A cyclical square-root model for the term structure of interest rates

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ABSTRACT
This paper presents a cyclical square-root model for the term structure of interest rates assuming that the spot rate converges to a certain time-dependent long-term level. This model incorporates the fact that the interest rate volatility depends on the interest rate level and specifies the mean reversion level and the interest rate volatility using harmonic oscillators. In this way, we incorporate a good deal of flexibility and provide a high analytical tractability. Under these assumptions, we compute closed-form expressions for the values of different fixed income and interest rate derivatives. Finally, we analyze the empirical performance of the cyclical model versus that proposed in Cox et al. (1985) and show that it outperforms this benchmark, providing a better fitting to market data.

1. Introduction

Through the time, modeling the term structure of interest rates (TSIR) has been the object of many studies and the aim of attention for economists and financial institutions. This paper proposes a cyclical square-root model where the instantaneous interest rate is pulled back to a certain time-dependent long term level characterized by an harmonic oscillator. Therefore, assuming a time-dependent mean reversion level will derive in a time-dependent spot rate volatility. Empirical evidence (see, for instance, Amin & Morton (1994) Chan, Karolyi, Longstaff, & Sanders (1992)) illustrated that interest rate volatility depends on the interest rates level. Then, it seems natural to model interest rate volatility using a similar functional form as that in the mean reversion level. Models proposed in the academic literature can be classified in endogenous and exogenous. Endogenous models make certain assumptions on the factors that drive the TSIR and their stochastic processes. The TSIR is fully characterized by these factors meaning that the current TSIR is output rather than an input of the model. Examples of one-factor models are Brennan and Schwartz (1980), Cox, Ingersoll, and Ross (1985), or Vasicek (1977) (CIR from now on). The main drawback of these models is the lack of empirical realism as they do not fit accurately the current TSIR and, consequently, do not price correctly fixed income assets. In order to cope with this problem, we can find two-factor models such as, for instance, Cox et al. (1985), Longstaff and Schwartz (1992), or Schaefer and Schwartz (1984) while Babbs and Nowman (1999), Balduzzi, Das, Foresi, and Sundaram (1996), Beaglehole and Temny (1991), Chen (1996), Dai and Singleton (2000), and Duffie and Kan (1996) introduced and analyzed different multi-factor models.

On the other hand, exogenous models consider the current TSIR as an input and derive future changes in interest rates avoiding intertemporal arbitrage opportunities. The first contribution was made by Ho and Lee (1986) who showed how to build a model consistent with the initial TSIR. Since this model has some disadvantages, their work has been extended by a number of authors such as Abaffy, Bertocchi, and Gnudi (2005), Black, Derman, and Toy (1990), Black and Karasinski (1991), Brigo, Mercurio, and Morini (2005), Heath, Jarrow, and Morton (1992), Hull and White (1990, 1993), and Mercurio and Moraleda (2000).


Derivative markets trade a huge volume of contracts on a daily basis and derivatives pricing has become an issue of utmost
importance. Despite the great progress in this matter, there is still a trade-off between analytical tractability and empirical accuracy. In this paper, we introduce a model where the mean reversion level and the spot rate volatility follow a cyclical process characterized by an harmonic oscillator. This cyclical model provides great flexibility to reflect the different shapes that the TSIR can exhibit empirically and provides a high analytical tractability, allowing an accurate fitting of the TSIR and constituting a powerful pricing tool. Under this framework, we analytically price zero-coupon bonds and different derivatives such as forward on bonds, European options on zero-coupon and coupon-bearing bonds, European bond forward options, swaps, swaptions, caps, floors, collars, and provide some risk management measures. Finally, we analyze the empirical performance of this model versus its natural benchmark, the CIR model. We show that, for the data set used in this analysis, the cyclical model outperforms this benchmark, providing a much better fitting to current data for every time horizon.

This paper is organized as follows. Section 2 introduces the cyclical model and its practical implications. Section 3 presents the general pricing partial differential equation and derives closed-form expressions for different derivatives. Section 4 presents the empirical analysis of the model. Finally, Section 5 summarizes the main findings and conclusions. Mathematical proofs are deferred to Appendix A.

2. A cyclical square-root model for the term structure

In this section, we propose our model and the specific functional form for each time-dependent parameter, and describe all the practical implications arising from this model.

Let \( r_t \) denote the instantaneous interest rate available at time \( t \) whose dynamics is

\[
dr_t = \mu_t \, dt + \sigma_t \, dW_t
\]

where \( W_t \) is a standard Wiener process and

\[
\mu_t = \kappa (\theta_t - r_t)
\]

\[
\sigma_t = \sigma \sqrt{\theta_t}
\]

where \( \kappa \in \mathbb{R}^+ \).

Consider the harmonic oscillator given as \( f(t) = A \sin(\varphi - \omega t) \), where \( A, \varphi, \) and \( \omega \) denote the amplitude, offset phase, and temporal frequency, respectively. This function provides a simple and flexible functional form to represent a cyclical behavior. In addition, working with an harmonic oscillator instead of a high-order polynomial provides a good deal of analytical tractability.

Departing from this harmonic oscillator, we assume that the mean reversion level, \( \theta_t \), and the volatility, \( \sigma_t^2 \), in Eqs. (2) and (3), are defined as

\[
\theta_t = A_0 \sin^2(\varphi - \omega t)
\]

\[
\sigma_t^2 = A_0 \sin^2(\varphi - \omega t)
\]

These specific expressions guarantee the positiveness of the mean reversion level and the interest rate volatility. It is clear that this model nests the CIR one taking \( \omega = 0 \) in Eqs. (4) and (5). Note that we incorporate two additional parameters, phase and frequency, with respect to the CIR model.

For square-root processes of this type, Cox et al. (1985) shows that the distribution function of interest rates is a rescaled non-central chi-square with \( \delta \) degrees of freedom. Note that, whenever \( \delta \) is not a positive integer, the distribution of \( r_t \) is unknown. Besides, the dimension of the process \( r_t \) is given by \( \delta = \frac{\ln a}{\ln \varphi} \). Eqs. (4) and (5) illustrate that both waves are in phase, then the model’s dimension can be represented as \( \delta = \frac{\ln a}{\ln \varphi} > 0 \).\(^2\)

\(^2\) Note that, if \( \sin(\varphi - \omega t) \) is equal to zero, then \( \delta \) becomes indeterminate. As this case would only occur for a infinitesimal period of time, we do not consider this possibility.

Our model guarantees the positiveness of interest rates. On this respect, Feller (1951) studied the Fokker–Plank–Kolmogorov equation for the transition density and showed that \( r_t > 0 \) if \( \delta > 2 \), however it can become null if \( \delta < 2 \) but will never become negative.

