Simple robust technique using time delay estimation for the control and synchronization of Lorenz systems

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ABSTRACT

This work presents two simple and robust techniques based on time delay estimation for the respective control and synchronization of chaos systems. First, one of these techniques is applied to the control of a chaotic Lorenz system with both matched and mismatched uncertainties. The nonlinearities in the Lorenz system is cancelled by time delay estimation and desired error dynamics is inserted. Second, the other technique is applied to the synchronization of the Lü system and the Lorenz system with uncertainties. The synchronization input consists of three elements that have transparent and clear meanings.

Since time delay estimation enables a very effective and efficient cancellation of disturbances and nonlinearities, the techniques turn out to be simple and robust. Numerical simulation results show fast, accurate and robust performance of the proposed techniques, thereby demonstrating their effectiveness for the control and synchronization of Lorenz systems.

1. Introduction

A chaotic system is sensitive to initial conditions, highly nonlinear, irregular, and complex. Chaotic behaviors have been studied extensively since the first classical chaotic attractor was introduced by Lorenz [1]. Since the pioneering research of controlling chaos [2], some of the research moved from the pure analysis of chaos to the control and synchronization of chaos. In most engineering systems, chaotic behavior is undesirable, and the goal of chaos control is to suppress or remove chaotic behavior, and to provide the system with stable and predictable behaviors. On the other hand, in the applications of secure communications, biological systems, chemical reactions, and information processing, prescribed chaotic behaviors are wanted, and the goal of chaos synchronization is to make the chaotic states of the system to track the desired chaotic trajectory.

As an example of chaotic systems to be controlled, the Lorenz system is popular because the Lorenz system is simple among many chaotic systems, yet captures many features of chaotic dynamics [3,4]. Various methods have been introduced to control or synchronize the Lorenz system. For example, bang–bang control [3], sliding mode control [4], feedback linearization [5], adaptive control [6–8], backstepping control [9,10], neural networks [11,12], and others in [13]. Recently, fusions of aforementioned control methods have been carried out to achieve more sophisticated control performance. For example, fuzzy logic and adaptive control is merged in [14,15]; adaptive and backstepping technique are merged in [16]; advantages of the adaptive control, neural network and sliding mode control are combined in [17]; fuzzy adaptive sliding mode is used in [18]; and adaptive neural-fuzzy-network control is proposed in [19].
To find out satisfactory solutions to chaos control and synchronization, in general, we should continue our research on the modeling and analysis of chaotic systems, and on fusing existing control methods, so that they become applicable to wider problem domain and provide more robust performance. In the mean time, when the parameters of chaotic systems are poorly understood, and if the practical implementation is considered, a viable alternative should be to pursue simplicity and transparency while preserving robustness. As for the simplicity, the control structure should have small number of terms in control input. As for the transparency, the control structure and the effect of tuning parameters should be transparent to designers.

As such a technique, we present a control technique using time delay estimation (TDE) method. The TDE was originally introduced to control robot manipulators [20–22]. The main idea of the TDE is to estimate unknown dynamics and disturbances by intentionally using time-delayed information. This estimation is used by our proposed technique to cancel the unknown dynamics and disturbances, while at the same time the desired dynamics is injected into the plant. The effectiveness of the TDE has consistently been crucial to the extraordinary robustness and simplicity demonstrated in [20–22] and other research works. Its effectiveness also motivates us to formulate the proposed technique for the control and synchronization of chaos dynamics, with expectations that the technique becomes simple in form, easy to implement, and yet robust.

There is another chaos control method intentionally using time-delay, which is called Pyragas method [23,24]. No prior goal (i.e., desired constant, periodic, or chaotic trajectory) can be specified in Pyragas method; the goal can only be achieved by trial-and-error in Pyragas method. On the contrary, the goal (desired constant, periodic, or chaotic trajectory) can be precisely specified and achieved through the proposed control using TDE technique.

