Abstract—The outage probability limit is a fundamental lower bound on the word error rate of coded communications systems. It is mainly determined by two parameters: the diversity order and the coding gain. With linear precoding, the maximum achievable coding rate yielding full-diversity can exceed the upper limit given by the standard Singleton bound. However, the effect of precoding on the coding gain is not well understood. This paper analyzes linear precoding from an information theoretical point of view and tries to optimize the coding gain. For discrete constellations, it is shown that constellation expansion together with one optimized precoding parameter is sufficient to approach the best outage achieved by a Gaussian alphabet, thus maximizing the coding gain.

I. INTRODUCTION

In many applications, such as frequency-hopping schemes (GSM, EDGE), the channel model is a block-fading (BF) channel [2], where for example due to a delay-constraint, a packet is transmitted through a finite number of channel states, each subject to independent flat fading. Also, the block fading channel model has proved its importance for designing good codes for cooperative communications [10], [13]. It has been proved that error-correcting codes achieve full-diversity on a cooperative relay channel if and only if they have full-diversity on the BF channel with two channel states. Hence, very good codes designed for the BF channel are an important starting point for designing codes for cooperative communications.

The outage probability limit is a lower bound on the average word error rate of coded systems [2], and is mainly determined by two parameters: its diversity gain and coding gain [16]. The maximum diversity order is given by the Singleton bound [12], [9], which can be improved when using linear precoding [10]. Linear precoding modifies the marginal distribution of components of a multidimensional constellation at the channel input. The design of these constellations has been extensively studied for uncoded transmission schemes [4], and few recent works studied coded transmission schemes [10], [8], [13].

It is well understood what is the effect of linear precoding on the diversity order. However, there is no insight in the effect on the coding gain of coded systems. Before designing practical coding schemes, it is important to understand the limits of the communications channel. Even in the simple case of a Gaussian channel input, a closed form expression for the outage probability is not known yet.

In this paper, the authors study linear precoding of discrete constellations from the point of view of outage probability. This analysis may serve as a basis for the design of practical coded systems with linear precoding. The paper is a first attempt in analyzing the effect of linear precoding on the outage probability by establishing simple upper bounds. These bounds are obtained by interpreting the outage probability as the probability that the fading point belongs to an outage volume in a real Euclidean space. For simplicity, the real space defined by the fading coefficients will be referred to as fading space in the remainder of this document.

II. SYSTEM MODEL

The transmitter output is a complex vector \( \mathbf{x} = [x(1), x(1), \ldots, x(B)] \) of length \( N \), where \( x(i,j) \) is the \( j \)-th component of block \( i \). The channel is memoryless with additive white Gaussian noise and multiplicative real fading. The fading coefficients are only known at the decoder side where the received signal vector is

\[
y(b) = \alpha_b \mathbf{x}(b) + \mathbf{w}(b), \quad b = 1, \ldots, B
\]

where \( y(b) = [y(b), y(b), \ldots, y(b)] \) is the received complex signal vector in block \( b \) and \( \mathbf{x}(b) = [x(b), x(b), \ldots, x(b)] \) is the \( b \)-th part of the transmitted vector. The noise vector \( \mathbf{w}(b) \) consists of independent noise samples which are complex Gaussian distributed, i.e., \( \mathbf{w}(b) \sim \mathcal{CN}(0, 2\sigma^2) \), where \( \gamma = \frac{1}{2\sigma^2} \) is the average signal-to-noise ratio. The fading coefficient \( \alpha_b \) is independent and identically distributed (i.i.d.) from block to block. For simplicity, we consider the fading coefficients to be Rayleigh distributed. Collecting the different components of \( \mathbf{x} \), transmitted on each of the fading gains, we obtain the vectors \( \mathbf{x}_n = \begin{bmatrix} x(1)_n, \ldots, x(B)_n \end{bmatrix}^T, \quad n = 1, \ldots, N \), where each vector belongs to a multidimensional constellation in the fading space, which we denote as \( \Omega \). For example, if \( x(i,j) \in M - QAM \), then \( \Omega = (M - QAM)^B \), where the exponent represents a Cartesian product. In presence of linear precoding as studied below, \( \Omega \) is a rotated version of \( (M - QAM)^B \). Assuming that the transmitter includes an error-correcting code of coding rate \( R_c \), the overall spectral efficiency is \( R = R_c \log_2(B) \), \( R_c \) is the coding rate yielding full-diversity on a Gaussian channel input, a closed form expression for the outage probability is not known yet.