3. Pricing

This section presents closed-form expressions for the price of zero-coupon bonds and, later, we analytically compute closed-form formulas for the prices of different securities.

Let \( P(r_t, t, T) \) denote the price at time \( t \) of a zero-coupon bond that pays $1 at maturity \( T \). Then, the bond price dynamics is given by the process

\[
dP_t = \mu_t (r_t, t, T) \frac{P_t}{P_t} (r_t, t, T) \, dt + \sigma_t (r_t, t, T) \frac{P_t}{P_t} (r_t, t, T) \, dW_t
\]

Applying Itô’s Lemma and using (1), it can be shown that

\[
\mu_t (r_t, t, T) = \frac{1}{P_t} \left( P_t + \mu_t P_t + \frac{1}{2} \sigma_t^2 P_t \right)
\]

\[
\sigma_t (r_t, t, T) = \sigma_t \frac{P_t}{P_t}
\]

where subscripts in \( P \) indicate the corresponding partial derivative. Applying standard no-arbitrage arguments, there exists a factor \( \lambda(t, r) \), called market price of risk, such that

\[
\mu_t (r_t, t, T) - r_t = \lambda(r_t, t, T) \sigma_t (r_t, t, T)
\]

Finally, some trivial algebra leads to the following partial differential equation (PDE)

\[
P_t (r_t, t, T) + (\mu_t - \lambda(r_t, t, T) \sigma_t) P_t (r_t, t, T) + \frac{1}{2} \sigma_t^2 P_{rr} (r_t, t, T)
\]

\[ - r_t P_{rr} (r_t, t, T) = 0 \]

that must be verified by the price of any derivative.

3.1. Bond pricing

Similarly to Cox et al. (1985), we consider a market price of risk such as

\[
\lambda(t, r) = \frac{\lambda \sqrt{\theta_t}}{\sigma_t}
\]

Using expressions (2), (3), and (11), the PDE (10) becomes

\[
P_t (r_t, t, T) + (\kappa (\theta_t - r_t) - \lambda \sqrt{\theta_t}) P_t (r_t, t, T) + \frac{1}{2} \sigma_t^2 P_{rr} (r_t, t, T)
\]

\[ - r_t P_{rr} (r_t, t, T) = 0 \]

The solution of this PDE, subject to the boundary condition \( P(r_t, T, T) = 1 \), \( \forall r_t \) is given by the following Proposition.

Proposition 1. The price at time \( t \) of a zero-coupon bond with maturity \( T \) and $1 face value is given by

\[
P(r_t, t, T) = A(\tau) e^{-B(\tau)l}
\]

where

\[
A(\tau) = \exp \left( \int_t^T \kappa \theta(t) \, dt \right)
\]

\[
B(\tau) = \frac{1}{4} \left( \frac{\lambda}{\sigma_t^2} \right) \left( (c_1 C(a, q, x) + MC(a, q, x)) \right) + \frac{1}{4} \left( \frac{\lambda}{\sigma_t^2} \right) \left( (c_1 M(a, q, x) + MC(a, q, x)) \right)
\]

\[
a = \frac{A_0^2 + (\kappa)^2}{4 \omega^2}
\]

\[
q = \omega - \omega \varphi
\]

\[
x = \varphi \omega - \omega \varphi
\]

\[
c_1 = \frac{\omega}{\omega - \omega \varphi}
\]

\[
\tau = T - t
\]

where \( \theta_t \) is given by (4), \( MC \) and \( MS \) represent the Mathieu cosine and sine function, respectively, and \( MCP \) and \( MSP \) represent the derivative
with respect to \( x \) of the Mathieu cosine and sine function, respectively. □

**Proof.** See Appendix A. □

**Remark 1.** An interesting approximation arises for \( q \approx 0 \), that is, periods of low volatility where the underlying frequency in the Mathieu function is relatively high. Satisfying this requirement will derive in

\[
MC(a, q, x) \approx \cos \left( \sqrt{a}x \right), \quad MS(a, q, x) \approx \frac{\sin \left( \sqrt{a}x \right)}{\sqrt{a}}
\]

In Fig. 1 we plot the discount function for four arbitrary set of parameters in the cyclical model against the CIR model. We can see the ability of our proposed model to fit different shapes of the term structure of bond prices, adding much more flexibility with the same analytical tractability as in the CIR model.

The following Corollary immediately arises.

**Corollary 1.** As a coupon bond can be interpreted as a portfolio of zero-coupon bonds, pricing of coupon bonds is straightforward applying Proposition 1. □

Replacing the bond price expression obtained in Proposition 1 into (7), (8) and using Eqs. (2)–(11), we get the next Corollary.

**Corollary 2.** The bond price dynamics under the no-arbitrage condition is given as

\[
dP(r_t, t, T) = \mu_p(r_t, t, T)P(r_t, t, T)dt + \sigma_p(r_t, t, T)P(r_t, t, T)dW_t
\]

where \( \mu_p(r_t, t, T) = r_t(1 - \kappa)B \), \( \sigma_p(r_t, t, T) = -B\sigma\sqrt{r_t} \). □

Note that, in a risk-neutral world, where \( \kappa = 0 \), the bond price process is a martingale. Under this framework, the TSIR is fully characterized considering the zero-coupon bond price \( P(r_t, t, T) \) given by Proposition 1, as stated in the following Corollary.

**Corollary 3.** The yield to maturity, \( R(r_t, t, T) \), is given by

\[
R(r_t, t, T) = -\frac{1}{\tau} \ln P(r_t, t, T), \quad \tau = T - t
\]

The short-term interest rate is defined as the instantaneous interest rate at time \( t \), that is,

\[
r_t = \lim_{\tau \to 0} R(r_t, t, T) = R(r_t, t, t)
\]

The instantaneous forward rate is given as

\[
f(r_t, t, T) = -\frac{\partial \ln P(r_t, t, T)}{\partial T}.
\]

Fig. 2 shows the TSIR for four different set of parameters in the cyclical model against the CIR model. We can see that our model is capable of replicating several different shapes of the TSIR such as upward sloping, downward sloping, humped, and inverted humped.

For illustrative purposes, Figs. 3 and 4 show how the TSIR responds to changes in the mean reversion speed and volatility in both models. In the CIR model, the higher the speed of mean reversion, the higher the interest rate while, in the cyclical model, the lower the mean reversion speed, the flatter the TSIR. Besides, in this model, there is a twist in the pattern due to the cyclical behavior. In Fig. 4, for both models, the higher the volatility, the lower the TSIR.

