Abstract. This paper deals with the asymptotic hyperstability of switched time-varying dynamic systems involving switching actions among linear time-invariant parameterizations in the feed-forward loop for any feedback regulator controller potentially being also subject to switching through time while being within a class which satisfies a Popov’s-type integral inequality. Asymptotic hyperstability is proven to be achievable under very generic switching laws if at least one of the feed-forward parameterization possesses a strictly positive real transfer function, a minimum residence time interval is respected for each activation time interval of such a parameterization and a maximum allowable residence time interval is simultaneously maintained for all active parameterization which are not positive real, if any. In the case that all the feed-forward parameterizations possess a common Lyapunov function, asymptotic hyperstability of the switched closed-loop system is achieved under arbitrary switching.

Keywords- hyperstability; asymptotic hyperstability; switched dynamic systems; switching laws.
large minimum time intervals are respected at certain stable active parameterizations. Such minimum residence time intervals depend on the active system parameterizations from the last testing for a minimum kept residence time lower-bound, [15-22]. The problem has been investigated for a variety of linear time-varying systems like, for instance, delay-free systems, time-delay systems or hybrid systems, [11-14], [17-18], or for impulsive controls, [31]. The topics of absolute stability and hyperstability are of a relevant interest still nowadays because of their theoretical importance and its wide range of applications including stabilization under parametrical dispersion of regulator components in either the absence or presence of delays, hybrid mixed continuous-time and digital systems or passivity issues. See, for instance, some recent related background in [24-30] and references there in. On the other hand, absolute stability has been also investigated for systems involving time-delays. See, for instance, [32] and references therein. The definition of absolute margins for missile guiding has been studied in [33]. Passivity, which is a more general property linked to asymptotic hyperstability, has been investigated in many papers. See, for instance, [34] and references there in.

This paper investigates conditions of asymptotic hyperstability of switched linear systems under regulation controls generated from nonlinear devices satisfying Popov’s- type integral inequalities. It is assumed that the switched dynamic system consists of: a) a set of linear parameterizations in the feed-forward loop, subject to switching through time to select the active parameterization on a certain time interval; and b) a, in general, nonlinear controller function taking values in a set of at most the same number of nonlinear feedback devices being also subject to switching. The following results are found:

1) if all the feed-forward loop linear parameterizations are strictly positive real (then their matrices of dynamics being all Hurwitz) and, furthermore, the corresponding matrices of dynamics possess a common Lyapunov function then asymptotic hyperstability (i.e. global asymptotic stability) of the switched dynamic system is achieved unconditionally for any switching law for any nonlinear controller device satisfying a Popov’s type integral inequality.

2) if there is at least one parameterization with strictly positive real transfer function then asymptotic hyperstability of the switched system might be achieved for switching laws satisfying generic constraints on the switching time instants. The remaining active parameterizations are not required to be either stable or with associate positive real transfer functions. Those constraints consist basically in respecting a minimum allowable residence time interval for a set of marked testing active parameterizations involving strictly positive real functions and, simultaneously, a maximum allowable residence time interval for the remaining active parameterizations. The minimum residence time for the active marked parameterizations depends on the parameters and residence time intervals from the last preceding test for a minimum residence time interval along the last current active marked parameterization.

The paper is organized as follows. Section 2 introduces the dynamic switched feed-back system to be then studied consisting of a feed-forward linear time-varying loop whose active parameterization takes values along a certain residence time within a set of time-invariant linear dynamic systems before each next switching and a feed-back regulation nonlinear loop which satisfies an integral Popov’s type
inequality through time. Such a nonlinear feedback law can also take values within a set of potential nonlinear devices by switching actions performed through time. Section 3 gives conditions for the input to the feed-forward loop to be bounded and to converge asymptotically to zero as time tends to infinity provided that the feedback loop satisfies a Popov’s-type inequality. The obtained results rely on the appropriate distribution through time of the active parameterizations selected through switching which possess strictly positive real transfer functions and maximum required allowable residence times of the parameterizations with no associated positive real functions. The results of this section are used in Section 4 to investigate conditions of asymptotic hyperstability (i.e. global asymptotic stability for any nonlinear feedback laws satisfying a Popov’s inequality) of the whole dynamic feedback switched system. It is found that switching can occur for arbitrary switching instants if all the selected active parameterizations in the feed-forward loop have positive real transfer functions and possess a common Lyapunov function. Otherwise, asymptotic hyperstability of the closed-loop system is guaranteed under sufficiency-type constraints of minimum residence time intervals for a set of marked parameterizations with strictly positive real transfer functions and of maximum allowable time intervals for active parameterizations which do not have positive real transfer functions. Section 5 presents some numerical illustrative examples.

2. Switched closed-loop system

Consider the \( n \)-dimensional single-input single-output switched nonlinear feedback dynamic system whose structure is:

\[
\dot{x}(t) = A_{\sigma(t)}(t)x(t) + b_{\sigma(t)}(t)u(t) ; \quad x(0) = x_0 \in \mathbb{R}^n
\]

(1)

\[
u(t) = -\varphi_{\sigma_0(t)}(\nu(t), t) ; \quad \nu(t) = c_{\sigma(t)}^T x(t) + d_{\sigma(t)} u(t) + d_{\sigma(t)} \varphi_{\sigma_0(t)}(\nu(t), t)
\]

(2)

for \( t \in \mathbb{R}_0^+ := \mathbb{R}_+ \cup \{0\} \), where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R} \), \( \nu(t) \in \mathbb{R} \) are, respectively, the state vector and scalar input, which is a feedback regulation control, and output where:

1) \( \sigma: \mathbb{R}_0^+ \to \overline{p} := \{1, 2, \ldots, p\} \) and \( \sigma_0: \mathbb{R}_0^+ \to \overline{p}_0 := \{1, 2, \ldots, p_0\} \subset \overline{p} \), \( p_0 \leq p \), for some given finite, and in general distinct, numbers \( p, p_0 \in \mathbb{N} \) of parameterizations of (1)-(2). The first one describes a switching law among such various constant parameterizations defined by the set of quadruples \( \{(A_i, b_i, c_i, d_i) : i \in \overline{p}\} \) of (1)-(2) of elements whose orders are compatible with the dimensionalities of the corresponding signals.

2) The function \( \sigma: \mathbb{R}_0^+ \to \overline{p} \) is defined as \( \sigma(t) = j = j(t) = j(t_i) ; \forall t \in [t_i, t_{i+1}) \) for some integer \( j \in \overline{p} \) for each \( t_i \in \{t_i\} \), and each integer \( i \in \mathbb{N}_0 \subseteq \mathbb{N} : = \mathbb{N} \cup \{0\} \), where \( STI = STI(\sigma) = \{t_i\} \) is the sequence of switching time instants generated from some given switching law \( SL = SL(\sigma), \) subject to \( t_{i+1} - t_i = T_i \geq 0 ; \forall i \in \mathbb{N} \). Such a function assigns at certain time intervals, a particular parameterization of the feed-forward part of the system which is modified at the switching time instants.
3) The function \( \sigma_0 : R_0^+ \rightarrow \overline{p}_0 \) is defined as \( \sigma_0(t) = j_0(t) = f_0(t_0) \); \( \forall t \in \left[ t_{i_0} , t_{i_0+1} \right] \) so that \( STI_0 = \{ t_0 \} \subseteq STI = \{ t_i \} \), with the switching constraint \( t \in STI_0 \Rightarrow t \in STI \) (the converse is not necessarily true), is the sequence of switching time instants of the feedback nonlinear part of the dynamic system. This means that the number of parameterizations of the nonlinear feedback device function is at most (but it can be lower) that of the linear feed-forward block. Also, even if they have the same number of parameterizations, the feedback switching activation does not necessarily occur for any switching time instant of the feed-forward loop. Also, any switching time of the feedback nonlinear device is also a switching time instant of the linear feed-forward it can have distinct ordering allocations within each sequence in the case that \( \sigma_0 \neq \sigma \). Furthermore, any switching time instant of the feedback part is always a switching time instant of the feed-forward part but the converse is not necessarily true. Such a function assigns for certain time intervals, a particular parameterization of the feedback nonlinear device of the system which is modified at the switching time instants.