This paper is organized as follows. In Section 2, we present TDE based control for regulations of the Lorenz system with matched and mismatched uncertainties. Section 3 presents simulation results for regulations of the Lorenz system. In Section 4, we design TDE based synchronizing technique for two different chaotic systems. Section 5 presents simulation results of the synchronization of two chaotic systems (the Lü system and the Lorenz system with parameter variation and disturbance). Finally, in Section 6, we give some concluding remarks.

2. Regulation of the Lorenz system using TDE

The classical Lorenz system is described as

\[
\begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - b x_3, \\
\end{align*}
\]

(1)

where \(x_1\) denotes the convective fluid motion, \(x_2\) denotes the horizontal temperature variation, and \(x_3\) denotes the vertical temperature variation; \(\sigma, b, \) and \(\rho\) are real positive parameters that represent the Prandtl number, a geometric factor, and the Rayleigh number, respectively.

The control of the Lorenz system is often realized by adding a control input \(u\) to the differential equation of state \(x_3\) [3]. A closed-loop experiment of the Lorenz equations with control input is given in [25]. Recently, bounded disturbance is considered in the differential equation of state \(x_1\), and the differential equation of state \(x_2\) to emulate more practical situation [4].

Then, the controlled Lorenz system is expressed by

\[
\begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 + d_1, \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 + d_2 + u, \\
\dot{x}_3 &= x_1 x_2 - b x_3, \\
\end{align*}
\]

(2a, 2b, 2c)

where \(d_1\) and \(d_2\) denote unknown disturbances, which are assumed to be continuous and bounded. As previous contributions [4,26], the control objective is to regulate \(x_1\) to a given constant \(x_{1r}\). Setting \(x_1(t) = x_{1r}\) in (1), we can obtain the other equilibrium points of the states, which are \(x_2(t) = x_{2r}, x_3(t) = x_{3r}/b\). Thus, the goal is to design a control input \(u\) in order to regulate to a specific point \(P_t = (x_{1r}, x_{2r}, x_{3r}) = (x_{1r}, x_{1r}, x_{3r}/b)\). The differential equation of state \(x_3\), (2c), is internally stable when \(x_1(t)\) and \(x_2(t)\) converge to \(x_{1r}\). So we will focus on the control of (2a) and (2b) from now on.

Defining the errors as \(e_1 = x_1 - x_{1r}, e_2 = x_2 - x_{2r}\), we can rewrite (2a) and (2b) as

\[
\begin{align*}
\dot{e}_1 &= -\sigma e_1 + \sigma e_2 + \eta_1, \\
\dot{e}_2 &= f(e_1, e_2, e_3) + u, \\
\end{align*}
\]

(3, 4)

where

\[
\begin{align*}
\eta_1 &= \sigma(x_{2r} - x_{1r}) + d_1, \\
f(e_1, e_2, e_3) &= \rho e_1 - e_2 - x_{1r} e_1 - x_{1r} e_3 - e_1 e_2 - x_{1r} e_3 - x_{1r} x_{3r} - x_{2r} + \rho x_{1r} + d_2. \\
\end{align*}
\]

(5, 6)

Note that since \(d_2\) was assumed to be continuous, \(f(e_1, e_2, e_3)\) becomes a continuous function, and because of this the approximation holds that \(f(e_1(t), e_2(t), e_3(t)) \approx f(e_1(t-L), e_2(t-L), e_3(t-L))\) provided that \(L\) is sufficiently small. In other words, \(f(e_1(t), e_2(t), e_3(t))\) can be estimated by using \(f(e_1(t-L), e_2(t-L), e_3(t-L))\). Let this estimation, the so-called TDE, be formally defined as
\[ \dot{f}(e_1, e_2, e_3)_{t-L} = f(e_1, e_2, e_3)_{t-L}, \]
\[ \text{where } \cdot \text{ denotes the estimated value of } \cdot, \text{ and } \cdot_{t-L} \text{ denotes time-delayed value of } \cdot. \]  
\[ f(e_1, e_2, e_3)_{t-L} \text{ can be expressed by} \]
\[ \dot{f}(e_1, e_2, e_3)_{t-L} = (e_2)_{t-L} - u_{t-L} \]
\[ \text{Clearly, it would be much easier and more efficient to evaluate } (e_2)_{t-L} - u_{t-L} \text{ than to estimate } f(e_1, e_2, e_3). \]