Figs. 5 and 6 reflect the response of the TSIR to different values of the mean reversion level parameter in both models. In the CIR model, the higher the mean reversion level, the higher the yield. In the cyclical model, it is harder to analyze this response as it depends on three parameters. Anyway, we observe that the lower the amplitude, the flatter and the lower the TSIR. When changing the temporal frequency, it seems clear that the higher the temporal frequency, the more humped the TSIR. Finally, for different offset phases, the curves occasionally crossover each other.

On the risk management side, we get the following Corollary.

**Corollary 4.** The two major bond risk measures, duration and convexity, are given as

- **Duration** measures the bond price sensitivity for a change in interest rates:
  
  \[
  \text{Duration} = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial r_t} = B(t, T)
  \]

- **Convexity** measures how duration changes with interest rates:
  
  \[
  \text{Convexity} = -\frac{1}{P(t, T)} \frac{\partial^2 P(t, T)}{\partial r_t^2} = B^2(t, T)
  \]

with \( B(t, T) \) as given by Proposition 1. □

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**Fig. 1.** Discount function for an arbitrary set of parameters. In all the cases, \( r_0 = 0.015, \kappa = 0 \). For the CIR model (light blue line) \( \theta = 0.1, \sigma = 0.005, \kappa = 0.1 \). For the cyclical model, we consider (a) \( \Lambda_0 = 0.2, \Lambda_0 = 0.001, \kappa = 0.1, \omega = 0.08, \phi = \pi \) (blue line), (b) \( \Lambda_0 = 0.1, \Lambda_0 = 0.005, \kappa = 0.15, \omega = 0.2, \phi = \pi /2 \) (red line), (c) \( \Lambda_0 = 0.08, \Lambda_0 = 0.002, \kappa = 0.15, \omega = 0.15, \phi = \pi /2 \) (black line), and (d) \( \Lambda_0 = 0.1, \Lambda_0 = 0.002, \kappa = 0.3, \omega = 0.10, \phi = \pi \) (green line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
3.2. Pricing of bond derivatives

This section derives the closed-form expression for the price of any interest rate derivative. For this endeavor, we will move from the risk-neutral to the $s$-forward measure, determining the corresponding probability law. First, consider a derivative whose payoff at time $s$ is given by $U_s(r_s)$. Applying one of the fundamental results of Heath et al. (1992), there exists a unique equivalent risk-neutral measure $\tilde{P}$ such that the value at time $t$ of this derivative $U_t$ can be represented as

$$U_t(r_t, t, s) = \tilde{E} \left[ e^{-\int_t^s r_u du} U_s(r_s) \right]$$

where $\tilde{E}$ denotes expectation with respect to the risk-neutral measure $\tilde{P}$, and the Wiener process under this measure is defined as

$$W_t = W_t + \int_0^t A_u du$$

Under this measure, applying (6), (9), and (13), the risk-neutral dynamics of the zero-coupon bond price is given as

$$dP(r_t, t, s) = r_t P(r_t, t, s) dt + \sigma_t P(r_t, t, s) d\tilde{W}_t$$

Hence, the discount bond price process is a martingale.

---

3. Obviously, if $U_t(r_t) = 1$, the previous bond price expression is obtained.
Applying Itô’s Lemma to $g(P(rt, t, s)) = \ln P(rt, t, s)$ and integrating from 0 to $t$, we get

$$P(rt, t, s) = P(r_0, 0, s) \times \exp \left( \int_0^t r_u du + \int_0^t \sigma_r(u, s) dW_u - \frac{1}{2} \int_0^t \sigma_r^2(u, s) du \right)$$

Hence, for each $s \in [t, T]$, the process $Z_t$ defined as

$$Z_t = \frac{P(rt, t, s)}{P(r_0, 0, s)} e^{-\int_0^t r_u du} = \exp \left( \int_0^t \sigma_r(u, s) dW_u - \frac{1}{2} \int_0^t \sigma_r^2(u, s) du \right)$$

is a martingale. Moreover, in line with Karatzas and Shreve (2012), we get...
Thus, by the Girsanov’s theorem, for each $s \in [t, T]$, there exists an equivalent $s$-forward measure $P^s$ such that

$$W_t^s = W_t + \int_0^t \sigma_p(u, s) du$$

where $W_t^s$ represent a standard Wiener process under $P^s$.

Karatzas and Shreve (2012) shows that, for a random variable $Y$, we get

$$E^s[Y|F_t] = \frac{1}{Z_t^s} E[Z_t^s Y|F_t]$$

Then, the following Proposition presents the equivalent change of measure.

**Proposition 2.** Under $\tilde{P}$, the value at time $t$ of any derivative $U_t(r_t, t, s)$ given by

$$U_t(r_t, t, s) = \tilde{E} \left[ e^{-\int_t^T \sigma_{\tilde{r}}(u, r_t) du} U_t(r_t) | r_t \right]$$

has an equivalent $s$-forward measure, $P^s$, such that

$$U_t(r_t, t, s) = P(r_t, t, s) \tilde{E} \left[ \frac{Z_t^s}{Z_t} U_t(r_t) | r_t \right] = P(r_t, t, s) \tilde{E} [U_t(r_t) | r_t]$$

where $\tilde{E}$ represents expectation under $\tilde{P}$ and

$$Z_t^s = e^{-\int_t^T \sigma_{\tilde{r}}(u, r_t) du}.$$ 

Now, we need to determine the probability law governing the random variable $r_t$. For such reason, consider the instantaneous forward rate given as

$$f(r_t, t, T) = D(t, T) + B_t(t, T) r_t$$

with

$$D(t, T) = -A_T(t, T) - \delta \int_t^T B_t(u, T) \sigma_u^2 du$$

Applying Itô’s Lemma and using the spot rate dynamics (1), we get

$$df(r_t, t, T) = \mu_t dt + \sigma_t dW_t$$

where $\mu_t = (B_T(t, T) - \kappa B_T(t, T)) r_t$. $\sigma_t = B_T(t, T) \sigma_t^{\sqrt{T_t}}$ Similarly to Heath et al. (1992), the following restriction on the forward rate drift is verified

$$\mu_t(\omega, t, T) = \sigma_t(\omega, t, T) \left[ \int_t^T \sigma_t(\omega, t, x) dx + A_t \right]$$

Now, replacing (14) into the forward rate process (17), we get

$$df(r_t, t, s) = \sigma_t(\omega, t, T) \left[ \int_t^T \sigma_t(\omega, t, x) dx + A_t \right] dt + \sigma_t(dW_t^s - A_t dt - \sigma_t dt)$$

Hence, using $\int_t^T \sigma_t(\omega, t, v) dv = \sigma_t(\omega, t, s)$ leads to

$$df(r_t, t, s) = \sigma_t(\omega, t, s) dW_t^s = B_t(t, s) \sigma_t^{\sqrt{T_t}} dW_t^s$$

Then, comparing Eqs. (17) and (18), we get

$$dr_t = \kappa (\theta_r - r_t) dt + \sigma_t^{\sqrt{T_t}} dW_t^s$$

where $\kappa = \frac{\sigma_t^{\sqrt{T_t}}}{\sigma_t^{\sqrt{T_t}}}$, $\theta_r = \theta_r$. Hence, under the $s$-forward measure, the instantaneous interest rate follows a CIR-type process with speed and level of mean reversion given by $\kappa$ and $\theta_r$, respectively. Then, standard methods applied in Cox et al. (1985) can be used.