4) The nonnegative real number \( T_r = T_r(\sigma) \) is the minimum residence (or dwelling) time interval at each parameterization and either \( p = 1 \) (i.e. the trivial case of a single parameterization of (1)-(2) with no switching) or \( j(t) \neq j(t_{i+1}) \) for all \( t \in STI \). If \( T_r = 0 \) then the switching law is unconditional in the sense that switching is fully arbitrary. In the linear case, unconditional switching is possible in a stable way if all the parameterizations possess a common Lyapunov function. Otherwise, a minimum residence time \( T_r > 0 \) is required at each parameterization so as to guarantee the stabilization of the linear time-varying switched system if all of them are stable or at least one should be stable, subject to a minimum residence time when such a stable parameterization is active, which depends of the whole sequences and respective active time intervals at the rest of parameterizations.

5) The set \( \overline{N} \) is a denumerable (proper or improper) subset of \( N \) that, if finite, describes switching processes with a finite number of switches among the \( p \) distinct parameterizations.

6) The control input is generated as the, in general, nonlinear output-feedback function \( u(t) = -\varphi_{\sigma_0(t)}(y(t), t) \) which is assumed to be piecewise continuous while satisfying the integral Popov’s -type inequality:

\[
\int_0^t \varphi_{\sigma_0(t)}(y(t), t) y(t) dt \geq -\gamma > -\infty \quad ; \text{some} \; \gamma \in R^+, \; \forall t \in R_0^+
\]  

(3)

Note that \( \varphi : \overline{p}_0 \times R^2 \rightarrow R \) defined by \( \varphi(t, y(t)) = \varphi_{\sigma_0(t)}(y(t), t) \) for \( \sigma_0(t) = j_0(t) = f_0(t_0) \); \( \forall t \in \left[ t_{i_0} , t_{i_0+1} \right] \) for some integer \( j_0 \in \overline{p}_0 \) for each \( t_{i_0} \in \left[ t_0 \right] \) is piecewise continuous within its

definition domain for any switching law \( \sigma_0 : R_0^+ \rightarrow \overline{p}_0 \subset \overline{p} \) if \( \varphi_{j_0} : R^2 \rightarrow R \); \( j_0 \in \overline{p}_0 \) are all piecewise continuous. It is not required in principle that the nonlinear devices be distinct for each distinct parameterization in the feed-forward loop. That is, some of the controller nonlinear functions can be
identical for different linear parameterizations of the feed-forward loop. The closed-loop system is displayed in Figure 1 below:

![Figure 1. Block diagram of the feedback nonlinear system](image)

Note that the switched feedback law \( u(t) = -\varphi_{\sigma(t)}(y(t), t) \) together with (3) implies that (3) itself is equivalent to:

\[
E(t) = \int_0^t (y(\tau)u(\tau))d\tau \leq y < \infty \quad \text{; some } y \in \mathbb{R}_+, \forall t \in \mathbb{R}_+^+
\]

where \( E(t) \) is an input-output energy measure of the feed-forward linear part of the closed-loop system.

3. Switching conditions for asymptotic convergence to zero of the input to the feed-forward loop

This section investigates parameterization switching sufficiency-type conditions for the input to the linear feed-forward loop to converge asymptotically to zero as time tends to infinity. The switching laws can involve parameterizations which are not strictly positive real being subject either a) to a finite number of switching actions; or b) to appropriate alternating with strictly positive real ones subject to maximum allowable residence times; or finally; c) to saturation- vanishing conditions of the input to the feed-forward linear loop. Note that the time-varying piecewise constant parameterization \( \{ A_{\sigma(t)}, b_{\sigma(t)}, c_{\sigma(t)}, d_{\sigma(t)} \} \) of the switched system (1)-(2) changes of values at time instants in \( STI \). It is well-known that, in the absence of switching , i.e. if \( p = 1 \) then the closed-loop system is said to be hyperstable if the linear transfer function \( G(s) = c^T (sI - A)^{-1}b + d \) is positive real, i.e. it belongs to the set \( PR \) of positive real functions fulfilling \( \text{Re} \ G(s) \geq 0 \) for \( \text{Re} s > 0 \) (condition of hyperstability of the linear feed-forward subsystem ) and the feedback law satisfies (3) (condition of hyperstability of the nonlinear feedback device) . If \( G(s) \) is strictly positive real, i.e. if it is in the set \( SPR \) of strictly positive real functions fulfilling \( \text{Re} G(s) > 0 \) for \( \text{Re} s \geq 0 \) then the closed-loop system is said to be asymptotically hyperstable. It is well-known that realizable positive real functions are either stable or critically stable of relative order (or relative degree) either zero or one. Their critically stable poles, if any, are single and with non-negative associate residues. Strictly positive real transfer functions are, in particular, stable.

In order to simplify the formalism, we will refer indistinctly to positive realness and strict positive realness either for transfer functions or for their state-space realizations being in particular associated with the various parameterizations of the switched system. The feedback system is said to be hyperstable (respectively, asymptotically hyperstable) if it is globally stable for any nonlinear output- feedback law...
satisfying Popov’s inequality (3). For any given switching rule \( \sigma : R^+ \rightarrow \bar{p} \), let us consider the impulse response of the feed-forward linear block \( g(t) = L^{-1}(G(s)) \); i.e. the Laplace anti-transform of \( G(s) \), denote the Fourier transform of a function \( f(t) \) as \( F(f) \), provided that it exists, and also define the subsequent auxiliary truncated input if the switching action never ends:

\[
u_{\sigma_0}(t_0) = \begin{cases} u(t), & t \in [t_{i_0}, t_{i_0}+1) \\ 0, & t \in (-\infty, t_{i_0}) \cup [t_{i_0}+1, \infty) \end{cases}; \quad \forall t \in R \quad (5.a)
\]

and

\[
u_{\sigma_0}(t) = \begin{cases} u(t), & t \in [t_q, \infty) \\ 0, & t \in (-\infty, t_q) \end{cases}; \quad t_q = t_{q_0}(t) \in STI_0, \quad \forall t \in R \quad (5.b)
\]

otherwise, i.e. if \( t_{q_0}(t) = t_q < \infty \) is the last switching time instant (i.e. if \( STI_0 \) has a finite cardinal) under the axiom \( t_0 (=0) \in STI \cap STI_0 \) for any \( \sigma : R_0^+ \rightarrow \bar{p} \), \( \sigma_0 : R_0^+ \rightarrow \bar{p}_0 \), and

\[
q = q(t) = \max\{ z \in N \cup \{0\} : t_q(t) (\in STI) \leq t \}; \quad q_0 = q_0(t) = \max\{ z \in N \cup \{0\} : t_{q_0}(t) (\in STI_0) \leq t \}
\]

so that \( t_q \) and \( t_{q_0} \) are, respectively, the last switching time instants of the feed-forward linear parameterization and of the nonlinear feedback device in the time interval \([0, t]\) under the switching rule \( \sigma : R_0^+ \rightarrow \bar{p} \). In the same way, given \( STI = \{ t_i \} \), we define the impulse response as

\[
g_\sigma(t) = g_\sigma(t_i); \quad \forall t \in [t_i, t_{i+1})
\]