The maximum achievable diversity order of this system is
where \( B \) decoding the precoding matrix \( P \) is not at the transmitter with an \( S \times S \) linear precoding at the transmitter with a \( P \) matrix of \( S \): 

\[
d_B(R_c) = 1 + |B(1 - R_c)|. \tag{2}
\]

The constraint on the coding rate \( R_c \) to have a diversity order of \( B \) is \( R_c \leq 1/B \). This bound can be improved by using linear precoding at the transmitter with an \( s \times s \) precoding matrix \( P \). The modified Singleton bound gives the maximum achievable diversity order of the system using linear precoding \[8\]:

\[
d_{BM}(R_c) = \begin{cases} 
  s \left( \frac{B}{s}(1 - R_c) \right) + 1 & \text{if } s \leq B \\
  B & \text{if } s > B
\end{cases} \tag{3}
\]

Here, we take \( s = B \), so that the maximum achievable diversity order of \( B \) is achieved when \( 0 < R_c < 1 \). The transmitted symbol vectors \( x_n \) can now be derived as follows:

\[
x_n = Pz_n, \tag{4}
\]

where \( z(b)_n \) is the \( n \)-th symbol of the \( b \)-th block generated by the modulator. It was shown in \[8\], that under iterative APP decoding the precoding matrix \( P \) has to be unitary. We restrict our study to real constellations, hence \( P \) is an orthogonal matrix. An \( S \times S \) real orthogonal matrix has \( \frac{1}{2}S(S-1) \) degrees of freedom. In this paper, we limit the scheme to two transmission blocks, \( B = 2 \), but any extension to larger \( B \) is possible. When \( s = 2 \), \( P \) is a rotation matrix where the rotation angle \( \theta \) is the degree of freedom. The identity matrix is a special case of \( P \) corresponding to \( \theta = 0 \).

Fig. 1 illustrates the effect of a rotation in the fading space of \( \alpha_1 \) and \( \alpha_2 \) when a BPSK modulator is used. When \( \alpha_1 \neq \alpha_2 \), \( \alpha, \Omega \) can be interpreted as the symbols of an unbalanced QPSK constellation in the fading space, where \( \alpha = [\alpha_1, \alpha_2] \).

III. OUTAGE PROBABILITY

The BF channel has a Shannon capacity that is essentially zero since the fading gain makes the mutual information a random variable which does not allow us to make the word error probability arbitrarily small under a certain spectral efficiency. The probability that mutual information be less than the transmitted rate is called information outage probability, it is denoted by \[15\] section 5.4,

\[
P_{out}(\gamma) = P(I(\alpha, \gamma, \theta) < R),
\]

where \( I(\alpha, \gamma, \theta) \) is the instantaneous mutual information as a function of the fading gains \( \alpha = [\alpha_1, \alpha_2] \), average SNR \( \gamma \), and rotation parameter \( \theta \). The word error probability on a coded BF channel is lower-bounded by \( P_{out} \) in the limit of large block length.

The instantaneous mutual information for a finite discrete constellation \( \Omega \) is given by Eq. \[5\] where the notation introduced in Section \[III\] is used. Note that when \( \theta = 0 \), \( I(\alpha, \gamma, \theta = 0) = 0.5 \times (I(\alpha_1, \gamma) + I(\alpha_2, \gamma)) \), where \( I(\alpha, \gamma) \) is simply the capacity of a point-to-point channel with fading coefficient \( \alpha \).

