Robust $\mathcal{H}_\infty$ Sensor Fault Diagnosis with Neural Network

Marcel Luzar, Marcin Witczak, and Christophe Aubrun*

Institute of Control and Computation Engineering, University of Zielona Góra, ul. Pogórna 50, 65-246 Zielona Góra, Poland
{m.luzar, m.witczak}@issi.uz.zgora.pl

Centre de Recherche en Automatique de Nancy
CRAN-UMR 7039, Nancy-Université, CNRS, F-54506 Vandoeuvre-les-Nancy Cedex, France
christophe.aubrun@cran.uhp-nancy.fr

Abstract. The paper deals with the problem of a robust fault diagnosis for Linear Parameter-Varying (LPV) systems with Recurrent Neural Network (RNN). The preliminary part of the paper describes the derivation of a discrete-time polytopic LPV model with RNN. Subsequently, a robust fault detection, isolation and identification scheme is developed, which is based on the observer and $\mathcal{H}_\infty$ framework for a class of non-linear systems. The proposed approach is designed in such a way that a prescribed disturbance attenuation level is achieved with respect to the sensor fault estimation error while guaranteeing the convergence of the observer.

Keywords: Fault diagnosis, fault identification, robust estimation, non-linear systems, observers, neural-networks

1 Introduction

The problem of fault diagnosis (FD) of non-linear industrial systems [4, 10, 8] has received considerable attention during the last three decades. Indeed, it developed from the art of designing a satisfactory performing systems into the modern theory and practice that it is today. Within the usual framework, the system being diagnosed is divided into three main components, i.e. plant (or system dynamics [14]), actuators and sensors. The paper deals with the problem of full fault diagnosis of sensors, i.e. apart from the usual two steps consisting of fault detection and isolation (FDI), the fault identification is also performed. This last step is especially important from the viewpoint of Fault-Tolerant Control (FTC) [9, 12], which is possible if and only if there is an information about the size of the fault being a result of fault identification (or fault estimation).

In this paper a robust fault estimation approach is proposed, which can be efficiently applied to realise the above-mentioned procedure. The proposed approach can be perceived as a combination of the linear-system strategies [2] and [11] for a class of non-linear systems [13]. The proposed approach is designed
in such a way that a prescribed disturbance attenuation level is achieved with respect to the fault estimation error while guaranteeing the convergence of the observer. The paper is organised as follows. On the beginning the derivation of a discrete-time polytopic LPV model with neural network is described. In section 3 the robust observer design procedure for sensor fault diagnosis is presented. Section 4 shows experimental results obtained with a laboratory multitank model. Finally, Section 5 conclude the paper.

2 Derivation of a discrete-time polytopic LPV model

The goal of this section is to present a neural state-space model that can be represent a general class of state-space models and can be easily transformed into a LPV one. The transformation method is derived by Lachhab et. all and presented in [6].

2.1 Recurrent Neural-Network Topology

Let us consider the following non-linear discrete-time LPV model:

\[
\begin{align*}
x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k, \\
y_{k+1} &= C(\theta_k)x_{k+1},
\end{align*}
\]

where \( A(\theta_k) \), \( B(\theta_k) \), \( C(\theta_k) \) are continuous mappings and \( \theta_k \) is a time-varying parameter. The dependence of the \( A, B, C \) on \( \theta_k \) represents a general LPV model; imposing that any of these matrices is parameter independent, i.e. fixed, will restrict the generality of the LPV model. The latter is indeed the case when RNN is transformed into the LPV model: \( A \) is parameter dependent but \( B \) and \( C \) are fixed. A general form of state-space neural network model is

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + A_1\sigma(E_1x_k) + B_1\sigma(E_2u_k), \\
y_{k+1} &= Cx_{k+1} + C_1\sigma(E_3x_{k+1}),
\end{align*}
\]

where \( x \in \mathbb{R}^n \) denotes the state vector, \( y \in \mathbb{R}^p \) the output and \( u \in \mathbb{R}^m \) the input vector. \( A, A_1, B, B_1, C, C_1, E_1, E_2 \) and \( E_3 \) are real valued matrices of appropriate dimensions and represent the weights which will be adjusted during the training stage of the RNN. The non-linear activation function \( \sigma(\cdot) \), which is applied elementwise in (3)–(4) is taken as a continuous, differentiable and bounded function. This RNN leads to a general form of the neural state-space model in the sense that if it is transformed into an LPV model in the form (1)–(2), the matrices \( A, B, C \) will be parameter dependent.

