Variational equations with constraints

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Abstract. In this paper we deal with a constrained variational equation associated with the usual weak formulation of an elliptic boundary value problem in the context of Banach spaces, which generalizes the classical results of existence and uniqueness. Furthermore, we give a precise estimation of the norm of the solution.

Key words: Constrained variational equations, Lax-Milgram theorem, elliptic boundary value problems.


1. Introduction

In the standard variational formulation of an elliptic boundary value problem, we can treat the essential boundary conditions as constraints. For instance, if $\Omega$ is an open and bounded set in $\mathbb{R}^n$ with a Lipschitz boundary $\Gamma$ and if $f_0 \in L^2(\Omega)$ and $g_0 \in H^{1/2}(\Gamma)$, then the elliptic boundary problem corresponding to the Poisson equation with non-homogeneous Dirichlet boundary condition

$$\begin{cases} -\triangle u = f_0 & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma \end{cases}$$

admits the following weak formulation:

Find $u \in H^1(\Omega)$ such that $u = g_0$ on $\Gamma$ and for all $v \in H^1_0(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f_0 v.$$

As usual, the identity $u = g_0$ on $\Gamma$ is understood by $\gamma_0(u) = g_0$, where $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\Gamma) \subset L^2(\Gamma)$ is the trace operator. We can write this variational formulation in an equivalent way by imposing weakly the boundary conditions. To be more precise, let $V := H^1(\Omega)$, let $W := H^{-1/2}(\Gamma)$, let $a : V \times V \rightarrow \mathbb{R}$ and $b : V \times W \rightarrow \mathbb{R}$ be the continuous bilinear forms given by

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v, \quad (u, v \in V)$$

and

$$b(v, w) := < \gamma_0(v), w >, \quad (v \in V, w \in W)$$

$< \cdot, \cdot >$ being the canonical bilinear form on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, and let $f : V \rightarrow \mathbb{R}$ and $g : W \rightarrow \mathbb{R}$ be the continuous linear forms defined by

$$f(v) := \int_{\Omega} f_0 v, \quad (v \in V)$$

and

$$g(w) := < g_0, w >, \quad (w \in W).$$

Taking into account that

$$Z := \{ v \in V : w \in W \Rightarrow b(v, w) = 0 \} = H^1_0(\Omega),$$

...
then the preceding weak formulation is just the following variational equation with constraints:

Find \( u \in V \) such that
\[
\begin{aligned}
z \in Z & \Rightarrow a(u, z) = f(z) \\
w \in W & \Rightarrow b(u, w) = g(w)
\end{aligned}
\]

More generally, the following problem is considered: let \( V \) and \( W \) be Hilbert spaces, let \( f : V \to \mathbb{R} \) and \( g : W \to \mathbb{R} \) be continuous linear functionals, let \( a : V \times V \to \mathbb{R} \) and \( b : V \times W \to \mathbb{R} \) be continuous bilinear forms and let \( Z := \{ v \in V : w \in W \Rightarrow b(v, w) = 0 \} \). Under these assumptions

find \( u \in V \) such that
\[
\begin{aligned}
z \in Z & \Rightarrow a(u, z) = f(z) \\
w \in W & \Rightarrow b(u, w) = g(w)
\end{aligned}
\]

In this paper, we characterize when this constrained variational equation admits a solution, in the more general framework of \( V \) being a reflexive Banach space and \( W \) a normed space, along the lines of the known classical results (see [1, §II.1 Proposition 1.1] or [2, Lemma 4.67]), which are nothing more than sufficient conditions guaranteeing that such a variational equation with constraints has a solution. Furthermore, we prove that for a certain solution (the solution under hypotheses of uniqueness) it is possible to obtain a concise estimation of its norm.

The vector spaces will always be considered as real vector spaces, although the results are equally valid in the complex case.

2. Constrained variational equations

First we evoke the following version of the Lax-Milgram theorem which, in fact, admits a more general formulation in the context of the locally convex topological vector spaces ([3, Theorem 1.2]).