A closed form expression for \( I(\alpha, \gamma, \theta) \) does not exist and \( I(\alpha, \gamma, \theta) \) is difficult to analyze because of mathematical expectations. Let us interpret this formula. The pdf of the mutual information depends on the probability distribution of \( \alpha_1 \) and \( \alpha_2 \). A good precoding matrix has to minimize the outage probability. One approach could be to choose \( \theta \) such that the mean of the pdf \( \mathbb{E}_{\alpha}[I(\alpha, \gamma, \theta)] \) is maximized, in the hope that this minimizes the area under its tail left. However, the ergodic mutual information \( \mathbb{E}_c[I(\alpha, \gamma, \theta)] \) for any precoding scheme quickly converges to \( m = \log_2 |\Omega| \), due to the limited spectral efficiency of a finite discrete constellation (see Fig. \[2\] for \( BPSK \times BPSK \)). Hence, the precoding matrix cannot be optimized for large \( \gamma \) with this method. Furthermore, all information with respect to the diversity order is lost.

A better approach is to interpret the outage probability in the fading space of \( \alpha_1 \) and \( \alpha_2 \). The outage probability is obtained by integrating the joint probability distribution \( p(\alpha_1, \alpha_2) \) over the volume \( V_o \), where \( V_o \) is the region of fading gains such that \( I(\alpha, \gamma, \theta) < R \). We say that the volume \( V_o \) is limited by an outage boundary \( B_0(V_o) \), defined by

\[
I(\alpha, \gamma, \theta) = R.
\]

Examples of such an outage boundary are given in Fig. \[4\].

Definition 1: We define \( \alpha_0 \) by the intersection between the outage boundary and the axis \( \alpha_2 = 0 \). More precisely \( I(\alpha_1 = \alpha_0, \alpha_2 = 0, \gamma, \theta) = R \). By convention, \( \alpha_0 = +\infty \) if the horizontal axis is an asymptote for the outage boundary.

Definition 2: We define \( \alpha_e \) as the intersection between the
I(α, γ, θ) = \frac{m}{2} - 2^{-m-1} E_{w(1),w(2)} \left( \sum_{x \in \Omega} \log_2 \left( \sum_{x' \in \Omega} \exp \left[ \frac{\sum_{b=1}^2 (-|α_b(x-b') - w(b)|^2 + |w(b)|^2)}{2σ^2} \right] \right) \right) \quad (5)

\begin{align*}
\text{Fig. 3. Examples of an outage boundary in the case of BPSK signaling, rotation angle is } \theta = 0, 10, 30 \text{ degrees, information rate is } R = 0.9 \text{ bpcu.}
\end{align*}

outage boundary and the line \( α_1 = α_2 \) (also known as the ergodic line). More precisely \( I(α_1 = α_2, α_e, γ, θ) = R \).

Definition 3: Outage boundary \( B^1 = B_0(V_0^1) \) is said to be upper bound outage boundary \( B^2 = B_0(V_0^2) \) if \( V_0^2 \subset V_0^1 \). The opposite holds for inner bounding.

The outage boundary and other objects in the fading space have simple interesting properties (without proof):

1) The ergodic line (defined by \( α_1 = α_2 \)) is insensitive to rotations. The constellation \( α, Ω \) forms a balanced QPSK constellation in the fading space, such that a rotation has no effect on the capacity.

2) The outage boundary is symmetric wrt. \( α_1 \) and \( α_2 \).

3) If an outage boundary upper bounds another outage boundary, then its outage probability is larger since \( P_{out}(γ, B_0(V_0)) = \int_{α \in V_0} p(α) dα \).

IV. BOUNDS ON THE OUTAGE PROBABILITY

Proposition 1: The outage boundary \( B_0(\text{gauss}) \) of a BF channel with a Gaussian input alphabet is upper bounded by the circular arc \( α_1^2 + α_2^2 = \frac{16^2-1}{2γ^2} \) and inner bounded by the circular arc \( α_1^2 + α_2^2 = \frac{4n-1}{2γ^2} \).

Proof: The proof is given for \( α_1 ≥ α_2 \), because of property 2 in the previous section.

When symbols \( z(b)_n \) generated by the modulator are independent and Gaussian distributed, then the vector \( z_n \) is circularly symmetric Gaussian, thus its pdf is invariant through a rotation. Hence \( I(α, γ, θ) = I(α, γ, θ) = 0, \) so that \( I(α, γ, θ) = 0.5 \ast (I(α_1, γ) + I(α_2, γ)) \).