If the convolution and Parseval’s theorems are jointly applied to (4) for zero initial conditions and extending the definition of the input on \( R \) with \( u(t) = 0 \) and \( g_\sigma(t) = 0 \) for \( t < 0 \), one gets:

\[
E(t) = \sum_{i=1}^{q(t)} \int_{t_i}^{t_i+1} g(\tau)u(\tau)d\tau + \int_{t_q}^{\infty} g(\tau)u(\tau)d\tau
\]

\[
= \sum_{i=1}^{q(t)} \left( \int_{t_{i-1}}^{t_i} \int_{0}^{t_q} g(\tau - \tau')u(\tau)d\tau' d\tau + \int_{t_{i-1}}^{t_i} \int_{t_q}^{\infty} g(\tau - \tau')u(\tau)d\tau' d\tau \right)
\]

\[
= \sum_{i=1}^{q(t)} \left( \int_{t_{i-1}}^{t_i} \int_{0}^{t_q} g_\sigma(t_{i-1})(\tau - \tau')u_\sigma_0(t_{i-1})(\tau) d\tau' d\tau + \int_{t_{i-1}}^{t_i} \int_{t_q}^{\infty} g_\sigma(t_1)(\tau - \tau')u_\sigma_0(t_{i-1})(\tau) d\tau' d\tau \right)
\]

\[
= \sum_{i=1}^{q(t)} \left( \int_{t_{i-1}}^{t_i} \int_{t_q}^{\infty} g_\sigma(t_{i})(\tau - \tau')u_\sigma(t_{i})(\tau) d\tau' d\tau + \int_{t_{i-1}}^{t_i} \int_{0}^{t_q} g_\sigma(t_{i})(\tau - \tau')u_\sigma(t_{i})(\tau) d\tau' d\tau \right)
\]
\[ E(t) \geq \frac{1}{2\pi} \sum_{\omega \in R_{0+}} \left( \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) \right)^2 \left| U_{\sigma_{(t)}}(j\omega) \right|^2 d\omega + \left( \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) \right)^2 \left| U_{\sigma_{(t)}}(j\omega) \right|^2 d\omega \]

\[ = \frac{1}{2\pi} \sum_{\omega \in R_{0+}} \left( \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) \right)^2 \left| U_{\sigma_{(t)}}(j\omega) \right|^2 d\omega + \left( \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) \right)^2 \left| U_{\sigma_{(t)}}(j\omega) \right|^2 d\omega \]

\( ; \forall t \in R_{0+} \). Now, consider the sequence of switching time instants until time \( t \) of the given switching law \( \sigma : R_{0+} \to \overline{P} \), \( STI(t) = STI \cap [0, t) \); \( \forall t \in R_{0+} \) and decompose it as the disjoint union \( STI(t) = STI_{p}(t) \cup STI_{n}(t) \cup STI_{z}(t) \), \( \forall t \in R_{0+} \) as follows:

\[ STI_{p}(\sigma, t) = \left\{ t_{i} \in STI(t) : \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) > 0 \right\} \]

\[ STI_{n}(\sigma, t) = \left\{ t_{i} \in STI(t) : \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) = - \max_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) < 0 \right\} \]

\[ STI_{z}(\sigma, t) = \left\{ t_{i} \in STI(t) : \min_{\omega \in R_{0+}} \text{Re} G_{\sigma_{(t)}}(j\omega) = 0 \right\} \]

\( STI_{p} = STI_{p}(\sigma), STI_{n} = STI_{n}(\sigma) \) and \( STI_{z} = STI_{z}(\sigma) \) are defined in the same way by including all the respective switching instants of the switching law \( \sigma : R_{0+} \to \overline{P} \), that is, the right-hand-side sets of the definitions in (8) modified for \( t_{i} \in STI \). Note that, by technical reasons towards a clear proof of Lemma 1 below, (8) are defined so that \( t \in STI \Rightarrow t \in STI(t) \) following the definition convention \( STI(t) = STI \cap [0, t) \). The first subsequent auxiliary result is concerned with the energy measure being
nonnegative for all time for the switching law $\sigma: R_0^+ \to \overline{\rho}$. The second one is referred to the non-negativity and boundedness of such a measure for all time.

**Lemma 1** (non-negativity of the input-output energy).

Define the switching-dependent amount:

$$g_{\sigma}(t) := \sum_{j \in STI_{(t_1)}} \min_{\omega \in R_0^+} \text{Re} G_{\sigma(j)}(j\omega) \left( \int_{t_j}^{t_j+1} u^2(t_j+\tau)d\tau \right) - \sum_{j \in STI_{(t_1)}} \max_{\omega \in R_0^+} \left| \text{Re} G_{\sigma(j)}(j\omega) \right| \max_{0 \leq \tau \leq t_j} u^2(\tau) $$

\[; \forall t_j \in STI \]  

where $t_{j+1} = t_j + T_j \; \forall t_j \in STI$ so that the $\sigma(j)$ parameterization of the feed-forward part of the system is active during a time interval $T_j$ in-between two consecutive switching time instants.

Then, the following properties hold:

(i) The input-output energy measure $E(t)$ is nonnegative for all time independent of the input $u: R_0^+ \to R$ if the switching law $\sigma: R_0^+ \to \overline{\rho}$ satisfies any of the two conditions below:

** (i.1) All the active parameterizations are positive real.

** (i.2) Any active parameterization in an interval $[t_{i-1}, t_i)$ which is not positive real is preceded by a strictly positive real one on $[t_{i-1}, t_i)$ while subject to a maximum (being potentially finite or infinity) residence time interval satisfying the constraint:

$$T_i = t_{i+1} - t_i < \frac{g_{\sigma}(t_i)}{\max_{\omega \in R_0^+} \left| \text{Re} G_{\sigma(t_i)}(j\omega) \right| \max_{0 \leq \tau \leq t_i} u^2(t_i + \tau)}$$ \[(10)\]

(ii) A necessary condition to guarantee that

$g_{\sigma}(t) = g_{\sigma}(t_i) + \mu(t_i) \min_{\omega \in R_0^+} \text{Re} G_{\sigma(t_i)}(j\omega) \left( \int_{t_i}^{t_i+1} u^2(\tau)d\tau \right) \left( 1 - \mu(t_i) \right) \max_{\omega \in R_0^+} \left| \text{Re} G_{\sigma(t_i)}(j\omega) \right| \max_{0 \leq \tau \leq t_i} u^2(\tau)$

\[; \forall t \in [t_i, t_{i+1}) \]  

is nonnegative is that the first parameterization after an arbitrary finite time is positive real in order to guarantee the non-negativity for all time of the input-output energy measure where $\mu: R_0^+ \to \{0,1\}$ is a binary indicator function of value $\mu(t_i) = 1$; $\forall t \in [t_i, t_{i+1})$ if $t_i \in STI_{p \cup STI}$ and $\mu(t) = \mu(t_i) = 0$ if $t_i \in STI_{(t)}$. A necessary condition to guarantee that (11) is nonnegative for all time, irrespective of the input, if the number of switches is finite is that the last active parameterization be positive real.