The control input is activated at \[ k \text{ and } k_2 \] are positive parameters to design. Substituting (7) and (8) into (9), we obtain the control input

\[ u = u_{t-L} - (e_2)_{t-L} - k_1 e_1 - k_2 e_2. \]

The gains \( k_1 \) and \( k_2 \) can be selected by any control design method for linear systems. For instance, pole-placement techniques may be used to determine \( k_1 \) and \( k_2 \), so that the eigenvalues of the following matrix may be located in the left half plane:

\[ \begin{bmatrix} -\sigma & \sigma \\ -k_1 & -k_2 \end{bmatrix}. \]

In order to prove the stability of the close loop system, (3), (4), and (9), we have selected a Lyapunov function as follows:

\[ V = 0.5 k_1 e_1^2 + 0.5 \sigma e_2^2. \]

Then, its derivative becomes

\[ \dot{V} = k_1 e_1 (-\sigma e_1 + \sigma e_2 + \eta_1) + \sigma e_2 [f(e_1, e_2, e_3) - f(e_1, e_2, e_3)_{t-L} - k_1 e_1 - k_2 e_2]. \]

Here, we define the time-delay estimation error \( \varepsilon \) as

\[ \varepsilon = f(e_1, e_2, e_3) - f(e_1, e_2, e_3)_{t-L}. \]

Because \( f(e_1, e_2, e_3) \) is a continuous function, \( \varepsilon \) is bounded for a small time delay \( L \), and \( \varepsilon \to 0 \) as \( L \to 0 \). Thus,

\[ \dot{V} = -\sigma k_1 e_1^2 + k_1 \eta_1 e_1 - \sigma k_2 e_2^2 + \sigma \varepsilon e_2 \leq -\sigma k_1 \left( e_1^2 - \frac{|\eta_1|}{\sigma} |e_1| \right) - \sigma k_2 \left( e_2^2 - \frac{|\varepsilon|}{k_2} |e_2| \right). \]

Since \( V \) is negative outside the set \( \{|e_1| < |\eta_1/\sigma|\} \cup \{|e_2| < |\varepsilon/k_2|\} \), the solutions are ultimately bounded. If \( \varepsilon = 0 \) (an ideal case as \( L \to 0 \)) and \( \eta_1 = 0 \) (i.e. no unmatched uncertainties), then the closed loop system is asymptotically stable.

3. Numerical experiment: regulation of the Lorenz system

The parameters of the Lorenz system are selected to be \( \sigma = 10, b = 8/3, \) and \( r = 28 \). The initial condition of the Lorenz system is \( (x_1, x_2, x_3) = (10, 10, 10) \). The fourth-order Runge–Kutta method is used to solve the systems with step size 0.0001 s. The time-delayed derivative \( (e_2)_{t-L} \) in the proposed controller (10) is calculated by numerical differentiation, as

\[ (e_2)_{t-L} = [(e_2)_{t} - (e_2)_{t-L}] / L. \]

The control input is activated at \( t = 5 \) s and the parameters of the controller (15) are set to be \( k_1 = 10, k_2 = 50, \) and \( L = 0.001 \). The regulation problem is designed as follows:

\[ (x_{1t}, x_{2t}, x_{3t}) = (8.5, 8.5, 27.1), \quad 0 \leq t < 10; \]
\[ (x_{1t}, x_{2t}, x_{3t}) = (12, 12, 54), \quad t \geq 10. \]

The simulation results are given in Figs. 1–3. Fig. 1 shows the response of the Lorenz system by the proposed control under a matched disturbance \( d_t = 0.5 \cos(5 \pi t) \). The state vector converges smoothly to the desired state, and the control input does not show any chattering. The peaks in the control input at \( t = 5 \) s and \( t = 10 \) s, which have been also observed in [8], come from the control effort to meet discontinuous shifts of desired regulation points at \( t = 5 \) s and \( t = 10 \) s.