Let us consider two points \( (α'_1, α'_2) \) and \( (α_1, α_2) \) in the fading space with the same module, i.e., \( α'_1^2 = α_1^2 + \delta \) and \( α'_2^2 = α_2^2 - \delta \). Consider \( \delta > 0 \). Then, \( 0.5 \ast (I(α_1, γ) + I(α_2, γ)) \leq 0.5 \ast (I(α'_1, γ) + I(α'_2, γ)) \) by Jensens’ inequality, since \( I(α_b, γ) \) is a concave function of the instantaneous SNR \( α_b^2 γ \). This proves that the mutual information is monotonically non-increasing when moving on a circular arc starting from point \( α_1 = α_2 \) on the ergodic line and ending on the horizontal axis \( α_2 = 0 \). Hence, on a circular arc:

- \( I(α_1, α_2) \mid α_1 = α_2 \geq I(α_1, α_2) \mid α_1 ≠ α_2 \)
- \( I(α_1, α_2) \mid α_2 = 0 \leq I(α_1, α_2) \mid α_2 ≠ 0 \)
- \( I(α_1, α_2) \mid α_1 = 0 \leq I(α_1, α_2) \mid α_1 ≠ 0 \)

Calculating \( α_0 \) and \( α_e \), as defined in Definitions 1 & 2 yields the radii of both circular arcs.

The outage boundary \( B_0(\text{gauss}) \) of a channel with Gaussian input alphabet is insensitive to precoding. Also, the outage probability of a channel with a Gaussian input is a lower bound of the outage probability of a channel with a discrete input \( [5] \) since \( B_0(\text{discrete}) \) inner bounds \( B_0(\text{discrete}) \).

Proposition 2: On a BF channel with a discrete input alphabet and without precoding, the outage boundary \( B_0(\text{discrete}) \) is upper bounded by a circular arc touching it at \( α_1 = α_0 \) (horizontal axis) and \( α_2 = α_0 \) (vertical axis). Also, \( B_0(\text{discrete}) \) is inner bounded by a circular arc touching it at \( α_1 = α_2 = α_e \).

Proof: The proof is similar to the Gaussian input case, because \( I(α_b, γ) \) is also a concave function of the instantaneous signal-to-noise ratio \( α_b^2 γ \).

Propositions [and 2] mainly stated that the outage boundary for a channel with a Gaussian input or a discrete input without precoding is upper bounded by a circular arc of radius \( α_0 \). This circular arc touches the outage boundary on both axis, at \( (α_1 = α_0, α_2 = 0) \) and \( (α_1 = 0, α_2 = α_0) \), where \( α_0 \) is given by Definition 1. This property still holds for a discrete alphabet with linear precoding. When precoding, the mutual information is given by Eq. [5], which cannot be split into two terms anymore. Consider a circular arc in the fading space \( α_1^2 + α_2^2 = c^2 \), where \( c \) is a positive constant. This has the following parametric representation:

\begin{align*}
α_1 &= c \cos(t), \quad t = 0..π/4 \quad (6) \\
α_2 &= c \sin(t), \quad t = 0..π/4 \quad (7)
\end{align*}

Inserting Eqs. (5) and (7) into Eq. (5) yields \( I(t, γ, θ) \) and after deriving \( I(t, γ, θ) \) with respect to \( t \), it can be shown that \( \frac{∂I(t, γ, θ)}{∂t} \big|_{t=0} = 0 \) and \( \frac{∂I(t, γ, θ)}{∂t} \big|_{0 < t < π/4} = 0 \). This is illustrated in Fig. 4 where \( I(t, γ, θ) \) is plotted for \( γ = 3dB \) and \( θ = 30 \) degrees. Physically, this can be interpreted as follows. As mentioned in section 1 \( α, Ω \) corresponds to an unbalanced QPSK constellation in the fading space. When moving on a circular arc, the average energy of this constellation is constant, because \( α_1^2 + α_2^2 = c^2 \) is constant. But the closer to the axes, the more unbalanced this QPSK constellation becomes. It is generally known that a good communication
scheme exploits all the available degrees of freedom in the channel equally, which is of a fundamental nature as we talk about channel capacity \[15\]. This concept is illustrated in Fig. 5 where the capacity of unbalanced QPSK constellations is plotted. It is clear that for a given SNR, the capacity is monotonically non-decreasing as the constellation becomes more unbalanced.