Previously a bit of notation: given two real vector spaces \( V \) and \( W \), a bilinear form \( \varphi : V \times W \to \mathbb{R} \) and \( v_0 \in V, w_0 \in W, \varphi(\cdot, w_0) \) stands for the linear form on \( V \)
\[
v \in V \mapsto \varphi(v, w_0) \in \mathbb{R}
\]
and analogously, \( \varphi(v_0, \cdot) \) stands for the linear form on \( W \)
\[
w \in W \mapsto \varphi(v_0, w) \in \mathbb{R}.
\]

In addition, given a real normed space \( V \), we shall write \( V^* \) to denote its topological dual space. Finally, \( (\cdot)_+ \) denotes the positive part, that is, for \( t \in \mathbb{R}, (t)_+ = \max\{t, 0\} \), and for a nonempty subset \( B \) of a set \( A \) and a mapping \( h : A \to \mathbb{R} \) we write \( h|_B \) for the restriction of \( h \) to \( B \).

**Proposition 2.1 ([3]).** Suppose that \( E \) is a real reflexive Banach space and that \( F \) is a real normed space, \( y_0^* \in F^* \), \( \varphi : E \times F \to \mathbb{R} \) is bilinear, and \( C \) is a nonempty convex subset of \( F \) such that for all \( y \in C \) we have that \( \varphi(\cdot, y) \in E^* \). Then

there exists \( x_0 \in E \) such that for all \( y \in C \), \( y_0^*(y) \leq \varphi(x_0, y) \)

if, and only if,

there exists \( \alpha > 0 \) such that for all \( y \in C \), \( y_0^*(y) \leq \alpha \| \varphi(\cdot, y) \| \).
Moreover, if one of these equivalent conditions is satisfied and for some \( y \in C \) we have that \( \varphi(\cdot, y) \neq 0 \), then
\[
\min \{ \| x_0 \| : x_0 \in E \text{ and for all } y \in C, \quad y_0(y) \leq \varphi(x_0, y) \} = \left( \sup_{y \in C, \varphi(\cdot, y) \neq 0} \frac{y_0(y)}{\| \varphi(\cdot, y) \|} \right). 
\]

In the next result we give conditions equivalent to the fact that the variational equation with constraints has a solution:

**Theorem 2.2.** Let \( V \) be a real reflexive Banach space, let \( W \) be a real normed space and let \( a : V \times V \to \mathbb{R} \) and \( b : V \times W \to \mathbb{R} \) be bilinear forms such that \( a \) is continuous and \( w \in W \Rightarrow b(\cdot, w) \in V^* \).

Let \( f \in V^* \) and \( g \in W^* \) and write
\[
Z := \{ v \in V : b(v, \cdot) = 0 \} \quad \text{and} \quad G := \{ v \in V : b(v, \cdot) = g \}.
\]

Then the following assertions are equivalent:

(i) The variational equation with constraints has a solution, that is,
\[
\text{there exists } u \in V \text{ such that } \begin{cases} z \in Z \Rightarrow a(u, z) = f(z) \\ w \in W \Rightarrow b(u, w) = g(w) \end{cases}. \tag{2.2.1}
\]

(ii) \( G \neq \emptyset \) and for some \( v \in G \) there exists \( \alpha > 0 \) such that
\[
z \in Z \Rightarrow f(z) - a(v, z) \leq \alpha \| a(\cdot, z) \|.
\]

(iii) \( G \neq \emptyset \) and for all \( v \in G \) there exists \( \alpha > 0 \) such that
\[
z \in Z \Rightarrow f(z) - a(v, z) \leq \alpha \| a(\cdot, z) \|.
\]

In addition, if one of these equivalent conditions is satisfied and there exists \( z \in Z \) such that \( a(\cdot, z) \neq 0 \), then we can take \( u \in V \) in (i) with
\[
\| u \| = \min_{v \in G} \left( \sup_{z \in Z, a(\cdot, z) \neq 0} \frac{f(z) - a(v, z)}{\| a(\cdot, z) \|} + \| v \| \right).
\]

**Proof.** Since statements (ii) and (iii) are clearly equivalent, we shall prove the equivalence between (i) and (iii).

Given that in both (i) and (iii) we are assuming that \( G \neq \emptyset \), let us fix \( v \in G \), so that \( G = v + Z \).
Then, the existence of a solution \( u \in G \) of the variational equation

\[
z \in Z \Rightarrow a(u, z) = f(z)
\]

is equivalent to the existence of \( z_v \in Z \) in such a way that \( u = v + z_v \) and

\[
a(z_v, \cdot)_{|Z} = f|_Z - a(v, \cdot)_{|Z}.
\]

