Existence and Stability of Almost-Periodic Solutions of Quasi-Linear Differential Equations with Deviating Argument

M. U. AKHMET
Department of Mathematics, Middle East Technical University
06531 Ankara, Turkey
marat@metu.edu.tr

(Received May 2003; revised and accepted August 2003)

Abstract—The paper is concerned with the existence of an almost-periodic solution of the system with deviating argument

\[
\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_k(t))),
\]

such that the associated homogeneous linear system satisfies exponential dichotomy and deviations of the argument are not restricted by any sign assumption. The exponential stability of the solution when the system is with delay argument is considered. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Quasi-Linear system, Almost-periodic solutions, Deviating argument, Stability.

1. INTRODUCTION AND PRELIMINARIES

Different aspects of the theory of almost-periodic solutions of quasi-linear differential equations with deviating argument, including applications, have been investigated by many authors [1-6]. The assumption that the associated linear equation has exponential dichotomy is one of the most important for the discussion. In [2], the problem was considered for functional differential equation when a matrix of coefficients is constant or periodic with respect to time variable \( t \), and the argument is delay. Paper [3] deals with the existence of almost-periodic solutions of a system with unique and constant deviation. The aim of the present paper is to investigate the problem for system (1), where deviations and the matrix of coefficients are almost-periodic functions. Moreover, we assume that the equation is of mixed type [7], that is, the derivative of \( x \) depends upon past as well as future values. One should emphasize that the general theory has not been considered for this type of equation as well as for equations with retarded argument or for systems of the neutral type [2,8].

Let \( \mathbb{N}, \mathbb{R} \) be sets of all natural and real numbers, respectively, and \( || \cdot || \) be the Euclidean norm in \( \mathbb{R}^n, n \in \mathbb{N} \). Let \( s \in \mathbb{R} \) be a positive number. We denote \( G_s = \{ x \in \mathbb{R}^n ||x|| \leq s \} \) and \( G_s^{k+1} = \)

M. U. Akhmet is previously known as M. U. Akhmetov. The author thanks anonymous referees for helpful criticism.

0893-9659/04/$ - see front matter © 2004 Elsevier Ltd. All rights reserved. Typeset by AM SG-TEX

doi:10.1016/j.aml.2003.08.012
that is, $G_{s}^{k+1}$ is $k$-times Cartesian product of $G_{s}$. Let a set $C_{0}(\mathbb{R})$ (respectively, $C_{0}(\mathbb{R} \times G_{H}^{k+1})$ for a given $H \in \mathbb{R}$, $H > 0$) be a set of all bounded and uniformly continuous on $\mathbb{R}$ (respectively, on $\mathbb{R} \times G_{H}^{k+1}$) functions. For $f \in C_{0}(\mathbb{R})$ (respectively, $C_{0}(\mathbb{R} \times G_{H}^{k+1})$) and $\tau \in \mathbb{R}$, the translate of $f$ by $\tau$ is the function $Q_{\tau}f = f(t + \tau), t \in \mathbb{R}$ (respectively, $Q_{\tau}f(t, z) = f(t + \tau, z)$, $(t, z) \in \mathbb{R} \times G_{H}^{k+1}$). A number $\tau \in \mathbb{R}$ is called $\epsilon$-translation number of a function $f \in C_{0}(\mathbb{R})$ ($C_{0}(\mathbb{R} \times G_{H}^{k+1})$), if $||Q_{\tau}f - f|| < \epsilon$, for every $t \in \mathbb{R}$, $(t, z) \in \mathbb{R} \times G_{H}^{k+1})$. A set $S \subset \mathbb{R}$ is said to be relatively dense, if there exists a number $l > 0$, such that $[a, a + l] \cap S \neq \emptyset$, for all $a \in \mathbb{R}$.

**Definition 1.** A function $f \in C_{0}(\mathbb{R})(C_{0}(\mathbb{R} \times G_{H}^{k+1}))$ is called an almost-periodic (almost-periodic in $t$ uniformly with respect to $z \in G_{H}^{k+1}$), if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists a respectively dense set of $\epsilon$-translations of $f$.