Define the state variable $X(t) = (x_1(t), \ldots, x_d(t))$ for an arbitrary number $d$ as the process generating $r_t = ||X(t)||^2$. The state variable dynamics for $x_1(t)$ is given by

$$dx_1(t) = \frac{1}{2} \sigma_t^{\sqrt{T_t}} B_t(s, t) dW_t^s(t)$$
Hence,
\[ r_s = \sum_{i=1}^{d} \left[ \int_{r_i}^{s} \sigma_u \sqrt{B_i(u,s)} dW_i(u) + X^i(u) \right]^2 \]
where \( W^i(t) = (W^i_1(t), \ldots, W^i_d(t)) \) is a d-dimensional Wiener process under \( P^r \).

Note that, under \( P^r \), the instantaneous forward rate can be represented as
\[ f(t,s) = ||X(t)||^2 + D(t,s) \]
Taking conditional expectations under \( P^r \) in (19) and using (15), (16) and (20), we get
\[ E_p[r_i|r_i] = D(t,s) + B_i(t,s) r_i \]
Taking \( \delta = \frac{\delta}{\delta/s} \) and defining \( \sigma = \frac{1}{\sigma/s} r_i \), we obtain
\[ E_p[\sigma|r_i] = \delta + \xi_1 \]
where
\[ \xi_1 = \frac{\delta B_i(t,s)}{D(t,s)} \]
Hence, \( \sigma \) follows a non-central chi-square distribution with \( \delta \) degrees of freedom and non-centrality parameter \( \xi_1 \).

Applying Proposition 2, the next proposition arises.

**Proposition 3.** The value at time \( t \) of any interest rate derivative with terminal pay-off \( U_i(r_i) \) is given by
\[ U_i(r_i,t,s) = P(r_i,t,s) E_p[U_i(\sigma)|r_i] \]
where \( P(r_i,t,s) \) is given by Proposition 1, \( E_p \) represents expectation under \( P^r \), and \( \sigma \sim \chi^2(\delta, \xi_1) \), with \( \xi_1 \) as given by (21).

After obtaining this general closed-form expression, we analyze several particular cases:

**3.2.1. Forward on zero-coupon bond**
Consider a forward contract expiring at time \( s \) written on a zero-coupon bond maturing at time \( T > s \) and \( S \) face value. Then, under the \( s \)-forward measure \( P^s \), the delivery price established at time \( t \) for this forward contract is given as
\[ F(r_i,t,s,T) = E_p[P(r_i,t,s,T)|r_i] \]
Then, using Proposition 3, the value of this bond forward is given as follows.

**Proposition 4.** The value at time \( t \) of a bond forward maturing at time \( s \), written on a zero-coupon bond expiring at time \( T > s \) and \( S \) face value is given by
\[ F(r_i,t,s,T) = E_p[P(r_i,t,s,T)|r_i] = \frac{A(s,T) e^{\delta(s-T)}}{(B(s,T)D(t,s) + 1)^2} \]
where \( A(s,T) \) and \( B(s,T) \) are given by Proposition 1, \( D(t,s) \) as given by (16), and
\[ \xi_1 = \frac{\delta B_i(t,s)}{D(t,s)} \]
\[ \xi_2 = \frac{1}{2 B_i(t,s) D(t,s) + 1} \]

**3.2.2. European option on zero-coupon bond**
Consider a European call option maturing at time \( s \) with strike \( K \), written on a zero-coupon bond that matures at time \( T > s \). Let \( c_i(r_i,s,T) \) denote the price at time \( t \) of an European call option. Then, the boundary condition of the PDE (12) is given by
\[ c_i(r_i,s,T) = \max\{P(r_i,s,T) - K, 0\} \]
Under the risk-neutral measure \( \hat{P} \), the price at time \( t \) of this call option is given by
\[ c_i(r_i,t,s,T) = \hat{E}_r [e^{-\int_{t}^{s} r_i dt} (P(r_i,s,T) - K)^{+} | r_i] \]
Then, using Proposition 3, we price this option as follows.

**Proposition 5.** The price at time \( t \) of a European call option with maturity \( s \) written on a zero-coupon bond expiring at time \( T > s \) face value is given by
\[ c_i(r_i,t,s,T) = \hat{E}_r [A(s,T) e^{-B(s,T) r_i} - K)^{+} | r_i] \]
\[ = \hat{E}_r [P(r_i,t,s)F(r_i,t,s,T) \chi^2(\rho_2, \delta, \xi_2) - P(r_i,t,s)K \chi^2(\rho_1, \delta, \xi_1) \]
where \( \chi^2(\cdot) \) denotes the non-central chi-square distribution function and
\[ \rho_1 = \frac{\delta}{B(s,T)D(t,s)} \ln \left( \frac{A(s,T)}{K} \right) \]
\[ \rho_2 = \frac{\rho_1}{B(s,T)D(t,s) + 1} \]
\[ \xi_1 = \frac{\delta B_i(t,s)}{D(t,s)} \]
\[ \xi_2 = \frac{\rho_1}{\rho_2} \xi_1 \]
with \( P(r_i,t,s), A(\cdot), \) and \( B(\cdot) \) as given in Proposition 1, \( F(r_i,t,s,T) \) as given by Proposition 4, and \( D(t,s) \) as given by Eq. (16).

**Proof.** See Appendix A.

**3.2.3. European option on coupon bond**
Consider a European call option that matures at time \( s \) and strike \( K \). The underlying asset is a coupon bond maturing at time \( T \) paying \( N \) coupons \( z_i \) at times \( j_i, i = 1, 2, \ldots, N \) where \( j_i > s, j_N = T \). The price of this coupon bond at time \( s \) is given as the sum of the corresponding zero-coupon bonds, that is,
\[ P(r_i,s,T) = \sum_{i=1}^{N} z_i P(r_i,s,j_i) \]
where \( P(r_i,s,j_i), i = 1, 2, \ldots, N \) is given by Proposition 1.