(iii) Assume that the first active parameterization after an arbitrary finite time is strictly positive real and that all non positive real parameterization, if any, satisfies the constraint of maximum residence time interval (10). Then, the input-output energy measure is positive for all $t > 0$. 

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Proof: Consider \( g_{\sigma}(t) \) defined in (11). It turns out that \( g_{\sigma}(0) = 0 \). Also, \( g_{\sigma}(t) \geq 0 \); \( \forall t \in [0, t_1] \) if \( \sigma(0) = 0 \) with \( t_1 \in STI \); i.e. if the first active parameterization at \( 0 \leq t_0 < t_1 \leq \infty \) is positive real since
\[
 g_{\sigma}(t) = g_{\sigma}(t_0) + \mu(t_0) \min_{\omega \in R_{0_1}} Re G_{\sigma(t_0)}(j\omega) \int_{t_0}^{t} u^2(\tau) d\tau = \min_{\omega \in R_{0_1}} Re G_{\sigma(t_0)}(j\omega) \int_{t_0}^{t} u^2(\tau) d\tau \geq 0; \forall t \in [t_0, t_1]
\]
(12)

Now, proceed by complete induction by assuming that \( g_{\sigma}(t) \geq 0 \) for \( t \in [0, t_i] \) and any given \( t_i \in STI \).
Thus, \( g_{\sigma}(t_i) \geq 0 \Rightarrow g_{\sigma}(t_i + \tau) \geq 0 \) for any \( \tau \in [0, t_i] \) by construction if: either

(a) \( t_i \in STI_0 \cup STI_\infty \) so that \( g_{\sigma}(t_i) \geq 0 \) so that \( g_{\sigma}(t) \geq 0 \); \( \forall t \in [0, t_{i+1}] \), then \( t_{i+1} \in STI \) may be any positive arbitrary time instant; or

(b) \( t_i \in STI_0 \cup STI_\infty \) so that \( g_{\sigma}(t_i) = 0 \) implies that \( g_{\sigma}(t) \geq 0 \); \( \forall t \in [0, t_{i+1}] \) and then \( t_{i+1} \in STI \) may be any positive arbitrary time instant (note that this part of Case b is included in Case a); or

(c) \( t_i = t_{i+1} - t_i < \max_{\omega \in R_{0_1}} Re G_{\sigma(t_i)}(j\omega) \max_{0 \leq \tau \leq t_i} u^2(t_i + \tau) \) if \( g_{\sigma}(t_i) > 0 \) and \( t_i \in STI_{\infty} \) which is subject to a maximum allowable guaranteed upper-bound except for the case of identically zero input on the current switching interval which allows an arbitrary next switching time instant \( t_{i+1} \in STI_{0} \cup STI_{\infty} \) since \( t_{i+1} \in STI_\infty \) since the function \( f(t, \delta) = \frac{g_{\sigma}(t_i)}{\max_{0 \leq \tau \leq \delta} u^2(t_i + \tau)} \) is nonnegative in the cases (b-c) (positive for the case c) and non-increasing, and then uniformly bounded on any definition domain \( [0, T_i] \) of nonzero measure, then there are always solutions in the incremental time argument \( \delta \) that satisfy the constraint \( 0 \leq \delta \leq f(t_i, \delta) \). Thus, Property (i) follows for (i.1) from case a and it follows for (i.2) from cases b-c. Property (ii) is proven as follows. Note that \( g_{\sigma}(t) \geq g_{\sigma}(t_0) = g_{\sigma}(0) > 0 \); \( \forall t \in [t_0 = 0, t_1] \) for any \( t_i \in STI \) since the first active parameterization after an arbitrary finite time is strictly positive real. From the cases (a)-(b) of the proof of Property (i), it follows that
\[
 g_{\sigma}(t) \geq g_{\sigma}(t_0) = g_{\sigma}(0) > 0; \forall t \in [t_0 = 0, t_1] \Rightarrow g_{\sigma}(t) \geq g_{\sigma}(t_1) \geq g_{\sigma}(t_0) = g_{\sigma}(0) > 0; \forall t \in [t_0 = 0, t^*]
\]
with \( t^* \in STI \) being the first switching time instant activating a positive real parameterization on \( [t^*, t^{**}] \) with \( t^{**} \) being the time instant of the next activation of a positive real parameterization. From the case (c) of the proof of Properties (i) for (i.1) and (i.2), it follows that \( g_{\sigma}(t^*) > 0 \Rightarrow g_{\sigma}(t) > 0 \);
\( \forall t \in [t_0 = 0, t^{**}] \). The subsequent time intervals of alternate activation of positive real and nonpositive real parameterizations are discussed in the same way leading to a complete induction proof of Property (iii). The two necessary conditions of Property (iii) follow directly by using simple contradiction arguments. □
Lemma 2 (uniform boundedness of the input-output energy measure). Assume that the switching law 
\( \sigma: R_0^+ \rightarrow \bar{\sigma} \) satisfies Lemma 1 and, furthermore, the nonlinear feedback device satisfies:
\[
\int_{t_i}^{t_i+\eta} \varphi_{\sigma_0(t)}(y(t), \tau) y(\tau) d\tau \geq -\gamma - \int_0^\eta \varphi_{\sigma_0(t)}(y(t), \tau) y(\tau) d\tau ; \forall t_i, t_{i+1} \in STI, \eta \in [0, T_i)
\]
where \( T_i = t_{i+1} - t_i \), and
\[
\int_{t_q}^{t_q+\eta} \varphi_{\sigma_0(t)}(y(t), \tau) y(\tau) d\tau \geq -\gamma - \int_0^\eta \varphi_{\sigma_0(t)}(y(t), \tau) y(\tau) d\tau ; \eta \in [0, \infty)
\]
if \( STI = \{ t_0, t_1, ..., t_q \} \), \( q < \infty \). Then, \( 0 \leq E(t) \leq \gamma < \infty \); \( \forall t \in R_0^+ \), i.e. the input-output energy measure is nonnegative and uniformly bounded for all time.

Proof: It follows directly from Lemma 1, the fact that (13) and (14) guarantee Popov’s inequality (3) and the equivalence of (3) and (4).