The matched nonlinear terms and disturbance in the chaotic Lorenz dynamic can be cancelled by the proposed control. Fig. 1(c) shows \( f(e_1, e_2, e_3), -f(e_1, e_2, e_3)_{t-L} = u_{t-L} - e_{t-L} \), and the TDE error \( f(e_1, e_2, e_3)_{t-L} - f(e_1, e_2, e_3)_{t-L} \). The small value (almost zero) of the TDE error displays that not only the TDE has been performed sufficiently accurate, but also nonlinear terms and disturbance has been cancelled almost completely, thereby demonstrating the robustness of the proposed technique.
Fig. 2 shows the response of the Lorenz system under the proposed control with matched disturbance $d_2 = 0.5 \cos(5 \pi t)$. The state vector is very close to the desired state and the control input is continuous. With bounded mismatched disturbance, however, the steady state describes a limit cycle in the time interval 13–18 s as shown in Fig 3(b).

4. Chaos synchronization using TDE

Consider the chaotic systems in the form of

$$\dot{x} = (A + \Delta A)x + f(x) + \Delta f(x) + u + d(t), \quad (17)$$

where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is a certain constant matrix, $f(x)$ is a smooth nonlinear vector function, and $u$ is the control input. $\Delta A$, $\Delta f(x)$, and $d(t)$ are the linear uncertain term, the nonlinear uncertain term, and bounded continuous disturbance vector term, respectively. The chaotic system can be rewritten as

$$\dot{x} = F(x) + u, \quad (18)$$

where

$$F(x) = (A + \Delta A)x + f(x) + \Delta f(x) + d(t). \quad (19)$$

Note that $F(x)$ is regarded as being composed of both certain terms and uncertain terms, and of both linear terms and nonlinear terms.

The desired chaotic system is given by

$$x_m = g(x_m), \quad (20)$$

where $x_m \in \mathbb{R}^n$ is the state vector of desired chaotic system and $g(x_m)$ is a smooth nonlinear vector function. Let $e = x - x_m$, then the chaotic system (17) and (20) are synchronous if $\lim_{t \to \infty} e(t) = 0$. The control objective is to force the error to vanish with a desired dynamics:
\[ e = A_e e, \tag{21} \]

where \( A_e \) is error system matrix, whose eigenvalues lie in the left half plane.

Subtracting (20) from (18), and substituting \( e \) of (21) with \( A_e e \), we can obtain

\[ u = A_e e + g(x_m) - F(x). \tag{22} \]

Fig. 2. The response of the Lorenz system under the proposed control with matched disturbance \( d_2 = 0.5 \cos(5\pi t) \) and mismatched disturbance \( d_1 = 0.5 \cos(5\pi t) \). The control input activates at time 5 s. (a) The state time response. (b) The phase portrait. (c) The nonlinear function \( f \) (dotted), its TDE error \( f - f t-L \) (solid), and \( u_{t-L} e t-L \) (dashed). (d) The control input \( u \).

Fig. 3. The error bound of system vector relates to equilibrium point at \( t \in [13, 18] \): (a) with matched disturbance \( d_2 = 0.5 \cos(5\pi t) \); (b) with matched disturbance \( d_2 = 0.5 \cos(5\pi t) \) and mismatched disturbance \( d_1 = 0.5 \cos(5\pi t) \).
A causal evaluation becomes possible under an assumption that $F(x)$ is continuous. For a sufficiently small $L$, holds the following:

\[ F(x) \cong F(x)_{t-L}. \]  

(23)

$F(x)_{t-L}$ can be calculated from (18) and expressed by

\[ F(x)_{t-L} = x_{t-L} - u_{t-L}. \]  

(24)

Then with combination of (22)–(24), the control input is given by

\[ u = u_{t-L} - x_{t-L} + g(x_m) + A_e e. \]  

(25)

The control input consists of three elements that have clear meanings: $u_{t-L} - x_{t-L}$ cancels given system dynamics; $g(x_m)$ injects a target chaos; and $A_e e$ shapes synchronization error dynamics, in particular the convergence characteristics.