Let \( c_i(r_i,s,j_i) \) denote the price at time \( t \) of this call option. Using (22), the boundary condition of the PDE (12) becomes now
\[ c_i(r_i,s,j_i,N) = \max\{\sum_{i=1}^{N} z_i P(r_i,s,j_i) - K, 0\} \]
Applying Proposition 3, the call option price is given as
\[ c_i(r_i,s,j_i,N) = \hat{E}_r [\hat{F}_r (\sigma, s, j_i,N) \max\{\sum_{i=1}^{N} z_i P(\sigma, s, j_i) - K, 0\} | r_i] \]
In line with Jamshidian (1989), we find \( K_i, i = 1, 2, \ldots, N \) such that
\[ \max\{\sum_{i=1}^{N} z_i P(\sigma, s, j_i) - K, 0\} = \sum_{i=1}^{N} z_i \max\{P(\sigma, s, j_i) - K, 0\} \]
where \( K_i = P(k_i, s, j_i) \) and \( k' \) is the solution of \( \sum_{i=1}^{N_n} c_i P(k', s, j_i) = K_4. \)

Hence, this option can be interpreted as a portfolio of European call options on zero-coupon bonds with “appropriate” strikes \( K_i \) as stated in the following Proposition.

**Proposition 6.** The price at time \( t \) of a European call option with maturity \( s \) on a coupon bond expiring at \( T \), paying coupons \( c_i \) at times \( j_i \), \( t = 1, 2, \ldots, N \) is given by

\[
c_i(r, s, j_i, K_i) = \sum_{i=1}^{N_n} c_i P(r, s, j_i, K_i)
\]

where \( c_i(r, s, j_i, K_i) \) is given by Proposition 5. \( \square \)

### 3.2.4. European bond forward option

Consider a European bond forward call option maturing at time \( s \) with strike \( K \), where the underlying asset is a bond forward contract with expiration date \( T_f \) written on a zero-coupon bond that matures at time \( T_b > T_f > t > s \) and \$1 face value. Let \( c_i(r, s, s, T_f, K) \) denote the price at time \( t \) of this call option. Then, the boundary condition for the PDE (12) is given by

\[
c_i(r, s, s, T_f, K) = \max \{ F(r, s, s, T_f) - K, 0 \}
\]

Under the risk-neutral measure \( \hat{P} \), the price at time \( t \) of this option is given as

\[
c_i(r, s, T_f, K) = \hat{E} \left[ e^{-\int_t^{T_f} r_s ds} (F(r, s, T_f) - K)^+ \right]^t
\]

Applying Proposition 3, the price of this option is given by the following Proposition.

**Proposition 7.** The price at time \( t \) of a European bond forward call option that matures at time \( s \) on a forward contract expiring at time \( T_f \) written on a zero-coupon bond maturing at time \( T_b \) and \$1 face value is given by

\[
c_i(r, s, T_f, K) = P(r, t, s) \hat{E} \left[ (F(r, s, T_f, T_b) - K)^+ \right]^t
\]

\[
= P(r, t, s) \Theta (r, s, s, T_f, T_b) \chi^2 (\rho, \delta, \xi, \xi_1)
\]

where \( \chi^2 (\cdot) \) denotes the non-central chi-square distribution function and

\[
\Theta (r, s, s, T_f, T_b) = \left( \mu_i^2 \delta_i \right)^2 A(T_f, T_b) e^{-\frac{\delta_i (1 - \delta_i)}{2}}
\]

\[
\mu_i^2 = \frac{1}{\nu} D(s, s, T_f) B(T_f, T_b) + 1
\]

\[
\delta_i = \frac{2}{\nu} \mu_i^2 \delta_i D(s, s, T_f) B(T_f, T_b) + 1
\]

\[
\rho_i^2 = 2 \left( 1 - \delta_i \right) \ln \left( \mu_i^2 \delta_i A(T_f, T_b) \right)
\]

with \( P(r, t, s), A(\cdot, \cdot), \) and \( B(\cdot, \cdot) \) as given in Proposition 1, \( D(\cdot, \cdot) \) as in (16), and \( \xi_1 \) as given by (21). \( \square \)

**Remark 2.** Note that \( P(r, t, s) \Theta (r, t, s, T_f, T_b) \) can not be interpreted as the forward price at time \( t \), but as the price at time \( t \) of an asset paying the forward price at time \( s \). \( \square \)

**Corollary 5.** For all the above cases, put option prices arise directly from the put-call parity. \( \square \)

---

4 Note that the existence of strikes \( K_i \) such that (23) has a solution is guaranteed as the bond price decreases with the instantaneous rate.}

### 3.3. Interest rate derivatives

In this subsection we focus our attention on pricing “pure” interest rate derivatives, that is, derivatives whose underlying is directly the interest rate. We start pricing FRAs, and then, we move to more complicated products such as swaps, caps, floors, and collars.

#### 3.3.1. Forward rate agreement

Consider a FRA with \$1 notional value and maturity \( s \), where the investor agrees to pay a fixed interest rate \( K \) and receive a floating rate with tenor \( T_f - s \). The floating rate is set at time \( s \) and the net cash-flow is received at time \( T_f > s \).

Then, under the risk-neutral measure \( \hat{P} \), the FRA value at time \( t \) is given by

\[
FRA_t(r, t, s, T_f, K) = \hat{E} \left[ e^{\int_t^{T_f} r_s ds} (F(r, s, T_f, s) - K)^+ \right]^t
\]

Applying Proposition 3, the value of this FRA is given by the following Proposition.

**Proposition 8.** The value at time \( t \) of a FRA with \$1 notional value and maturity \( s \), paying a fixed rate \( K \) and receiving a floating rate with tenor \( T_f - s \) is given by

\[
FRA_t(r, s, T_f, K) = P(r, t, T_f) \hat{E} (F(r, s, T_f, s) - K)^+ = P(r, t, T_f) \left[ B(s, T_f) D(t, s) \left( 1 + \frac{\xi_1}{\sigma} \right) - \frac{\ln(A(s, T_f))}{T_f - s} - K \right]
\]

with \( P(r, t, T_f) \) and \( B(t, s) \) as given in Proposition 1, \( D(t, s) \) as given by (16), and \( \xi_1 \) as given in (21). \( \square \)

#### 3.3.2. Interest rate swap and swaption

An interest rate swap can be interpreted as either the difference between two coupon bonds or a portfolio of FRAs. Hence, swap valuation is a straightforward application of Proposition 1 or 8. Moreover, swaptions can be valued applying Proposition 6.

