

Corollary 27. There are modular lattices $L$ which are cover preserving embeddable into a partition lattice, but whose dual $L^*$ is not.

Proof. Recall that by Theorem 6 the lattice $L_1$ of Fig. 1 is not cp partition embeddable since its unit element is of type (3.3). However, all $M_3$-elements of the dual lattice $L_1^*$ happen to have type $\leq (2.1)$. Whence $L_1^*$ is cp partition embeddable by Corollary 25. □

9. Cp partition representability versus cp $k$-linear representability

We first survey the main facts about $k$-linear representations of modular lattices and then compare them with our results about cover preserving partition representations.

Let $k$ be an arbitrary field. A $k$-linear representation of a modular lattice $L$ is a nonconstant homomorphism $\phi : L \rightarrow L(k^*)$. Two $k$-linear representations $\phi : L \rightarrow L(k^*)$ and $\phi' : L \rightarrow L(k^*)$ are isomorphic if there is a vector space automorphism $f : k^* \rightarrow k^*$ such that $\phi'(a) = f(\phi(a))$ for all $a \in L$. A $k$-linear representation $\phi : L \rightarrow L(k^*)$ is decomposable if there is a proper decomposition $k^* = V_1 \oplus V_2$ such that $\phi(a) = (\phi(a) \cap V_1) \oplus (\phi(a) \cap V_2)$ for all $a \in L$. A modular lattice $L$ is $k$-linear if there is an injective $k$-linear representation $\phi : L \rightarrow L(k^*)$.

Let us recall the classical representability results for complemented (i.e. geometric) modular lattices. For such lattices the concepts 'simple', 'subdirectly irreducible' and 'directly irreducible' coincide. The simple complemented modular lattices $L$ with $\delta(L) \geq 3$ are precisely the subspace lattices of nondegenerate projective spaces. It follows from the coordinatization of projective spaces satisfying Desargue's law that each simple complemented modular lattice $L$ with $\delta(L) \geq 4$ is isomorphic to $L(k^*)$ for some uniquely determined field $k$. Moreover, for any $K$-linear representation $\phi : L \rightarrow L(K^*)$ one has $\text{char}(K) = \text{char}(k)$.

Which modular lattices $L$ are $k$-linear for all fields $k$? By the above, such a lattice $L$ does not contain as a sublattice the subspace lattice of a nondegenerate projective plane. Hence $L$ must be 2-distributive by (4). In a remarkable paper Jónsson and
Nation [10] showed that conversely each 2-distributive modular lattice $L$ is (cp) $k$-linear for any field $k$ with $|k| > |L|$. In [9] this result was improved to the following. Each 2-distributive modular lattice $L$ is $k$-linear over any field $k$, and cp $k$-linear for $|k| > o(L) := \text{order of } L$. Moreover, such a cp representation $\phi : L \rightarrow L(k^n)$ can be easily obtained from any base of lines $(J(L), A)$.

Namely, by Lemma 5 it suffices to construct a $k$-linear matroid $\{e_p \mid p \in J(L)\}$ which satisfies (1) and (2). Fix a coatom $a$ of $I$ and assume by induction that such a matroid exists for $J(a)$. Now consider the localization $(J(1/a), A(1/a))$ of the base $(J(L), A)$. By (14) and (16) this space is connected and acyclic (!), so one is tempted to extend the matroid $\{e_p \mid p \in J(1/a)\}$ in the following way (take Fig. 7 as a prototype): Assign an arbitrary vector $e_i \in L(k^n) - \langle e_p \mid p \in J(1/a) \rangle$ to the join irreducible $1 \in J(1/a)$. Since $|k| > o(L)$ one can choose pairwise independent vectors $e_2, e_3, e_4 \in \langle e_1, e_{13} \rangle - \{e_1, e_{13}\}$. Analogously, choose $e_5, e_6 \in \langle e_1, e_{15} \rangle - \{e_1, e_{15}\}$ respectively $e_8, e_9 \in \langle e_1, e_{16} \rangle - \{e_1, e_{16}\}$. Continue, e.g., by choosing a vector $e_7 \in \langle e_6, e_{13} \rangle - \{e_6, e_{13}\}$, then pairwise independent vectors $e_{10}, e_{11}, e_{12} \in \langle e_9, e_{14} \rangle - \{e_9, e_{14}\}$, etc. In this way one clearly obtains a $k$-linear matroid $\{e_p \mid p \in J(L)\}$ which satisfies (1). However, showing (2) is nontrivial and amounts essentially to the (nonmatroidal) proof of [9, Theorem 5.1]. Note that different bases $(J(L), A)$ can yield non-isomorphic matroids $(J(L), -)$. Also observe that this induction over prime quotients does not yield appropriate graphic matroids for lattices of order 2 (otherwise this paper would be shorter!).

Which modular lattices $L$ are uniquely $k$-linear for all fields $k$? Here we call a $k$-linear lattice $I$ uniquely $k$-linear if up to isomorphism there are precisely $s(L)$ indecomposable $k$-linear representations $\phi_i : L \rightarrow L(k^n)$ (such that $\ker(\phi_i) = \theta_i$, where the $\theta_i$, $1 \leq i \leq s(L)$, are the maximal congruences of $L$). It is shown in [9, Corollaries 4.3, 5.4] that the lattices which are uniquely $k$-linear for each field $k$ are precisely the acyclic modular lattices of order 2 (shortly called ‘acyclic’ in [9]). Such lattices are necessarily 2-distributive, i.e. the assumption of 2-distributivity in Corollary 26 is actually redundant. Each modular lattice satisfies the inequality $|J(L)| \geq 2\delta(L) - s(L)$ and the acyclic modular lattices are characterizable as those modular lattices for which equality holds [9, Theorem 6.4]. Thus, by Dilworth’s Theorem, the dual of an acyclic lattice is acyclic. It is natural to ask which acyclic lattices admit a base $(J(L), A)$ of form (8) (i.e. a chain). By Lemma 21(a) such a lattice $L$ has no $M_2$-elements of type (3.0), whence all $M_2$-elements are of type 0, 1 or (2.0). However, this condition is generally not sufficient for a base of form (8). See [8, 9, 17] for further interesting properties of acyclic modular lattices.

Observe that for 2-distributive modular lattices $L$ the above inequality $|J(L)| \geq 2\delta(L) - s(L)$ can be sharpened to $|J(L)| \geq 2\delta(L) - s(L) + r^*(L)$, with equality iff $L$ is moreover of order 2. This follows from Lemma 19.

Let $\mathcal{P}$ denote the class of modular lattices which are cover preserving embeddable into a partition lattice. Then $\mathcal{P}$ properly contains the class of acyclic modular lattices (Theorem 23), and is properly contained in the class of all 2-distributive modular lattices of order 2 (Theorem 6). Recall that the cp $k$-linear representability of lattices from $\mathcal{P}$ is established in Corollary 9 without the use of 2-distributivity.
To be concrete, consider, e.g., the lattice $L_1$ of Fig. 1. By the above it is $cp$ embeddable into $L(k^*)$ for each field $k$. But by Theorem 6 there is no $cp$ embedding into a partition lattice since the unit element of $L_1$ is of type (3.3s). Dropping any doubly irreducible $p \in J(L_1)$, the lattice $L_1 - \{p\}$ becomes $cp$ partition embeddable, but is not acyclic. Dropping another doubly irreducible $q \in J(L)$ results in the acyclic lattice $L_1 - \{p, q\}$.

References