Denote by $\mathcal{A}P(\mathbb{R})$ ($\mathcal{A}P(\mathbb{R} \times G_{H}^{k+1})$), the set of all such functions. The following assumptions will be needed throughout the paper.

\begin{enumerate}
    \item[(C1)] $A(t) \in \mathcal{A}P(\mathbb{R})$ is an $n \times n$ matrix, $\tau_{j} \in \mathcal{A}P(\mathbb{R})$, $j = 1, k$.
    \item[(C2)] $f \in \mathcal{A}P(\mathbb{R} \times G_{H}^{k+1})$, for every $s \in \mathbb{R}$, $s > 0$.
    \item[(C3)] $\exists l \in \mathbb{R}$, $l > 0$, such that
\end{enumerate}

$$
||f(t, z_{1}) - f(t, z_{2})|| \leq l \sum_{j=0}^{k} ||z_{1}^{j} - z_{2}^{j}||,
$$

where $z_{i} = (z_{i}^{1}, \ldots, z_{i}^{k}) \in \mathbb{R}^{n(k+1)}$, $i = 1, 2$.

**Lemma 1.** (See [9].) If $f \in \mathcal{A}P(\mathbb{R} \times G_{H}^{k+1})$ and $\psi \in \mathcal{A}P(\mathbb{R})$, $\psi : \mathbb{R} \rightarrow G_{H}^{k+1}$, then, $f(t, \psi(t)) \in \mathcal{A}P(\mathbb{R})$ and $\text{mod}(f(t, \psi(t))) \subset \text{mod}(f, \psi)$.

**Lemma 2.** If $\phi \in \mathcal{A}P(\mathbb{R})$, then, $\phi(t - \tau_{j}(t)) \in \mathcal{A}P(\mathbb{R})$, $j = 1, k$, and $\text{mod}(\phi(t - \tau_{j}(t))) \subset \text{mod}(\phi, \tau_{j})$.

**Lemma 2** follows immediately from Lemma 1, and since $\phi(t - x) \in \mathcal{A}P(\mathbb{R} \times G_{H})$.

Denote $F_{f, \phi}(t) = f(t, \phi(t), \phi(t - \tau_{1}(t)), \ldots, \phi(t - \tau_{k}(t)))$, where $\phi(t) \in C_{0}(\mathbb{R})$. Using Lemmas 1 and 2, it is easy to check that the following lemma is valid.

**Lemma 3.** If $f \in \mathcal{A}P(\mathbb{R} \times G_{H}^{k+1})$ and $\phi(t) \in \mathcal{A}P(\mathbb{R})$, then, $F_{f, \phi}(t) \in \mathcal{A}P(\mathbb{R})$ and $\text{mod}(F_{f, \phi}(t)) \subset \text{mod}(f, \phi, \tau_{j}, j = 1, k)$.

### 2. EXPONENTIAL DICHOTOMY AND ALMOST-PERIODIC SOLUTIONS

Let

$$
\frac{dx}{dt} = A(t)x
$$

be the associated homogeneous linear system of (1). Assume that (2) satisfies exponential dichotomy. That is, there exists a projection $P$ and positive constants $\sigma_{1}, \sigma_{2}, K_{1}, K_{2}$, such that

\begin{align*}
    \|X(t)PX^{-1}(s)\| & \leq K_{1}\exp(-\sigma_{1}(t-s)), \quad t \geq s, \\
    \|X(t)(I-P)X^{-1}(s)\| & \leq K_{2}\exp(\sigma_{2}(s-t)), \quad t \leq s,
\end{align*}

where $X(t)$ is a fundamental matrix of (2).

Let

$$
G(t, s) = \begin{cases} 
    X(t)PX^{-1}(s), & \text{if } t \geq s, \\
    X(t)(P-I)X^{-1}(s), & \text{if } t \leq s,
\end{cases}
$$

be a Green's function of (2).
LEMMA 4. (See [9].) If \( \psi \in \mathcal{AP}(\mathbb{R}) \), then,

\[
\phi(t) = \int_{-\infty}^{\infty} G(t,s) \psi(s) \, ds
\]

is the almost-periodic function and \( \text{mod}(\phi) \subset \text{mod}(A, \psi) \).

THEOREM 1. Assume that Conditions (C1)–(C3) are valid, equation (2) satisfies exponential dichotomy and

\[
\lim_{k \to \infty} \left( \frac{K_1}{\sigma_1} + \frac{K_2}{\sigma_2} \right) < 1.
\]

Then, there exists a unique solution \( \phi(t) \in \mathcal{AP}(\mathbb{R}) \) of (1) and \( \text{mod}(\phi) \subset \text{mod}(A, f, \tau_j, j = 1, k) \).