#### 3.3.3. Cap, floor, and collar

A cap (floor) contract guarantees to its holder a pay-off if a certain floating interest rate is above (below) a specified rate, the cap (floor) level. Similarly to swaps, caps and floors involve a series of regular payments, usually referred as caplets or floorlets. Therefore, a cap (floor) can be interpreted as a portfolio of caplets (floorlets).

Consider a caplet written on a floating rate with \$1 face value and maturity \( s \). If the caplet is exercised, the investor pays a fixed interest rate \( K \) and receives a floating rate with tenor \( T_f - s \). The floating rate is set at time \( s \) and the net cash-flow is received at time \( T_f > s \).

Under the risk-neutral measure \( \hat{P} \), the price at time \( t \) of this caplet is given by

\[
Caplet_t(r, s, T_f, T_s) = \hat{E} \left[ e^{\int_t^{T_f} r_s ds} (R(r, s, T_s) - K)^+ \right]^t
\]

Under the s-forward measure \( \hat{P} \), the caplet price is given by the following Proposition.

**Proposition 9.** The price at time \( t \) of a caplet written on the floating rate with \$1 face value and tenor \( T_f - s \) is given as

\[
Caplet_t(r, s, T_f, T_s) = \hat{E} \left[ e^{\int_t^{T_f} r_s ds} (R(r, s, T_s) - K)^+ \right]^t
\]

\[
= P(r, t, T_f) \left[ B(s, T_f) d(T_s, s) \left( 1 + \frac{\xi_1}{\sigma} \right) - \frac{\ln(A(s, T_f))}{T_f - s} - K \right]
\]

\[
\times \left[ \delta + \xi_1 - 2e^{\frac{\xi_1}{2} \sum_{n=0}^{\infty} \left( \frac{\xi_1}{2} \right)^n \frac{n! \Gamma \left( \frac{1}{2} + n + 1.5 \right)}{n! \Gamma \left( \frac{1}{2} + n \right)} \right]
\]

\[
- \left( \frac{\ln(A(s, T_f))}{T_f - s} + K \right) P(r, t, T_f) \left[ 1 - \chi^2 (\rho, \delta, \xi_1) \right]
\]
where \( \chi^2(\cdot) \) denotes the non-central chi-square distribution function, \( I(\cdot) \) represents the Gamma function, and

\[
\rho = \left( K + \frac{\ln(A(s, T_s))}{T_s - s} \right) \delta(T_s - s) D(t, s) B(s, T_s)
\]

with \( P(r_t, t, T_s) \) and \( B(t, s) \) as given in Proposition 1, \( D(t, s) \) as given by (16), and \( \xi_1 \) as given by (21).

Proof. See Appendix A. □

In order to price a floorlet, same type of calculations as in this Proposition can be used. Alternatively, we can use the caplet–floorlet parity.

Cap, floor, and collar prices are a straightforward application of these results.

4. Empirical analysis

In this section we analyze the empirical performance of the cyclical model versus the CIR model. In other words we estimate the gain provided by the incorporation of the harmonic oscillators into the pricing formula. In the pricing section we have computed analytical expressions for interest rates and for bond and derivative prices. Hence, we could use realized data on either of these variables to evaluate the empirical performance of the cyclical model. On this regard, we consider the yield to maturity time series for four different alternatives, namely, 3-months, 1-, 5-, and 10-years maturities. In this way we can directly relate the yield to maturity with the zero-coupon bond price as stated in Corollary 3. Moreover, applying Proposition 1, we have that the yield to maturity is given by the following expression

\[
R(r_t, \tau) = \frac{B(\tau)}{\tau} r_t - \ln A(\tau), \quad \tau = T - t
\]

We opt for using interest rate data because a good fit of the discount function for future cash flows should suggest a potentially good fit of bond prices, and later, of interest rate derivative prices. Alternatively, it is unclear that a relatively good fit of derivative prices might necessarily imply a good fit of the discount factor. Obviously, the empirical analysis of the cyclical model for derivative prices is an interesting issue and is left for further research.

The data set used for this study consists of daily yields obtained from US Treasury (constant maturity) Bills. In more detail, we consider four maturities, 3-months, 1-, 5-, and 10-years, from February 1, 2013 up to February 11, 2014. Each time series consists of 257 daily observations. Fig. 7 presents the time series graph for each maturity. We can see that the 3-month yield provides very low values during the whole period, in fact it becomes null on September 26, 2013, while the maximum yield does not exceed 0.14%. The 1-year rate displays a similar behavior with very low rates oscillating between 0.09% and 0.17%. Finally, the 5- and 10-year yields present a quite alike picture (the linear correlation between both series is 0.9852) but with different mean level.

The fitting or calibrating problem can be formulated as a problem of minimizing the sum of squared pricing errors in the form of a regression model, that is

\[
SR(x) = \sum \left( R(r_t, \tau) - \frac{B(\tau)}{\tau} r_t + \ln A(\tau) \right)^2
\]

where the terms \( A(\tau) \) and \( R(t, \tau) \) are model specific and the error term, \( u_t \), can be interpreted as the approximation error in the practical implementation of the pricing formula. We estimate each model solving a non-linear optimization problem that searches for the values of each model parameters that minimize the sum of squared residuals in (24), \( SR(\hat{x}) = \sum \left( R(r_t, \tau) - \frac{B(\tau)}{\tau} r_t + \ln A(\tau) \right)^2 \).

For the CIR model, the values of the structural parameters are given by \( x = (\theta, \sigma, \kappa) \), while the cyclical model is defined by the structural parameters \( b = (A_0, A_1, K, \omega, \phi) \).