For stability and identifiability proofs of the proposed RNN reader is refereed to [6].
2.2 Simplification and Assumptions

The Recurrent Neural-Network (RNN) model described by (3)–(4) can be simplified by removing any of the sigmoidal layers. This can be done according to *a priori* information about the identified system. The LPV model (1)–(2) of multitank system, which was derived from a physical laboratory model and used in experiments presented in Section 4, has only matrix $A$ parameter dependent. Thus, a specific transformation proposed by [5] is used to simplify such a model. Removing the sigmoidal layers from the input and the output paths means that the resulting LPV model has parameter dependence in matrix $A$ only. Moreover, for a practical implementation this simplified RNN is modified as shown in Fig. 1: the outputs instead of the states are taken as input to sigmoidal layer. This modification facilitates the implementation of LPV controllers designed based on this model. The modified RNN is represented as

$$x_{k+1} = Ax_k + Bu_k + A_1\sigma(E_1Cx_k),$$

$$y_{k+1} = C x_{k+1}. \tag{5}$$

![Fig. 1. Simplified state-space recurrent neural network](image)

3 Sensor fault diagnosis

Neural model described by (5)–(6) can be modified to following state-space form:

$$x_{k+1} = Ax_k + Bu_k + g(x_k) + W_1w_k,$$

$$y_{k+1} = C x_{k+1} + L_s f_{s,k} + W_2w_{k+1}, \tag{8}$$

where $x_k \in \mathbb{X} \subset \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^r$ stands for the input, $y_k \in \mathbb{R}^m$ denotes the output, $f_{s,k} \in \mathbb{R}^m$ stands for the sensor fault. While $w_k \in l_2$ is an exogenous disturbance vector with $W_1 \in \mathbb{R}^{n \times n}$, $W_2 \in \mathbb{R}^{m \times n}$ being its
distribution matrices while

\[ l_2 = \{ w \in \mathbb{R}^n | \| w \|_{l_2} < +\infty \}, \| w \|_{l_2} = \left( \sum_{k=0}^{\infty} \| w_k \|^2 \right)^{\frac{1}{2}}. \quad (9) \]

The main objective of this section is to provide a detailed design procedure of the robust observer, which can be used for sensor fault diagnosis. In other words, the main role of this observer is to provide the information about the sensor fault. Indeed, apart from serving as a usual residual generator (see, e.g., [14]), the observer should be designed in such a way that a prescribed disturbance attenuation level is achieved with respect to the sensor fault estimation error while guaranteeing the convergence of the observer.

Let us define the matrix \( X \) be partitioned in such a way that

\[
X = \begin{bmatrix}
x_1^T \\
\vdots \\
x_m^T
\end{bmatrix}
\]

(10)

where \( x_j \) stands for the \( j \)th row of \( X \). Let us also denote \( X^j \) as the matrix \( X \) without the \( j \)th row and \( y^j \) as a vector \( y \) without the \( j \)th element.

The sensor fault diagnosis will be realised by a set of \( m \) observers of the form:

\[
\hat{x}_{k+1} = A\hat{x}_k + g(\hat{x}_k) + K_j \left( y^j_k - C^j \hat{x}_k \right), \quad j = 1, \ldots, m, \quad (11)
\]

while the \( j \)th output (for \( L_{s,k} = I \)) is described by

\[
y_{j,k} = c_j^T x_k + w_{2,j}^T w_k + f_{j,k}. \quad (12)
\]

Thus:

\[
f_{j,k} = y_{j,k} - c_j^T x_k - w_{2,j}^T w_k, \quad (13)
\]

and an \( j \)th fault estimate is

\[
\hat{f}_{j,k} = y_{j,k} - c_j^T \hat{x}_k. \quad (14)
\]