Let $P$ denote a positive definite solution to the Lyapunov equation

\[ PA_e + A_e^T P = -I, \]  

(26)

which exists as $A_e$ is stable. The Lyapunov function is defined as $V = \frac{1}{2}e^T Pe$. The derivative is

\[
\dot{V} = e^T Pe + e^T Pe = e^T (P(x - x_m) + (x - x_m)^T) Pe = e^T (P[F(x) + u - g(x_m)] + [F(x) + u - g(x_m)]^T) Pe
\]

\[
= e^T [P[F(x) + u_{t-L} - x_{t-L} + g(x_m) + A_e e - g(x_m)] + [F(x) + u_{t-L} - x_{t-L} + g(x_m) + A_e e - g(x_m)]^T] Pe
\]

\[
= e^T (PA_e + A_e^T P)e + 2e^T Pe \leq -||e||^2 + 2||e||||Pe||.
\]  

(27)

where $e = F(x) - F(x)_{t-L}$. Since $F(x)$ is a continuous function, $e$ is bounded for a small time delay $L$. Moreover, $e \to 0$ as $L \to 0$. Define $\delta = 2||Pe||$, then (27) becomes

\[
\dot{V} \leq -||e||(||e|| - \delta).
\]  

(28)

The $e(t)$ is ultimately bounded with respect to the disc $D_\delta = \{e : ||e|| < \delta\}$, which is centered at the origin and acts as ellipsoid. Thus, practical synchronization [28] (or approximate output synchronization, $\lim_{t \to \infty} ||e(t)|| < \delta$ [29]) is achieved through the proposed control. The size of $D_\delta$ can be adjusted by choosing the design matrices $A_e$ (because $A_e$ determines $P$ in (26)). This will be verified through simulations in the following section.

Incidentally, the $A_e$ can be selected as negative definite diagonal matrix as $A_e = diag\{-K_1, -K_2, -K_3\}$ where $K_1, K_2, K_3$ are positive gains. Thus, shaping the synchronization error dynamics is very easy because $K_1, K_2, K_3$ can be tuned independently.

5. Chaos synchronization between Lorenz and Lü systems

The desired chaotic system (Lü system) [27] is given by

\[
\begin{align*}
\dot{x}_{m1} &= a(x_{m2} - x_{m1}), \\
\dot{x}_{m2} &= -x_{m1}x_{m3} + cx_{m2}, \\
\dot{x}_{m3} &= x_{m1}x_{m2} - bx_{m3}
\end{align*}
\]  

(29)

where $x_{m1}, x_{m2}$ and $x_{m3}$ are state variables. The parameters of Lü system are selected to be $a = 35, b = 3$, and $c = 20$. The initial values of the desired chaotic system are $x_{m1}(0) = -10, x_{m2}(0) = -5$, and $x_{m3}(0) = 5$. The state variables and phase portrait of the uncontrolled Lü system are shown in Fig. 4.

![Phase Portrait of Lu System](image_url)

**Fig. 4.** The Lü system. (a) State variables. (b) Phase portrait.
The Lorenz system to be controlled is as follows:
\[
\begin{align*}
\dot{x}_1 &= \left(r + \delta r\right)x_2 - x_1 x_3 + d_1 + u_1, \\
\dot{x}_2 &= \left(r + \delta r\right)x_1 - x_2 x_3 + d_2 + u_2, \\
\dot{x}_3 &= x_1 x_2 - \left(b + \delta b\right)x_3 + d_3 + u_3,
\end{align*}
\]
(30)
where the notations of \(x_1, x_2, x_3, r, b, d_1, d_2, \) and \(d_3\) are the same as in (1) and (2). \(\delta r, \delta \sigma, \delta b\) denote the corresponding perturbation of the parameters \(\sigma, r, b\), respectively. The parameters are set as follows: \(\sigma = 10, r = 28, b = 8/3; \delta \sigma = 0.1, \delta r = 0.1, \delta b = 0.2;\) and \(d_1 = 0.3 \sin x_2, d_2 = 0.1 \cos x_1, \) and \(d_3 = 0.2 \sin(3x_2)\). The initial values of the Lorenz system are set \(x_1(0) = 0.2, x_2(0) = 0.6,\) and \(x_3(0) = 1.\) The state variables and phase portrait of the uncontrolled Lorenz system are shown in Fig. 5.