Table 1 presents two panels including the mean and standard deviation of the estimates for the parameters of the cyclical and
Table 1
Parameters estimates for the cyclical and CIR models.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>3-Month</th>
<th>1-Year</th>
<th>5-Year</th>
<th>10-Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_h$ ($\times$ 100)</td>
<td>0.0941 (0.0007)</td>
<td>0.2897 (0.0029)</td>
<td>2.8167 (0.1560)</td>
<td>5.5725 (0.3728)</td>
</tr>
<tr>
<td>$A_x$</td>
<td>0.0100 (0.0111)</td>
<td>0.0100 (0.0061)</td>
<td>0.0958 (0.0006)</td>
<td>0.0188 (0.0844)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>11.8393 (0.7696)</td>
<td>7.4316 (0.6626)</td>
<td>5.4321 (0.0558)</td>
<td>0.4993 (0.0527)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>4.0557 (0.0662)</td>
<td>8.0238 (0.0727)</td>
<td>2.6405 (0.1105)</td>
<td>1.3039 (0.0015)</td>
</tr>
<tr>
<td>$\sum \min</td>
<td>S</td>
<td>/C_0$</td>
<td>9.3977 $\times$ 10^{-6}</td>
<td>4.8067 $\times$ 10^{-6}</td>
</tr>
<tr>
<td>$\sum</td>
<td>u</td>
<td>/C_2$</td>
<td>0.0393</td>
<td>0.0264</td>
</tr>
<tr>
<td>$\beta$ ($\times$ 100)</td>
<td>0.0687 (0.0027)</td>
<td>0.1432 (0.0020)</td>
<td>1.2893 (0.0157)</td>
<td>3.0374 (1.4177)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0250 (0.0904)</td>
<td>0.1127 (0.0104)</td>
<td>0.0100 (0.0035)</td>
<td>0.0408 (3.6893)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>4.9596 (0.2260)</td>
<td>2.0937 (0.1014)</td>
<td>2.4596 (0.0029)</td>
<td>1.6478 (0.0187)</td>
</tr>
<tr>
<td>$\sum \min</td>
<td>S</td>
<td>/C_0$</td>
<td>1.6080 $\times$ 10^{-5}</td>
<td>6.3736 $\times$ 10^{-6}</td>
</tr>
<tr>
<td>$\sum</td>
<td>u</td>
<td>/C_2$</td>
<td>0.0421</td>
<td>0.0334</td>
</tr>
</tbody>
</table>

This table presents the mean and standard deviation (in parenthesis) of the estimated parameters over the sample period ranging from February 1, 2013 to February 11, 2014. For this period, $\sum \min |S|/C_0$ and $\sum |u|/C_2$ aggregate over time the least-squares pricing errors and the absolute value of the pricing errors, respectively.

CIR models, respectively, for each maturity and over the whole sample period. Each panel also shows two measures of the ability of each model to fit the observed data, namely, $\sum \min |S|/C_0$, the minimized numerical value of the objective function, and $\sum |u|/C_2$, the sum of the absolute value of pricing errors. We can see that the cyclical model provides a more precise adjustment to actual market data, reducing the aggregate sum of square residuals of the 3-month, 1-, 5-, and 10-year maturities by 42%, 25%, 70%, and 81%, respectively.

Fig. 8 compares graphically the pricing errors (in absolute value) of both models for every maturity and during the whole sample period. We can see that most of the time the cyclical model outperforms the CIR one. Moreover, whenever the CIR model provides lower pricing errors than the cyclical model, the gap between both models is quite narrow. However, there are many days where the cyclical model provides much lower pricing errors and the gap with the CIR model is quite large.

It is worth to mention that, for the regression problem, we have assumed the approximation provided in Remark 1. For each regression problem the values of the parameter $q$ obtained for the four maturities under analysis are $-7.5986 \times 10^{-3}$, $-1.9456 \times 10^{-3}$, $-1.7529 \times 10^{-4}$, and $-0.0014$, respectively. To evaluate the validity of this approximation, we consider the value of $q$ that is furthest from zero, that is, $q = -0.0014$. Fig. 9 shows the comparison between the Mathieu cosine and sines versus their respective approximation functions. We can see that, in both cases, the approximation is excellent, in fact we are not able to spot any visual difference.

Fig. 8. Time series evolution and pricing errors in absolute value for each model during the whole sample period, February 1, 2013 to February 11, 2014.
5. Conclusions

This paper has presented a cyclical square-root model for the TSIR assuming that the mean reversion level of interest rates and the spot rate volatility follow a cyclical behavior modeled by a harmonic oscillator functional form. This specification preserves the analytical tractability and provides high flexibility, allowing the model to capture a variety of different shapes of the discount factor and the TSIR. We have included several examples where, for arbitrary sets of parameters, our model is capable of replicating different yield curve shapes such as upward sloping, downward sloping, humped, and inverted humped. In addition, we have presented a thoroughly analysis on how the TSIR responds to different values of each parameter, illustrating an interesting insight of the model main features. Moreover, fixing the temporal frequency to zero, the cyclical model nests the original one presented in Cox et al. (1985).

Under this framework, we have derived closed-form expressions for the prices of bonds, forward on bonds, European options on zero-coupon and coupon-bearing bonds, European bond forward options, swaps, swaptions, caps, floors, and collars.

Finally, we have analyzed the empirical performance of the cyclical model versus its natural benchmark, the CIR model. Our findings show that the cyclical model provides a much better fit and empirical accuracy. In this sense, our model fulfills a real necessity providing a powerful and simple tool for pricing and risk management purposes and for empirical issues and should be of special interest for traders, financial institutions, and risk managers.

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Appendix A. Appendix of Proofs

Proof of Proposition 1

To solve Eq. (12), we guess an exponential-affine functional form for the bond price

$$P(r_t, t, T) = A(t, T) e^{-B(t, T)r_t}$$

with terminal conditions $A(T, T) = 1, B(T, T) = 0$. Then,

$$P_t = \frac{A_t}{A} P - B r_t P, \quad P_r = -BP, \quad P_{rr} = B^2 P$$

where arguments have been omitted and subscripts in functions $P, A$, and $B$ denote partial derivatives. Replacing these expressions into (12), we get

$$\frac{A_t}{A} - B r_t - B[K(h_t - r_t) - \lambda_t r_t] + \frac{1}{2} \sigma_t^2 r_t B^2 - r_t = 0$$

with boundary condition $A(T, t) = 1, B(t, T) = 0$. Since this equation is linear in $r_t$, we obtain the following system of ordinary differential equations (ODEs)

$$B_t - (\lambda_t + \kappa) B - \frac{1}{2} \sigma_t^2 B^2 + 1 = 0$$

(25)

$$A_t - k h_t A B = 0$$

(26)
Applying standard theory for Ricatti-type equations and defining $\tau = T - t$, the solution of (25) is given as $B(t) = e^{\frac{\tau}{\mu(t)}}$, where $\nu(t)$ and $u(t)$ are solutions of the system

$$-\nu'(t) + u'(t) - \kappa \nu(t) = 0$$
$$-u'(t) + \lambda u(t) + \frac{1}{2} \sigma_t^2 \nu(t) = 0$$

(27)

(28)

Replacing the derivative of (27) into (28), we obtain the second-order ODE

$$\nu''(t) + b \nu'(t) + e(t) \nu(t) = 0$$

where $b = \kappa - \lambda$, and

$$e(t) = -\frac{\lambda}{2} \kappa^2 - \frac{1}{2} A_0 \sin^2(\varphi - \omega T + \omega \tau)$$