The fault estimation error \( \varepsilon_{f_{j,k}} \) of the \( j \)th sensor is

\[
\varepsilon_{f_{j,k}} = f_{j,k} - \hat{f}_{j,k} = -c_j^T x_k + c_j^T \hat{x}_k - w_{2,j}^T w_k = -c_j^T e_k - w_{2,j}^T w_k, \quad (15)
\]

while the state estimation error is:

\[
e_{k+1} = A e_k + s_k - K_j C^j e_k - K_j W_2 w_k + W_1 w_k, \quad (16)
\]

\[
e_{k+1} = (A - K_j C^j) e_k + s_k - W w_k, \quad (17)
\]

\[
e_{k+1} = A_1 e_k + s_k - W w_k. \quad (18)
\]

where

\[
s_k = g(x_k) - g(\hat{x}_k). \quad (19)
\]
Note that both \( e_k \) and \( e_{f,k} \) are non-linear with respect to \( e_k \). To settle this problem within the framework of this paper, the following solution is proposed. Using the Differential Mean Value Theorem (DMVT) [15], it can be shown that

\[
g(a) - g(b) = M_x(a - b),
\]

with

\[
M_x = \begin{bmatrix}
\frac{\partial g_1}{\partial x}(c_1) \\
\vdots \\
\frac{\partial g_n}{\partial x}(c_n)
\end{bmatrix},
\]

where \( c_1, \ldots, c_n \in \text{Co}(a, b), c_i \neq a, c_i \neq b, i = 1, \ldots, n \). Assuming that

\[
\tilde{a}_{i,j} \geq \frac{\partial g_i}{\partial x_j} \geq a_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n,
\]

it is clear that:

\[
M_x = \{ M \in \mathbb{R}^{n \times n} | \tilde{a}_{i,j} \geq m_{x,i,j} \geq a_{i,j}, \ i, j = 1, \ldots, n, \}
\]

Thus, using (20), the term \( A_1 e_k + s_k \) in (18) can be written as

\[
A_1 e_k + s_k = (A + M_{x,k} - K^j C^j) e_k
\]

where \( M_{x,k} \in M_x \).

From (24), it can be deduced that the state estimation error can be converted into an equivalent form

\[
e_{k+1} = A_2(\alpha)e_k - \hat{W}w_k,
\]

\[
A_2(\alpha) = \tilde{A}(\alpha) - K^j C^j,
\]

which defines an LPV polytopic system [1] with

\[
\tilde{\mathcal{A}} = \left\{ \tilde{A}(\alpha) : \tilde{A}(\alpha) = \sum_{i=1}^{N} \alpha_i \tilde{A}_i, \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0 \right\},
\]

where \( N = 2^n \). Note that this is a general description, which does not take into account that some elements of \( M_{x,k} \) maybe constant. In such cases, \( N \) is given by \( N = 2^{n-c} \) where \( c \) stands for the number of constant elements of \( M_{x,k} \).

The problem of \( H_\infty \) observer design [7] is to determine the gain matrix \( K \) such that

\[
\lim_{k \to \infty} e_k = 0 \quad \text{for} \ w_k = 0,
\]

\[
||e_f||_2 \leq \omega ||w||_2 \quad \text{for} \ w_k \neq 0, \ e_0 = 0.
\]
The general framework for designing robust observer is:
\[
\Delta V_k + \varepsilon_{f,k}^T \varepsilon_{f,k} - \mu^2 w_k^T w_k < 0, \quad k = 0, \ldots, \infty,
\]
with
\[
V_k = e_k^T P(\alpha) e_k
\]
Consequently, it can be shown that:
\[
\Delta V_k + \varepsilon_{f,k}^T \varepsilon_{f,k} - \mu^2 w_k^T w_k =
\]
\[
e_k^T (A_2(\alpha)^T P(\alpha) A_2(\alpha)) e_k +
\]
\[
e_k^T (A_2(\alpha)^T P(\alpha) W) w_k +
\]
\[
w_k^T \left( W^T P(\alpha) A_2(\alpha) \right) e_k +
\]
\[
w_k^T \left( W^T P(\alpha) W \right) w_k < 0
\]
By defining
\[
v_k = [e_k^T, w_k^T]^T,
\]
the inequality (31) becomes
\[
\Delta V_k + \varepsilon_{f,k}^T \varepsilon_{f,k} - \mu^2 w_k^T w_k = v_k^T M_V v_k < 0,
\]
where
\[
M_V = \begin{bmatrix}
A_2(\alpha)^T P(\alpha) A_2(\alpha) - P(\alpha) + c_j c_j^T & A_2(\alpha)^T P(\alpha) W + c_j w_{2,j} \\
W^T P(\alpha) A_2(\alpha) + w_{2,j} c_j^T & W^T P(\alpha) W + w_{2,j} w_{2,j} - \mu^2 I
\end{bmatrix}.
\]
The following two lemmas can be perceived as the generalisation of those presented in [1].