Remark 1. Note that Lemma 1 is formulated for the feed-forward linear part irrespective of the feedback law. Note also that the sufficient-type conditions of Lemma 1 guarantee that the switching law 
\( \sigma: R_0^+ \rightarrow \bar{\sigma} \) leads to a nonnegative input-output energy measure for all time under the necessary condition of fulfilment of such sufficient ones that the first active parameterization at the finite initial switching time instant \( t_0 \) be positive and that the last one be also positive real if there is a finite number of switchings (otherwise, sufficiency conditions fail in both cases). In this last case, the last active parameterization is required to be globally stable in order to guarantee the boundedness of the input-output energy for any admissible piecewise continuous input satisfying Popov’s inequality (3) under Lemma 2 holds. This follows easily from a simple counter example to the negation of the property. Assume the linear control \( u(t) = y(t) = -\varphi(y(t), t) \) that trivially satisfies Popov’s inequality that leads to 
\( E(t) - E(t_q) = \int_{t_q}^t y^2(\tau) d\tau \rightarrow \infty \) as \( t \rightarrow \infty \) for nonzero initial conditions if the last active parameterization on \( [t_q, t) \) is unstable.

Remark 2. Any switching law \( \sigma: R_0^+ \rightarrow \bar{\sigma} \) of the feed-forward loop involving only the activation of parameterizations with positive real transfer functions satisfies all the positivity conditions of Lemma 1. This result is a direct generalization of a well-known previous one for single parameterizations.

The following result gives conditions that guarantee that the input to the feed-forward loop is bounded and converges asymptotically to zero as time tends to infinity. This is a very important preliminary result to then guarantee asymptotic hyperstability under supplementary sufficient-type conditions related to switching rules on the parameterizations of the feed-forward linear loop.

Lemma 3. Assume a switching law \( \sigma: R_0^+ \rightarrow \bar{\sigma} \) of the feed-forward loop such that all \( \varphi_j: R^2 \rightarrow R ; j \in \bar{\sigma}_0 \) are piecewise continuous for any active parameterizations while satisfying (13)-(14) guaranteeing the integral Popov’s inequality (3). Then, the following properties hold:
(i) If all the active parameterizations are strictly positive real then the input is uniformly bounded for all time and, furthermore, \( \exists \lim_{t \to \infty} u(t) = 0 \).

(ii) Assume that the first active parameterization after an arbitrary finite time is strictly positive real and that all active non positive real parameterization, if any, is preceded by a strictly positive real one while satisfying the constraint of maximum allowable residence time interval (10) for any active non positive real active parameterization. Then, the input to the feed-forward loop is uniformly bounded for all time and, furthermore, \( \exists \lim_{t \to \infty} u(t) = 0 \).

(iii) Assume that the switching law activates infinitely many times strictly positive real parameterizations and that first active parameterization after an arbitrary finite time is strictly positive real. Assume also that the system feed-forward loop of any active parameterization on \( [0,\infty) \) which is not positive real has no pole at \( s = 0 \) and satisfies the saturation-vanishing input constraint \( |u(t)| \leq K e^{-\lambda t}; \forall t \in [t_i, t_{i+1}) \) with \( t_i \in STI_n \) for some real constants \( \lambda > 0 \) and \( K > 0 \) subject to \( \infty > \lambda > \max \left( \lambda_0, \frac{\ln T_i}{2T_i} \right) \) with \( T_i = t_{i+1} - t_i \) for some prefixed \( \lambda_0 \in R_+ \). Then, the input to the feed-forward loop is uniformly bounded for all time and, furthermore, \( \exists \lim_{t \to \infty} u(t) = 0 \).

**Proof:** (i) It all the parameterizations are strictly positive real then, one gets from (4), (9) and (12) since \( STI(t) = STI(t); \forall t \in R_{0^+} \):

\[
0 \leq \min_{t_j \in STI, \omega \in R_{0^+}} Re G_{\sigma(t_j)}(j\omega) \sum_{t_j \in STI(t_i)} \left( \int_{t_{j-1}}^{t_j} u^2(\tau) d\tau \right) \\
\leq g_{\sigma}(t) \sum_{t_j \in STI(t_i)} \min_{\omega \in R_{0^+}} Re G_{\sigma(t_j)}(j\omega) \left( \int_0^{T_{j-1}} u^2(\tau) d\tau \right) \leq \gamma < \infty; \forall t_i \in STI 
\]

(15)

\[
0 \leq \sum_{t_i \in STI} \min_{\omega \in R_{0^+}} Re G_{\sigma(t_i)}(j\omega) \left( \int_{t_i}^{t_{i+1}} u^2(\tau) d\tau \right) \leq \liminf_{t \to \infty} g_{\sigma}(t) \leq \limsup_{t \to \infty} g_{\sigma}(t) \leq \gamma < \infty
\]

\[
0 \leq g_{\sigma}(t) = g_{\sigma}(t_i) + \min_{\omega \in R_{0^+}} Re G_{\sigma(t_q)}(j\omega) \left( \int_{t_q}^{t_{i+1}} u^2(\tau) d\tau \right) \leq \gamma < \infty, \forall t \geq t_q (\in STI) \text{ if } STI \cap (t_q, \infty) = \emptyset
\]

(16)

The last property implies that there is a finite number of active parameterizations, all being strictly positive real. Note that all of them are finite for all \( s \in C \) since strictly positive real transfer functions cannot possess critical poles, then they are integrator-free. First, assume that \( \min_{\omega \in R_{0^+}} Re G_{\sigma(t)} \geq d > 0 \);

i.e. the relative order of the strictly positive real transfer function is zero so that it has the same number of zeros and poles. Note that the above amounts are, furthermore, strictly bounded from below by zero if the input \( u : R_{0^+} \to R \) is piecewise continuous and non-identically zero. Furthermore, if the switching action never ends then \( \text{card}(STI) = \infty \) and \( t_i (\in STI) \to \infty \). Thus, \( \lim_{t_i (\in STI) \to \infty} \int_{t_i}^{t_i + \tau} u^2(\tau) d\tau = 0 \);

\[ \forall \tau \in [0, T_i) \] and since the input is piecewise continuous and the sequence of integrals \( \int_{t_i}^{t_i + \tau} u^2(\tau) d\tau \)
has infinitely many elements, it follows that \( \exists \lim u(t) = 0 \) since \( \min_{t_i \in STI, \omega \in R_0^+} ReG_{\sigma(t_i)} > 0 \) (otherwise the sum of infinitely many elements in (15) can not be bounded which leads to a contradiction). If switching ends in a finite time instant \( t_q \in STI \) for some \( q \in N_0 \) then \( \text{card}(STI) = q + 1 < \infty \) and there is no switching for \( t > t_q \) so that \( \lim_{t \to \infty} \int_{t}^{t_q} u^2(\tau) d\tau = 0 \) and, again, \( \exists \lim u(t) = 0 \) since \( \int_{t_q}^{t} u^2(\tau) d\tau \) is a strictly increasing function of time for any \( 0 \leq t_q < \infty \) and any nonzero input, then contradicting (16), unless \( \exists \lim u(t) = 0 \).