PROOF. Let \( \Psi = \{ \psi \in \mathcal{AP}(\mathbb{R}) \mid \text{mod}(\psi) \subset \text{mod}(A, f, \tau_j, j = 1, k) \} \) be a complete metric space with sup-norm \( \| \cdot \|_0 \). Define an operator \( \Theta \) on \( \Psi \), such that

\[
\Theta(\psi(t)) = \int_{-\infty}^{\infty} G(t,s) F_{f, \psi}(s) \, ds.
\]

Lemmas 3 and 4 imply that \( F_{f, \psi} \in \Psi \) and \( \Theta : \Psi \to \Psi \). If \( \psi_1, \psi_2 \in \Psi \), then,

\[
\| \Theta(\psi_1(t)) - \Theta(\psi_2(t)) \| \leq \left\| \int_{-\infty}^{t} X(t) P X^{-1}(s) (F_{f, \psi_1}(s) - F_{f, \psi_2}(s)) \, ds \right\|
+ \left\| \int_{t}^{\infty} X(t) (I - P) X^{-1}(s) (F_{\psi_1}(s) - F_{\psi_2}(s)) \, ds \right\|
\leq \lim_{k \to \infty} \left( \frac{K_1}{\sigma_1} + \frac{K_2}{\sigma_2} \right) \| \psi_1(t) - \psi_2(t) \|_0.
\]

Thus, \( \Theta \) is a contractive operator and there exists a unique almost-periodic solution of the equation

\[
\psi(t) = \int_{-\infty}^{\infty} G(t,s) F_{f, \psi}(s) \, ds,
\]

which is the solution of (1). The theorem is proved.

REMARK 2.1. Apparently, the general problem of existence of solutions for equations of mixed type has not been considered yet. Even for the case of advanced argument, there are certain difficulties if we try to define a solution for increasing \( t \) [10,11]. Hale remarked in [7] that “these equations seem to dictate that boundary conditions should be specified in order to obtain a solution in the way as one does for elliptic partial differential equations”. We regard the boundedness of the solution on \( \mathbb{R} \) as a boundary condition in the proof of Theorem 1. Similar arguments were used in [12] to investigate various problems for ordinary and functional differential equations. Authors of [13] used this method to prove existence of a bounded solution for the equation of advanced type when \( t \geq t_0 \).

3. EXPONENTIAL STABILITY

This section is concerned with the problem of exponential stability of the almost-periodic solution of the system with delay argument. Then, all solutions of system (1) are continuous. Denote by \( X(t, s) = X(t) X^{-1}(s) \), the Cauchy matrix of (2). We will need the following assumptions.

(C4) \( \exists \{a, b\} \subset \mathbb{R}, b \geq 1, a > 0 \), such that \( \| X(t, s) \| \leq b \exp(-a(t-s)), t \geq s. \)

(C5) \( \text{blk} < a. \)

Assume that \( \tau_j(t) \geq 0, j = 1, k \) and denote \( \tau_0 = \max\{\sup_j \tau_j(t), j = 1, k\} > 0, \)

\( m(l) = 1 - \exp(\sigma \tau_0) \text{blk}(k+1)(a-\sigma)^{-1}, \) where \( \sigma \in \mathbb{R}, 0 < \sigma < a, \) is fixed.

(C6) \( m(l) > 0. \)

Conditions (C1)–(C6) and Theorem 1 imply that there exists a unique solution \( \xi_0(t) \in \mathcal{AP}(\mathbb{R}) \) of (1). Denote \( C_0([t_0 - \tau, t_0]) \) the set of all continuous on \([t_0 - \tau, t_0]\) functions and let \( x(t, \phi) \), \( \phi(t) \in C_0([t_0 - \tau, t_0]) \) be a solution of (1), such that \( x(t, \phi) = \phi(t), t \in [t_0 - \tau, t_0]. \)
DEFINITION 2. The solution $\xi_0(t)$ is called uniformly exponentially stable, if there exists a number $\sigma \in \mathbb{R}$, $\sigma > 0$, such that for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon)$, such that the inequality
$$\max_{t_0 - \tau_0 \leq t \leq t_0} ||\phi(t) - \xi_0(t)|| < \delta$$
implies
$$||x(t, \phi) - \xi_0(t)|| < \varepsilon \exp(-\sigma(t - t_0)),$$
for all $t \geq t_0$.

REMARK 3.1. The definition of exponential stability [11] is slightly changed in Definition 2, so that the coefficient of exponential decay can be arbitrarily small.

THEOREM 2. Assume that $(C_1)-(C_6)$ are valid. Then, the almost-periodic solution $\xi_0(t)$ of (1) is uniformly exponentially stable.

PROOF. One can see that $v(t) = x(t, \phi) - \xi_0(t)$ is a solution of the equation
$$\frac{dv}{dt} = A(t) v + w(t, v(t)) + v(t - \tau_1(t)), \ldots, v(t - \tau_k(t)),$$
where $w$ satisfies
$$||w(t, v_1) - w(t, v_2)|| \leq l \sum_{j=0}^{k} ||v_j^1 - v_j^2||,$$