Finally, we show that achieving full diversity is not an issue for rotated modulations. Optimizing the coding gain is the key issue, this optimization is described in the next section. Proposition 3 is valid for linear precoding with a rotation angle \( \theta \) taking any value in \( 0 \ldots \pi/2 \) (except those possible singular values for which the projection of \( \Omega \) on one axis yields \( M' \) points where \( \log_2 M' < R \), i.e., full diversity cannot be achieved for that rate \( R \) with this singular angle value due to a symmetry property in the constellation.

**Proposition 3:** For any coding rate \( 0 < R_c < 1 \), the outage probability of a block-fading channel with a discrete input alphabet and linear precoding \( (s = B = 2) \) exhibits a double diversity order, i.e., \( P_{\text{out}}(\gamma) \propto 1/\gamma^2 \).

**Proof:** Let us find an expression for \( \alpha_0 \). Let \( S \) be the projection of \( \Omega \) on the axis \( \alpha_2 = 0 \). Then \( S \) has some capacity formula \( C(S) = f(\alpha_0^2 \gamma) \). Since the transmitted rate is \( R \) on the outage boundary, then
\[
\alpha_0^2 = f^{-1}(R) / \gamma.
\]

Now, thanks to the circular arc upper bounding \( B_0 \), we have
\[
P_{\text{out}} \leq P(\alpha_1^2 + \alpha_2^2 < \alpha_0^2) \leq \alpha_0^4.
\]

Because the fading gains \( \alpha_1 \) and \( \alpha_2 \) are Rayleigh distributed, we can write \( P(\alpha_1^2 + \alpha_2^2 < \alpha_0^2) \leq \alpha_0^4 \), so finally
\[
P_{\text{out}} \leq \alpha_0^4 \propto 1/\gamma^2.
\]

V. OPTIMIZATION OF THE OUTAGE PROBABILITY OF PRECODED DISCRETE CONSTELLATIONS

In the previous section, we found that outage boundaries of block fading channels with precoded discrete constellations are upper bounded by a circular arc touching the outage boundaries on the axes. Hence, the circular arc corresponds to an upper bound on the outage probability which is minimized by optimizing the unbalanced cases, i.e., \( \alpha_1 = 0 \) or \( \alpha_2 = 0 \). This has some pleasant consequences:

- A circular arc with a fixed center at the origin is determined through one variable, i.e., its radius. This renders an easy optimization where only one parameter has to be optimized. Therefore, \( 2 \times 2 \) orthogonal precoding matrices are sufficient to extremely approach an inner bound on the outage probability, i.e., the outage probability with Gaussian inputs.
- On the axes, the rotated QPSK constellation \( \alpha, \Omega \), becomes a 4-PAM constellation. Increasing the constellation size from BPSK to 4-PAM will render a rotated 16-QAM constellation \( \Omega \) in the fading space, which is smashed into a 16-PAM constellation when \( \alpha_1 = 0 \) or \( \alpha_2 = 0 \). This 16-PAM constellation will need less energy than a 4-PAM to achieve the same rate, which indicates the road to approach the Gaussian outage boundary.

Indeed, Fig. 6 shows the great improvement of constellation expansion combined with the optimization\( ^4 \) of \( \theta \) such that its capacity on the axes is maximized.

The outage probability of the proposed multidimensional constellations is shown in Fig. 7. This confirms that constellation expansion together with the optimization of one parameter is sufficient to achieve the outage probability with a Gaussian input alphabet.

VI. CONCLUSIONS AND FUTURE WORK

We have studied the effect of linear precoding on the coding gain of the outage probability of block fading channels. We have analyzed the outage boundaries in the fading space and established simple upper bounds which yield an easy optimization of the outage probability for a discrete constellation.

\( ^4 \)In the case a \( 4 \times 4 \) precoding matrix would have been used, than there would be 6 optimization parameters.

\( ^5 \)This is a low complexity optimization because the simulation only involves a point-to-point channel with one fading state.
One optimization parameter has shown to be sufficient to extremely approach the outage probability corresponding to a Gaussian input. In future work, we wish to elaborate on the multidimensional constellation design and the design of error-correcting codes with linear precoding.

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