But \( Z \) is reflexive, because the bilinear form \( b \) is continuous in the first variable. Hence Proposition 2.1 guarantees that the preceding assertion is equivalent to

there exists \( \alpha > 0 \) such that

\[
z \in Z \Rightarrow f(z) - a(v, z) \leq \alpha \|a(\cdot, z)_{|Z}\|.
\]

In view of the arbitrariness of \( v \in G \), we have established the equivalence (i) \( \Leftrightarrow \) (iii). Moreover, given \( v \in G \), as Proposition 2.1 ensures that \( z_v \in Z \) can be chosen with

\[
\|z_v\| = \left( \sup_{z \in Z, a(\cdot, z)_{|Z} \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)_{|Z}\|} \right) + \|v\|
\]

the triangular inequality gives that \( u \in V \) in (i) can be taken in such a way that

\[
\|u\| \leq \sup_{z \in Z, a(\cdot, z)_{|Z} \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)_{|Z}\|} + \|v\|.
\]

Since \( v \) is any element in \( G \), we arrive at

\[
\|u\| \leq \inf_{v \in G} \left( \sup_{z \in Z, a(\cdot, z)_{|Z} \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)_{|Z}\|} + \|v\| \right),
\]

but \( u \in G \) and

\[
\|u\| = \sup_{z \in Z, a(\cdot, z)_{|Z} \neq 0} \frac{f(z) - a(u, z)}{\|a(\cdot, z)_{|Z}\|} + \|u\|,
\]

so it follows that

\[
\|u\| = \min_{v \in G} \left( \sup_{z \in Z, a(\cdot, z)_{|Z} \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)_{|Z}\|} + \|v\| \right)
\]

which is the announced equality. \( \square \)

Obviously, in Theorem 2.2 \( \|a(\cdot, z)_{|Z}\| \) is the norm of \( a(\cdot, z)_{|Z} \) in \( Z^* \). Furthermore, observe that in view of Proposition 2.1 we have that

\[
G \neq \emptyset \Leftrightarrow \text{there exists } \beta > 0 \text{ such that for all } w \in W, \ g(w) \leq \beta \|b(\cdot, w)\|.
\]

In the Lax-Milgram type result established in Proposition 2.1, one can indeed suppose that \( F \) is a real Hausdorff locally convex space (the same techniques as in [3]). Thus, in Theorem 2.2 we can assume that \( W \) is such a topological space.

With regard to uniqueness we can establish a very easy technical characterization:
Lemma 2.3. Under the hypotheses and notation of Theorem 2.2, if the variational equation with constraints (2.2.1) has a solution, it is unique if, and only if,

\[ z \in Z \text{ and } a(z, \cdot)_{|Z} = 0 \Rightarrow z = 0. \]  

(2.3.1)

Proof. To start assume that \( a \) satisfies the non-degeneration condition (2.3.1). Given \( v \in G \), as \( G = v + Z \), then if (2.2.1) admits two solutions \( u_1 \) and \( u_2 \), we have that, for some \( z_1, z_2 \in Z \),

\[ u_1 = v + z_1 \text{ and } u_2 = v + z_2, \]

hence

for all \( z \in Z \),

\[ a(z_1 - z_2, z) = 0. \]

By hypothesis it follows that \( z_1 = z_2 \), that is, \( u_1 = u_2 \).

And conversely, if there exists \( z_0 \in Z \setminus \{0\} \) such that

\[ z \in Z \Rightarrow a(z_0, z) = 0, \]

then given a solution \( u \) of (2.2.1), \( u + z_0 \) is clearly a solution as well, which is different than \( u \). \( \square \)

In the next results we introduce some more restrictive hypotheses than those of Theorem 2.2, but they guarantee uniqueness of solution and moreover simplify the control of the norm of the solution:

Corollary 2.4. Let \( V \) be a real reflexive Banach space, let \( W \) be a real normed space and let \( a : V \times V \to \mathbb{R} \) and \( b : V \times W \to \mathbb{R} \) be bilinear forms such that \( a \) is continuous and satisfies (2.3.1) and

\[ w \in W \Rightarrow b(\cdot, w) \in V^*. \]