The synchronization of the Lü system and the Lorenz system under the proposed control. (a) Phase portrait of (desired) Lü system and the controlled Lorenz system. (b) Control inputs. (c) Dynamics synchronization errors \((t \in [0, 1])\). (d) Dynamics synchronization errors \((t \in [1, 20])\).

Fig. 5. The uncontrolled Lorenz system with disturbances and parameter variations. (a) State variables. (b) Phase portrait.

Fig. 6. The synchronization of the Lü system and the Lorenz system under the proposed control. (a) Phase portrait of (desired) Lü system and the controlled Lorenz system. (b) Control inputs. (c) Dynamics synchronization errors \((t \in [0, 1])\). (d) Dynamics synchronization errors \((t \in [1, 20])\).
From Fig. 6(a), we can see that the controlled Lorenz system (response system) exactly synchronizes with the Lü system (drive system). Control inputs in Fig. 6(b) exhibit no chattering. The dynamic synchronization errors are rapidly suppressed (the settling time is less than 0.1 s), and the steady states error are very small (less than 0.25%), as shown in Fig 6(c) and (d).

It is instructive to check whether each element in (31) functions as intended. Fig. 7 clearly shows that the term $F(x)_{t-1}$ can effectively estimate the nonlinearity $F(x)$, and $F(x)_{t-1}$ can effectively cancel most of the nonlinearity $F(x)$. Fig. 8 shows the performance of the element to correct synchronization error. The larger absolute values of the eigenvalues of $A_e$ become, the smaller errors one can obtain. As to the element responsible for injecting Lü dynamics, one

![Fig. 7. The nonlinear terms and their estimation errors. (a) $F(x)_{1}$ and its estimation error. (b) $F(x)_{2}$ and its estimation error. (c) $F(x)_{3}$ and its estimation error.](image)

![Fig. 8. The synchronization of the Lü system and the Lorenz system under the proposed control with different control gains: (a) with $A_e = \text{diag}(-50, -50, -50)$ and (b) with $A_e = \text{diag}(-250, -250, -250)$.](image)
can easily observe its performance in Fig. 6(c) and (d) and confirm that the injected dynamics are achieved within the accuracy plotted in the figures. Thus, simulation results demonstrate the effective functionalities of the three elements in (31):

\[ u = u_{L} - \dot{x}_{L} + g(x_{m}) + A_{e}e \]  
\[ \text{cancellation of Lorenz system dynamics} \quad \text{injection of Lu dynamics} \quad \text{shaping synchronization errors dynamics} \]  

(31)

6. Conclusion

The proposed control and synchronization method are simple, transparent, and robust. The control (regulation) technique is developed by cancelling the nonlinearities in the Lorenz system by TDE and inserting desired error dynamics. The simulation results show that the proposed regulation technique is effective for cancellation of matched uncertainties. A uniform ultimate boundedness can be achieved through proposed regulation technique with matched and unmatched uncertainties.

The synchronization technique consists of three elements that have clear meanings: a system dynamics cancelling element (by using TDE), a target chaos injection element, and an element to shape the dynamics of synchronization error. Through simulation studies and their final analysis, it was confirmed that the three elements function as intended. The synchronization input (31) has transparent structure and is easy to tune; specifying driving dynamics (in this example, Lu system) is all that is necessary for a user to do. The Lorenz system is controlled to achieve Lu system, and the simulation results show the fast, accurate, and robust performance of the proposed synchronization technique.

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