Next, assume that \( \min_{\omega \in R_0^+} ReG_{\sigma(t)} > 0 \) and \( \lim_{\omega \to \infty} ReG_{\sigma(t)} = 0 \), i.e. the relative order is one so that the number of the poles minus that of the zeros is one. In this case, define a strictly decreasing nonnegative real function \( \omega_0: R_0^+ \to R_0^+ \), that is, \( \omega_0 = \omega_0(\epsilon) \to +\infty \) as \( \epsilon \to 0^+ \). Thus, (15) leads to:

\[
0 \leq \min_{t_j \in STI(t_j)} ReG_{\sigma(t)}(j\omega) \sum_{t_j \in STI(t_j)} \left( \int_{t_j}^{t_j+1} u^2(\tau) d\tau \right) = \sum_{t_j \in STI(t_j)} \left( \int_{t_j}^{t_j+1} u^2(t_j + \tau) d\tau \right) \leq \sum_{t_j \in STI(t_j)} \min_{\omega \in R_0^+} ReG_{\sigma(t)}(j\omega) \left( \int_{t_j}^{t_j+1} u^2(t_j + \tau) d\tau \right) 
\leq \gamma_0 - 2 \sum_{t_j \in STI(t_j)} \int_{0}^{\infty} ReG_{\sigma(t)}(j\omega) U_{\omega_0(t_j)}(j\omega) d\omega < \infty \quad \forall t_j \in STI
\] (17)

for any positive finite real constants \( \epsilon, \zeta \), and \( \gamma_0 \) so that

\[
\infty > \gamma_0 > \lim_{\epsilon \to 0^+} \inf_{\omega \in R_0^+} \left( \gamma + \zeta + 2 \sum_{t_j \in STI(t_j)} \int_{0}^{\infty} ReG_{\sigma(t)}(j\omega) U_{\omega_0(t_j)}(j\omega) d\omega \right) = \gamma + \zeta > 0
\]

since Popov’s inequality (3) implies that \( \int_{0}^{\infty} \varphi_{\sigma_0}(t) y(t) d\tau \geq -\gamma y \geq -\gamma_0 > -\infty \) for any finite real constant \( \gamma_0 \geq \gamma \) and \( \lim_{\epsilon \to 0^+} \omega_0(\epsilon) = +\infty \). Thus, \( \exists \lim u(t) = 0 \) remains valid if the strictly positive real transfer functions have relative order equal to one. Property (i) has been proven.

Property (ii) follows from similar considerations as those in the proof of Property (i) from (9) and (12) under the maximum allowable residence time constraint (10) for any active non positive real active parameterizations provided that the first active parameterization after an arbitrary finite time is strictly positive real.

Property (iii) is proven by noting that: 1) \( t_i \in STI \) does not need to be accounted for in Popov’s inequality or equivalent feed-forward loop since the minimum real value of its associate feed-forward transfer function is zero; 2) one gets the following relations for parameterizations which are not positive real under no critical pole at \( s = 0 \), what implies \( \left| ReG_{\sigma(t)}(j\omega) \right| < \infty \) together with the input saturation –vanishing constraint \( |u(\tau)| \leq K e^{-\lambda t_i} \), \( \forall t \in [t_i, t_{i+1}] \) for \( t_i \in STI \):

\[
-\sum_{t_i \in STI_n} \int_{t_i}^{t_{i+1}} y(t) \varphi_{\sigma_0}(t) y(t_1) d\tau = \sum_{t_i \in STI_n} \int_{t_i}^{t_{i+1}} u(t) y(t) d\tau \leq \frac{1}{2\pi} \sum_{t_i \in STI_n} \max_{\omega \in R_0^+} ReG_{\sigma(t)}(j\omega) \int_{0}^{\infty} U_{\omega_0(t)}(j\omega) d\omega
\]
\[ \leq \max_{\omega \in \mathbb{R}_0^+, \tau \in STI_n} \left| \text{Re} G_{\sigma (t_1)} (j \omega) \right| \sum_{t_j \in STI_n} \int_{t_j}^{t_{j+1}} u^2 (\tau) d\tau \]

\[ \leq K^2 \max_{\omega \in \mathbb{R}_0^+, \tau \in STI_n} \left| \text{Re} G_{\sigma (t_1)} (j \omega) \right| \sum_{t_j \in STI_n} T_i e^{-2 \lambda t_j} \leq \frac{K^2}{1 - e^{-2 \lambda_0 t}} < \infty \]  

(18)

Since a (finite) \( \lambda \) - constant fulfilling \( \infty > \lambda > \max \left( \lambda_0, \max_{t_j \in STI_n} \ln T_j / 2 T_j \right) \) exists since \( \max_{t_j \in STI_n} \ln T_j / 2 T_j < \infty \),
equivalently, \( \max_{t_j \in STI_n} \left( T_i e^{-2 \lambda t_j} \right) < 1 \), for \( T_j \in (0, \infty) \) and \( \lim \sup_{T_j \to \infty} \max \left( \lambda_0, \max_{t_j \in STI_n} \ln T_j / 2 T_j \right) < \infty \), where \( t^* = \min (t : t \in STI_n) \). The combination of (15)-(17) with Popov’s inequality equivalent versions (3) and (4), guaranteed under (13)-(14), Lemma 2 and (6) yields after separating the bounded contribution of non strictly positive real parameterizations from the strictly positive real ones of zero relative order:

\[ 0 < \min_{t_j \in STI_p} G_{\sigma (t_j)} \int_{0}^{t_j} u^2 (\tau) d\tau < E (t) \leq \gamma + \frac{K^2}{1 - e^{-2 \lambda_0 t}} < \infty \]  

(19)

Since the switching law possesses infinitely many strictly positive real active parameterizations by hypothesis, it follows that \( \int_{0}^{t_j} u^2 (\tau) d\tau \) cannot be a strictly increasing function of time, since, otherwise, a contradiction would follow from (19), thus, \( \exists \lim_{t \to \infty} u (t) = 0 \). For the case of relative order equal to one, the proof follows closely to the corresponding case in the proof of Property (i).

**Remark 3.** Note that Lemma 3 (ii) is useful for the case of switching laws when there are infinitely many active strictly positive real parameterizations and those which are not strictly positive real can be active either a finite number of times or infinitely many times while being subject to a maximum residence time constraint. On the other hand, Lemma 3 (iii) admits infinitely many non strictly positive real parameterizations in the switching law but subject to a time-dependent saturation-vanishing input constraint in the feed-forward loop while excluding transfer functions with poles at the origin. In particular, positive real transfer functions which are not strictly positive real with simple poles at the origin are excluded of the switching law as a result.

**4. Asymptotic hyperstability of the nonlinear switched system**

This section gives some results on asymptotic hyperstability for switching laws among different linear parameterizations in the feed-forward loop for nonlinear feedback devices subject also to switching laws and satisfying Popov’s inequality (3). The following known result for linear switched systems will be used. It is known for switchings among stable parameterizations that if a set of matrices of dynamics has a common Lyapunov function then stability is preserved under arbitrary switching. However, in general, if a common Lyapunov function does not exist for all the parameterizations, it is needed that a minimum residence time be respected at each active stable parameterization. The duration of such a residence time depends on the previous active parameterizations, the corresponding stability abscissas and the time intervals of activity of those ones in the switching law. It is only required to guarantee global stability under a switching law that at least one parameterization be stable with a sufficient large residence time.
related to the preceding active configurations and the various corresponding residence times accounted from the last time instant that a stable parameterization was active.

**Theorem 1.** The following properties hold:

(i) Let $A = \{A_i : i \in \overline{p}\}$ be a set of $p$ Hurwitz matrices. Then all matrices in $A$ have a common Lyapunov function if and only if $\sum_{i=1}^{p} \left(A_i X_i + X_i A_i^T\right) < 0$ for any given $n$-matrices $X_i \succeq 0$; $i \in \overline{p}$ ("\succeq" denoting matrix positive semidefiniteness). The open-loop switched system (1), i.e. $u = 0$, is globally asymptotically stable for any given arbitrary switching law $\sigma: R_{0+} \rightarrow \overline{p}$.