Setting $u(t) = g(t)M(t)$, expression (29) becomes

$$g(\tau)M''(t) + (2g'(t) + bg(t))M'(t) + (g''(t) + bg'(t))M(t) = 0$$

that represents a Mathieu's differential equation if $2g'(t) + bg(t) = 0$. Then, we get

$$g(\tau) = c e^{\tilde{\kappa}(\tau)}$$

with arbitrary constant $c$. Then, we obtain

$$\nu(t) = e^{\tilde{\kappa}(t)}(c_1MC(a,q,x) + c_2MS(a,q,x))$$

(30)

where $MC$ and $MS$ represent the Mathieu cosine and sine functions, respectively, and

$$a = -\frac{A_0 (\varphi + \omega \tau)^2}{4 \omega^2}, \quad Q = -\frac{A_0}{8 \omega}, \quad \chi = \varphi - \omega T + \omega \tau$$

The boundary condition $B(0) = 0$ implies $\nu(0) = 0$. Then, choosing $c_2 = 1$ in (30) implies

$$c_1 = \frac{MS(a,q,\varphi - \omega T)}{MC(a,q,\varphi - \omega T)}$$

Substituting (30) and its derivative into (27), we get

$$u(t) = e^{\tilde{\kappa}(t)}\left[\frac{1}{2} (\varphi + \omega \tau)(c_1MC(a,q,x)MS(a,q,x)) + \omega(c_1MCP(a,q,x) + MSP(a,q,x))\right]$$

(31)

where $MCP$ and $MSP$ represent the derivatives with respect to $x$ of the Mathieu cosine and sine functions, respectively. Therefore, using expressions (30) and (31), we get $B(t)$. Finally, Eq. (26) immediately provides $A(t; T) = \exp\left\{-\int_0^T \kappa \theta B(t; T) dt\right\}$. □

Proof of Proposition 5

From Proposition 3, we know

$$c_s(r_s, T, K) = P(r_s, T, s)E\left[A_s(T) e^{-\int_0^T (\frac{d}{D}(s) - K) d(T,s)} - K \right] \left| r_s \right.$$ 

where $E$ represents expectation with respect to the $s$-forward measure $P_s^r$. Hence,

$$c_s(r_s, T, K) = P(r_s, T, s) \int_0^\infty A_s(T) e^{-\int_0^T (\frac{d}{D}(s) - K) d(T,s)} - K \right] d\mathcal{X}(s,\delta,\xi_1)$$

$$= P(r_s, T, s) \int_0^\infty A_s(T) e^{-\int_0^T (\frac{d}{D}(s) - K) d(T,s)} - K \right] d\mathcal{X}(s,\delta,\xi_1)$$

$$-K P(r_s, T, s)\mathcal{X}(\rho_1,\delta,\xi_1)$$

where $\mathcal{X}(\cdot)$ denotes the non-central chi-square distribution function and

$$\hat{\xi}_1 = \frac{\partial \mathcal{X}(T,T)}{\partial T}$$

$$\rho_1 = \frac{\int B(s, T) D(t, s) \ln \left( \frac{A(s, T)}{K} \right)}{\int B(s, T) D(t, s)}$$

Using the expression for the density function of a non-central chi-square distribution, the integral becomes

$$P(r_s, T, s)A_s(T)$$

$$\times \int_0^\infty e^{-\frac{1}{2} \sigma^2 T} e^{-\frac{1}{2} \xi_1^2} - \frac{\xi_1^2}{2 T} \sum_{n=0}^{\infty} \frac{\sigma^n \xi_1^n}{n! T^{\frac{1}{2} + n}} \frac{d\sigma}{d\xi_1^n}$$

Considering the change of variable $y = (\frac{\delta}{2} B(s, T) D(t, s) + 1)\sigma$ and defining

$$\rho_2 = \frac{\int B(s, T) D(t, s)}{\int B(s, T) D(t, s) + 1} \xi_1 = \frac{\rho_1}{\rho_2}$$

we get

$$P(r_s, T, s)F(r_s, T, s, T)\mathcal{X}(\rho_2,\delta,\xi_2)$$

with $F(r_s, T, s, T)$ as given by Proposition 4. □

Proof of Proposition 9

From Proposition 3, we know

$$Caplet_t(r_s, T, T_s, T, K) = P(r_s, T, T) E\left[(R(s, T, \delta)) - K \right] | r_s$$

$$= P(r_s, T, T) \int_{\rho_1}^{\infty} \frac{B(s, T) D(t, s)}{\delta(T_s - s)} \sigma d\mathcal{X}(s,\delta,\xi_1)$$

$$- \left( \frac{\ln(A_s(T_s))}{T_s - s} + K \right) P(r_s, T_s) \left[1 - \mathcal{X}(\rho,\delta,\xi_1) \right]$$

where $E$ represents expectation with respect to the $s$-forward measure, $\mathcal{X}(\cdot)$ denotes the non-central chi-square distribution function, $\xi_1$ as given by (21), and

$$\rho = \left( K + \ln(A_s(T_s)) \right) \frac{\delta(T_s - s)}{D(s)B(s, T_s)}$$

Using the expression for the density function of a non-central chi-square distribution, the integral becomes

$$\frac{D(t, s)B(s, T_s)}{\delta(T_s - s)} \left( \frac{\xi_1 - 2 e \cos(\frac{1}{2} \sum_{n=0}^{\infty} \frac{\xi_1^n}{n! T^{\frac{1}{2} + n}}) d\sigma}{\xi_1^2} \right)$$

Note that

$$\int_0^\infty e^{-\frac{1}{2} \xi_1^2} - \frac{\xi_1^2}{2 T} \sum_{n=0}^{\infty} \frac{\sigma^n \xi_1^n}{n! T^{\frac{1}{2} + n}} \frac{d\sigma}{d\xi_1^n}$$

where $\Gamma(x, q) = \int_0^q e^{-x} e^{t-1} dt$ and $\Gamma'(x, q) \equiv \Gamma(x, \infty)$. Using the Taylor expansion of the exponential function and straightforward algebra, (32) becomes

$$\frac{D(t, s)B(s, T_s)}{\delta(T_s - s)} \left[ \delta + \xi_1 - 2 e \cos(\frac{1}{2} \sum_{n=0}^{\infty} \frac{\xi_1^n}{n! T^{\frac{1}{2} + n}}) \frac{n}{n! T^{\frac{1}{2} + n}} \right] d\xi_1.$$ □

References