Let \( g \in W^* \), let us take

\[ Z := \{ v \in V : b(v, \cdot) = 0 \} \]

and suppose that there exist constants \( \lambda, \beta > 0 \) with

\[ z \in Z \Rightarrow \lambda \| z \| \leq \| a(\cdot, z)_{|Z} \| \]  

(2.4.1)

and

\[ w \in W \Rightarrow g(w) \leq \beta \| b(\cdot, w) \|. \]  

(2.4.2)

Then, for all \( f \in V^* \) the corresponding constrained variational equation admits a unique solution, that is,

\[ \text{there exists a unique } u \in V \text{ such that } \begin{cases} 
z \in Z & \Rightarrow a(u, z) = f(z) \\
w \in W & \Rightarrow b(u, w) = g(w) \end{cases}. \]

Besides, the solution \( u \) satisfies the following a priori estimate:

\[ \| u \| \leq \frac{\| f \|}{\lambda} + \left( 1 + \frac{\| a \|}{\lambda} \right) \min_{v \in G} \| v \| \]

\[ \leq \frac{\| f \|}{\lambda} + \left( 1 + \frac{\| a \|}{\lambda} \right) \beta. \]
Proof. The existence of a solution follows from Proposition 2.1 and Theorem 2.2 and its uniqueness from Lemma 2.3. Furthermore, Theorem 2.2 ensures that the solution \( u \) satisfies the identity
\[
\|u\| = \min_{v \in G} \left( \sup_{z \in Z, \ a(\cdot, z) \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)\|} + \|v\| \right).
\]
But, as a consequence of (2.4.1),
\[
\min_{v \in G} \left( \sup_{z \in Z, \ z \neq 0} \frac{f(z) - a(v, z)}{\|a(\cdot, z)\|} + \|v\| \right)
= \min_{v \in G} \left( \sup_{z \in Z, \ z \neq 0} \frac{f(z)}{\|a(\cdot, z)\|} + \|v\| \right)
\leq \inf_{v \in G} \left( \sup_{z \in Z, \ z \neq 0} \frac{f(z) + \|a\|\|v\|\|z\|}{\lambda\|z\|} + \|v\| \right)
= \frac{\|f\|}{\lambda} + \left( 1 + \frac{\|a\|}{\lambda} \right) \inf_{v \in G} \|v\|,
\]
and since by Proposition 2.1 and (2.4.2)
\[
\inf_{v \in G} \|v\| = \min_{v \in G} \|v\|
= \sup_{w \in W} \frac{g(w)}{\|b(\cdot, w)\|}
\leq \beta,
\]
we have the announced bound.

Remark 2.5. When the bilinear form \( b \) is nondegenerate in the second variable \((b(\cdot, w) = 0 \Leftrightarrow w = 0)\), the norm in \( W \)
\[
|w| := \|b(\cdot, w)\|, \quad (w \in W),
\]
allows us to arrive at an explicit expression of \( \min_{v \in G} \|v\| \), in view of Proposition 2.1:
\[
\min_{v \in G} \|v\| = |g|.
\]
Consequently, the bound for the norm of the solution of the constrained variational equation in Corollary 2.4 takes the form
\[
\|u\| \leq \frac{\|f\|}{\lambda} + \left( 1 + \frac{\|a\|}{\lambda} \right) |g|.
\]
In particular, if we assume a stronger condition than (2.4.2), specifically, that there exists \( \beta > 0 \) such that
\[
w \in W \Rightarrow \|w\| \leq \beta \|b(\cdot, w)\|,
\]
then \( b \) is nondegenerate in the second variable and moreover
\[
|g| \leq \beta |g|,
\]
and hence we have the stability estimate
\[ \|u\| \leq \left( \frac{\|f\|}{\lambda} + \left( 1 + \frac{\|a\|}{\lambda} \right) \beta \|g\| \right). \]

The assumption (2.5.1) is referred to as the Babuška-Brezzi condition (see [4, 5], and [6] for some recent progress). Note that it can be equivalently reformulated as
\[ \inf_{w \in W} \sup_{v \in V, v \neq 0} \frac{b(v, w)}{\|v\|} > 0, \]
which also originates the terminology inf-sup condition for (2.5.1).