(ii) The matrices in $A$ do not have a common Lyapunov function if and only if there is a set of $n$-matrices $X_i \succeq 0$; $i \in \overline{p}$ such that $\sum_{i=1}^{p} \left(A_i X_i + X_i A_i^T\right) = 0$.

(iii) The matrices in $A$ do not have a common Lyapunov function if at least one non-Hurwitz matrix exists in the set $A = A \cup \{A_i^{-1} : i \in \overline{p}\}$.

(iv) The set $A = \{A_i : i \in \overline{p}\}$ of Hurwitz matrices has a common Lyapunov function only if $\sum_{i=1}^{p} (\alpha_i A_i + \beta_i A_i^{-1})$ is a Hurwitz matrix; $\forall \alpha_i, \beta_i \in R_{0+}$ such that $\sum_{i=1}^{p} (\alpha_i A_i + \beta_i A_i^{-1}) \in R_{1+}^{n \times n}$.

(v) The set $A = \{A_i : i \in \overline{p}\}$ of Hurwitz matrices has a common Lyapunov function if

$$V(x(t)) = x^T(t)P_x(t)$$

where $P = P^T > 0$ is a positive definite real $n$-matrix if

$$\|A_i - A_k\| < \frac{\lambda_{\text{min}}(Q_k)\lambda_{\text{max}}(A_k) - \epsilon}{K_k^2 \lambda_{\text{max}}(Q_k)} = \frac{\lambda_{\text{max}}(A_k^T P + PA_k)\lambda_{\text{max}}(A_k) - \epsilon}{K_k^2 \lambda_{\text{min}}(A_k^T P + PA_k)}$$

$; \forall i \in \overline{p}$ for any given $k \in \overline{p}$, for any given arbitrary real constant $\epsilon \in (0, \lambda_{\text{max}}(A_k)]$ and some testable real constant $K_k \geq 1$, where $\lambda_{\text{max}}()$ and $\lambda_{\text{min}}()$ stand for the $\lambda()$-matrix and $Q_k = -(A_k^T P + PA_k)$. The open-loop system (1)-(2) is globally exponentially stable for any arbitrary switching law $\sigma: R_{0+} \rightarrow \overline{p}$, $\forall p \in N$.

**Proof:** Properties (i)-(iv) are given in [23-24]. Property (v) is proven as follows: Since $A_k$ is Hurwitz then it satisfies a Lyapunov equation $A_k^T P + PA_k = Q_k = -Q_k^T < 0$ for any given $k \in \overline{p}$ and $Q_k = Q_k^T > 0$ and $P = P^T = \int_0^\infty e^{A_k^T \tau} Q_k e^{A_k \tau} d\tau > 0$ satisfying the Lyapunov equation since for any real constant $\epsilon \in (0, \lambda_{\text{max}}(A_k)]$ ($\epsilon \in [0, \lambda_{\text{max}}(A_k)]$) if all the eigenvalues of $A_k$ are distinct) and some testable real constant $K_k \geq 1$, one has:

$$\|P\| = \lambda_{\text{max}}(P) = \int_0^\infty e^{A_k^T \tau} Q_k e^{A_k \tau} d\tau \leq K_k^2 \lambda_{\text{max}}(Q_k) \int_0^\infty e^{-2(\lambda_{\text{max}}(A_k) - \epsilon)\tau} d\tau \leq \frac{K_k^2 \lambda_{\text{max}}(Q_k)}{2(\lambda_{\text{max}}(A_k) - \epsilon)}$$

Then,
\[
A_i^T P + PA_i = A_k^T P + PA_k + (A_i^T - A_k^T)P + P(A_i - A_k) = -Q_i = -Q_k^T < 0; \quad \forall i \in \overline{\mathcal{I}}
\]  
(22)
is guaranteed if
\[
2\lambda_{\text{max}}(P)\|A_i - A_k\|_2 \leq \frac{K^2 \lambda_{\text{max}}(Q_k)}{\lambda_{\text{max}}(A_k)} \|A_i - A_k\|_2 < -\lambda_{\text{max}}(A_k^T P + PA_k) = \lambda_{\text{min}}(Q_k)
\]
\[
\Leftrightarrow \|A_i - A_k\|_2 < \frac{\lambda_{\text{min}}(Q_k) \|A_i^T P + PA_k\|_2}{K^2 \lambda_{\text{max}}(Q_k)}; \quad \forall i \in \overline{\mathcal{I}}
\]  
(23)and any given \( j \in \overline{\mathcal{I}} \) and Property (v) follows since
\[\dot{V}(x(t)) = -x^T(t)Q_{\sigma(t)}(t)x(t) < 0 \quad \text{if} \quad x(t) \neq 0 \quad \text{with} \quad Q_{\sigma(t)}(t) = Q_k > 0 \quad \text{for some} \quad k \in \overline{\mathcal{I}}; \quad \forall t \in \overline{\mathcal{R}}_{0+} \quad \text{and any given switching law} \quad \sigma: \overline{\mathcal{R}}_{0+} \rightarrow \overline{\mathcal{I}}. \]

The following result basically establishes in its simpler form that, if there is at least a stable parameterization and the residence time is subject to a minimum residence for each time interval where such a parameterization is active, then the open-loop system (1) is globally exponentially stable.

**Theorem 2.** Let \( \lambda_{ij} : j \in \overline{\mathcal{I}} \) be the spectrum of \( A_i \); \( \forall i \in \overline{\mathcal{I}} \) and let \( K_i(\in \mathbb{R}_+) \geq 1, \quad \rho_i \in \mathbb{R}; \quad i \in \overline{\mathcal{I}} \) be constants such that:

\[
\max Re \lambda_{ij} \leq -\rho_i (\max Re \lambda_{ij} < -\rho_i \quad \text{if there is some eigenvalue of} \quad A_i \quad \text{of multiplicity larger than one for} \quad \forall i \in \overline{\mathcal{I}})
\]

any \( i \in \overline{\mathcal{I}} \), and
\[
\|e^{A_i t}\| \leq K_i e^{-\rho_i t}; \quad \forall i \in \overline{\mathcal{I}}.
\]

Consider the open-loop system with a switching law \( \sigma: \overline{\mathcal{R}}_{0+} \rightarrow \overline{\mathcal{I}} \) generating the following sequence of switching time instants:

\[
\text{STI} = \text{STI}(\sigma) = \left\{ t_{i_0}, t_{i_0+1}, ..., t_{i_0+i_1} = t_1^*, t_{i_0+i_1+1}, ..., t_{i_0+i_1+i_2} = t_2^*, ..., t_{i_0+...+i_j} = t_j^*, .... \right\}
\]  
(24)with \( \{i_j\} \) being a finite or infinite strictly increasing sequence of nonnegative integers subject to \( i_0 = 0 \) and \( i_{j+1} - i_j \leq \xi < \infty \) under the following assumptions:

A.1. The set \( \mathcal{A} \) contains at least one Hurwitz matrix.