**Lemma 1.** The following statements are equivalent
1. There exists \( X \succ 0 \) such that
\[
V^T X V - W \prec 0 \tag{35}
\]
2. There exists \( X \succ 0 \) such that
\[
\begin{bmatrix}
-W & V^T U^T \\
U V & X - U - U^T
\end{bmatrix} \prec 0 \tag{36}
\]
**Proof.** Applying the Schur complement to (2) gives
\[
V^T U^T (U^T + U - X)^{-1} U V - W \prec 0 \tag{37}
\]
Substituting \( U = U^T = X \) yield
\[
V^T X V - W \prec 0 \tag{38}
\]
Thus, (1) implies (2). Multiplying (36) by \( T = \begin{bmatrix} I & V^T \end{bmatrix} \) on the left and by \( T^T \) on the left of (36) gives (35), which means that (2) implies (1) and hence the proof is completed.
Lemma 2. The following statements are equivalent

1. There exists $X(\alpha) > 0$ such that
   \[ V(\alpha)^T X(\alpha) V(\alpha) - W(\alpha) < 0. \tag{39} \]

2. There exists $X(\alpha) > 0$ such that
   \[
   \begin{bmatrix}
   -W(\alpha) & V(\alpha)^T U^T \\
   U V(\alpha) X(\alpha) - U - U^T 
   \end{bmatrix} < 0. \tag{40}
   \]

Proof. The proof can be realised by following the same line of reasoning as the one of Lemma 1.

It is easy to show that that (40) is satisfied if there exist matrices $X_i > 0$ such that
\[
\begin{bmatrix}
-W_i & V_i^T U_i^T \\
U_i V_i X_i - U - U^T 
\end{bmatrix} < 0, \quad i = 1, \ldots, N. \tag{41}
\]

Theorem 1. For a prescribed disturbance attenuation level $\mu > 0$ for the fault estimation error (15), the $H_\infty$ observer design problem for the system (7)–(8) and the observer (11) is solvable if there exists matrices $P_i > 0$ ($i = 1, \ldots, N$), $U$ and $N$ such that the following LMIs are satisfied:
\[
\begin{bmatrix}
-P + c_j c_j^T & c_j w_{2,j}^T & A_{2,i}^T U^T \\
-w_{2,j} c_j & w_{2,j} w_{2,j}^T - \mu^2 I & W_i^T U^T \\
U A_{2,i} & U W & P_i - U - U^T 
\end{bmatrix} < 0. \tag{42}
\]

where
\[
UA_{2,i} = U(\dot{A}_i - KC) = U \dot{A}_i - NC. \tag{43}
\]

Proof. Observing that the matrix (34) must be negative definite and writing it as
\[
\begin{bmatrix}
A_{2}(\alpha)^T \\
W^T 
\end{bmatrix} P(\alpha) \begin{bmatrix}
A_{2}(\alpha)^T \\
W^T 
\end{bmatrix} + \begin{bmatrix}
-P(\alpha) + c_j c_j^T & c_j w_{2,j}^T & A_{2,i}^T U^T \\
w_{2,j} c_j & w_{2,j} w_{2,j}^T - \mu^2 I & W_i^T U^T \\
U A_{2,i} & U W & P_i - U - U^T 
\end{bmatrix} < 0. \tag{44}
\]

and then applying Lemma 2 and (41) leads to (42), which completes the proof.

Finally, the design procedure boils down to solving LMI (42) and then $K = U^{-1} N$.