Bearing in mind that conditions (2.3.1) and (2.4.1) are satisfied when \( a \) is coercive on \( Z \times Z \), we deduce the following immediate consequence, which is well-known (see [1, §II.1 Proposition 1.1] or [2, Lemma 4.67]) in the particular case of \( V \) and \( W \) being Hilbert spaces, even though in those references it is not established that \( \inf_{v \in G} \|v\| \) is attained, nor any estimation of such a minimum:

**Corollary 2.6.** Assume that \( V \) is a reflexive real Banach space, \( W \) is a real normed space, \( f \in V^* \), \( g^* \in W^* \) and \( a : V \times V \rightarrow \mathbb{R} \) and \( b : V \times W \rightarrow \mathbb{R} \) are bilinear forms in such a way that \( a \) is continuous and
\[ w \in W \Rightarrow b(\cdot, w) \in V^*. \]
Suppose in addition that, taking
\[ Z := \{ v \in V : b(v, \cdot) = 0 \} \quad \text{and} \quad G := \{ v \in V : b(v, \cdot) = g \}, \]
there exists \( \lambda > 0 \) such that
\[ z \in Z \Rightarrow \lambda \|z\|^2 \leq a(z, z) \]
and that \( G \neq \emptyset \). Then
\[ \text{there exists a unique } u \in V \text{ such that } \begin{cases} z \in Z \Rightarrow a(u, z) = f(z) \\ w \in W \Rightarrow b(u, w) = g(w) \end{cases}. \]
Furthermore, the a priori estimate
\[ \|u\| \leq \frac{\|f\|}{\lambda} + \left( 1 + \frac{\|a\|}{\lambda} \right) \min_{v \in G} \|v\| \]
is valid. In particular, if there exists \( \beta > 0 \) such that \( w \in W \Rightarrow \|w\| \leq \beta \|b(\cdot, w)\| \), then the norm of \( u \) satisfies the following estimation:
\[ \|u\| \leq \frac{\|f\|}{\lambda} + \left( 1 + \frac{\|a\|}{\lambda} \right) \beta \|g\|. \]

Let us take up again the constrained variational equation associated with the elliptic boundary value problem considered in the Introduction. Then, maintaining the notation used throughout this paper, it holds that \( G \neq \emptyset \) because \( g_0 \in H^{1/2}(\Gamma) \). In addition, as a consequence of the Poincaré-Friedrichs inequality, \( a \) is coercive on \( Z \), and so Corollary 2.6 applies.
In Theorem 2.2 and Lemma 2.3 we characterize when the variational equation with constraints admits a unique solution, for two fixed functionals \( f \in V^* \) and \( g \in W^* \). Finally, in view of Corollary 2.4, Lemma 2.3, Remark 2.5 and the uniform boundedness theorem it is not difficult to derive the following result, for arbitrary functionals \( f \in V^* \) and \( g \in W^* \):

**Corollary 2.7.** Let \( V \) be a real reflexive Banach space, let \( W \) be a real normed space and let \( a : V \times V \rightarrow \mathbb{R} \) and \( b : V \times W \rightarrow \mathbb{R} \) be bilinear forms such that \( a \) is continuous and

\[
w \in W \Rightarrow b(·, w) \in V^*.
\]

Let \( Z := \{v \in V : b(v, ·) = 0\} \). Then, for all \( f \in V^* \) and for all \( g \in W^* \)

there exists a unique \( u \in V \) such that

\[
\begin{align*}
z \in Z &\Rightarrow a(u, z) = f(z) \\
w \in W &\Rightarrow b(u, w) = g(w)
\end{align*}
\]

if, and only if,

\[
z \in Z \text{ and } a(z, ·)|_Z = 0 \Rightarrow z = 0
\]

and

\[
\begin{align*}
z \in Z &\Rightarrow \lambda \|z\| \leq \|a(·, z)|_Z\| \\
w \in W &\Rightarrow \|w\| \leq \beta \|b(·, w)\|
\end{align*}
\]

In addition, if one of these equivalent conditions is satisfied, we have the following stability estimate:

\[
\|u\| \leq \frac{\|f\|}{\lambda} + \left(1 + \frac{\|a\|}{\lambda}\right) \beta \|g\|.
\]

The main tool in the proof of our results is the generalization of the Lax-Milgram theorem that appears in Proposition 2.1. We refer to [7] for other general inequality formulations.

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**References**