4 Experimental results

To verify proposed approach, a sensor fault detection was implemented for multitank system. The considered multi-tank system (Fig. 2) is designed for simulating the real industrial multi-tank system in the laboratory conditions [3]. The multi-tank system can be efficiently used to practically verify both linear
and non-linear control, identification and diagnostics methods. The multi-tank system consists of three separate tanks placed each above other and equipped with drain valves and level sensors based on a hydraulic pressure measurement. Each of them has a different cross-section in order to reflect system nonlinearities. The lower bottom tank is a water reservoir for the system. A variable speed water pump is used to fill the upper tank. The water outflows the tanks due to gravity. The considered multi-tank system has been designed to operate with an external, PC-based digital controller. The control computer communicates with the level sensors, valves and a pump by a dedicated I/O board and the power interface. The I/O board is controlled by the real-time software, which operates in a Matlab/Simulink environment.

![Multi-tank system](image)

**Fig. 2.** Multi-tank system

The simplified RNN presented in Section 2 is implemented in Neural Network Time Series Tool (ntstool) provided by MATLAB and trained with classical Levenberg-Marquardt algorithm.

Let the initial condition for the system and the observer be:

\[ x_0 = [0.1, 0.2, 0.3]^T, \quad \hat{x}_0 = \mathbf{0.00001}, \]

while the input and the exogenous disturbance are:

\[ u_k = 1, \quad w_k \sim N(0, 0.01 I). \]

Although measurements from all water level sensors are available, in experiments measurements from two of three sensor were taken into account in order to
simulate insensitivity of the system for third faulty sensor. To present the results of sensor fault diagnosis, the following set of faults scenarios for \( i^{th} \) sensor were introduced:

1. Incipient fault

\[
\begin{align*}
  f_{s_1,k} &= \begin{cases} 
    -0.005(k - 300), & \text{for } 500 \geq k \geq 300, \\
    0, & \text{otherwise,}
  \end{cases} \\
  f_{s_2,k} &= \begin{cases} 
    -0.005(k - 1000), & \text{for } 1500 \geq k \geq 1000, \\
    0, & \text{otherwise,}
  \end{cases} \\
  f_{s_3,k} &= \begin{cases} 
    -0.005(k - 1800), & \text{for } 2100 \geq k \geq 1800, \\
    0, & \text{otherwise,}
  \end{cases}
\end{align*}
\]

2. 20\% decrease in the accuracy

\[
\begin{align*}
  f_{s_1,k} &= \begin{cases} 
    -0.2y_{k}, & \text{for } 500 \geq k \geq 300, \\
    0, & \text{otherwise.}
  \end{cases} \\
  f_{s_2,k} &= \begin{cases} 
    -0.2y_{k}, & \text{for } 1500 \geq k \geq 1000, \\
    0, & \text{otherwise.}
  \end{cases} \\
  f_{s_3,k} &= \begin{cases} 
    -0.2y_{k}, & \text{for } 2100 \geq k \geq 1800, \\
    0, & \text{otherwise.}
  \end{cases}
\end{align*}
\]

Figure 3 presents results of fault identification for each type of fault for the first sensor for the nominal case \((\hat{x}_0 \neq x_0 \text{ and } u_k \neq 0)\). Moreover, figure 6 shows the evolution of \( \|e_k\| \) for first and second sensor which confirms that condition (28) is satisfied.

![Fault identification for incipient and decrease fault in first sensor](image)

**Fig. 3.** Fault identification for incipient and decrease fault in first sensor

Similar results of fault identification were obtained for second and third sensor and shown in figures (4)-(5).
Fig. 4. Fault identification for incipient and decrease fault in second sensor

Fig. 5. Fault identification for incipient and decrease fault in third sensor

Fig. 6. Evolution of $\|e_k\|$ (for $k = 0, \ldots, 20$) for first and second sensor

5 Conclusions

The paper deals with the problem of robust sensor fault estimation with neural networks. In particular, a combination of the celebrated generalised observer
scheme with the robust $H_\infty$ approach is proposed to settle the problem of robust fault diagnosis. The proposed approach is designed in such a way that a prescribed disturbance attenuation level is achieved with respect to the sensor fault estimation error while guaranteeing the convergence of the observer. The final part of the paper is concerned with a comprehensive case study regarding the multi-tank system. The achieved results show the performance of the proposed approach, which confirm its practical usefulness.

References