A.2. \( \text{STI} \supseteq \text{STI}^* = \left\{ t_i^* : i \in \overline{N}^* \cup \{0\} \right\} \) for some \( \overline{N}^* = \{1, 2, ..., N^*\} \subset \mathcal{N} \) is a set of marked switching time instants chosen provided that either a minimum residence time constraint is guaranteed if \( N^* = \infty \) for each current marked active parameterization \( \sigma(t_i^*); \quad \forall i \in \overline{N}^* \cup \{0\} \), being necessarily stable, according to:

\[
T_x^* = \left\{ t_i^* : i \in \overline{N}^* \cup \{0\} \right\} \quad \text{satisfies} \quad \sum_{i=0}^{N^*-1} \ln K_i \left( \frac{\phi_{i_0+j}^{i_0+j} e^T}{\sigma(t_{i_0+j}^{i_0+j} e^T)} \right) - \sum_{i=0}^{N^*-1} \rho_i \left( \frac{\phi_{i_0+j}^{i_0+j} e^T}{\sigma(t_{i_0+j}^{i_0+j} e^T)} \sigma(t_{i_0+j}^{i_0+j} e^T) \right) \left[ \ln \delta \right] < 0
\]  
(25)for some real constant \( \delta \in (0, 1) \), or \( N^* < \infty \) with the last active parameterization \( \left\{ t_{N^*}, \infty \right\} \) being Hurwitz. Thus, the open-loop switched system is globally exponentially stable.
Proof. Note that the marked switching time instants $STI^* = \{t_*^i: i \in \overline{N}^* \cup \{0\}\}$ and their corresponding residence time instants are related with the values of $STI$ as follows:

$$t_{k+1}^* - t_k^* = t_{k+1}^{j,i} - t_k^{j,i} = \sum_{i,j=1}^{t_{k+1}^* - t_k^*} + \sum_{i,j=0}^{T_k^* - t_k^*} ; T_k^* = \sum_{j=0}^{t_k^* + 1} - \sum_{j=0}^{t_k^*}$$

; $\forall i \in STI, \forall t_k^* \in STI^*$ (26)

Note that for identically zero input the unforced solution of (1) satisfies on the time interval under some active stable parameterization with a Hurwitz matrix of dynamics $A\in \mathcal{A}\{A_i: i \in \mathcal{P}\}$ leading to $x(t_k^* + \tau) = e^{A\big(\tau - t_k^*\big)} x(t_k^*)$;

$$\forall \tau \in \left[0, T_{j,i}^{\infty}\right]$$

and the stability abscissa of the Hurwitz matrix $A\big(\tau - t_k^*\big)$ being

$$-\rho_{\big(\tau - t_k^*\big)} < 0.$$ Thus, one gets the following condition if the marked active residence time within

$$\left[t_k^*, t_{k+1}^*\right]$$

is large enough to satisfy the minimum residence time constraint (25):

$$\left\| x(t_{k+1}^*) \right\| \leq e^{-\rho_{\big(\tau - t_k^*\big)} T_{j,i}^{\infty}} \left( \prod_{i,j=0}^{t_{k+1}^* - t_k^*} K^{t_{j,i}^{t_k^*}} \right) e^{-\rho_{\big(\tau - t_k^*\big)} T_{j,i}^{\infty}} \leq \delta < 1 \quad (26)$$

and for any bounded initial conditions, the sequence $\left\{x(t_k^*)\right\}$ converges exponentially to zero for identically zero control provided that there are infinitely many switches (i.e. if switching does not end in finite time so that $\overline{N}^* = \infty$). Also, $x(t_k^* + \tau) = e^{A\big(\tau - t_k^*\big)} x(t_k^*) \rightarrow 0$ as $x(t_k^*) \rightarrow 0$;

$$\forall \tau \in \left[0, T_{j,i}^{\infty}\right].$$ Note also that $x(t_{N^*}^* + \tau) = e^{A\big(\tau - t_{N^*}^*\big)} x(t_{N^*}^*) \rightarrow 0$ at exponential rate as $\tau \rightarrow \infty$ is bounded for any bounded initial conditions since $t_{N^*}^* < \infty$, since $N^* < \infty$, implies that $x(t_{N^*}^*)$ is bounded. Thus, the open-loop switched system is globally exponentially stable. □

Note that Theorem 2 does not require explicitly that the residence time constraint has to be kept for each stable active parameterization or for that being stable if there is just one stable. The test can be applied at
finite time intervals of lengths being subject to prescribed upper-bounds. This can translate in the marked residence time resulting increased as such testing time intervals become larger. Now, Theorems 1-2 on global exponential stability of the open-loop system are combined with Lemma 3 to generate some asymptotic hyperstability-type results for the closed-loop system (1)-(2) subject to a Popov’s type integral inequality (3) for all time.

**Theorem 3.** Consider a switched feedback system (1)-(2) with the nonlinear feedback device satisfying a Popov’s inequality constraint (13), eventually both constraints (13)-(14) in the case of a finite number of switches. The following properties hold:

(i) Assume that all the transfer functions of the feed-forward loop being built with the various matrices of the set \(\mathbf{A}\) are strictly positive real (then also Hurwitz) and, furthermore, (20) holds, under the necessary and sufficient condition \(\sum_{i=1}^{p} (A_i X_i + X_i A_i^T) < 0\) for any given n-matrices \(X_i \geq 0; i \in \mathbb{P}\). Then, the closed-loop system is asymptotically hyperstable for any switching law \(\sigma: \mathcal{R}_{0+} \to \mathbb{P}\), for any \(p \in \mathbb{N}\), such that the nonlinear feedback device satisfies Lemma 2, i.e. Popov’s inequality (13) , eventually (13)-(14) if it generates only a finite number of switches. If, in particular, \(\phi_{\sigma(t)}(t) = \phi(t) \); \(\forall t \in \mathcal{R}_{0+}\), i.e. the nonlinear device does not depend on switching and satisfies Popov’s inequality (3), then the closed-loop system is unconditionally asymptotically hyperstable for any arbitrary switching law \(\sigma: \mathcal{R}_{0+} \to \mathbb{P}\).

(ii) Assume that all the matrices of \(\mathbf{A}\) are Hurwitz with their associate transfer functions being strictly positive real. If \(\sum_{i=1}^{p} (A_i X_i + X_i A_i^T) = 0\) for some n-matrices \(X_i \geq 0\), \(i \in \mathbb{P}\), or if \(\sum_{i=1}^{p} (\alpha_i A_i + \beta_i A_i^{-1})\) is not Hurwitz for some \(\alpha_i, \beta_i \in \mathcal{R}_{0+}\) subject to \(\sum_{i=1}^{p} (\alpha_i A_i + \beta_i A_i^{-1}) \in \mathcal{R}^{en+}\), then the closed-loop switched system (1)-(2) is not unconditionally asymptotically hyperstable for a switching-independent feedback nonlinear device \(\phi_{\sigma(t)}(t) = \phi(t)\); \(\forall t \in \mathcal{R}_{0+}\) any given switching law \(\sigma: \mathcal{R}_{0+} \to \mathbb{P}\) even if the transfer functions of all the parameterizations of the feed-forward loop are strictly positive real.

(iii) Consider a switched system (1)-(2) possessing (at least) one strictly positive real parameterization of the feed-forward loop under some switching law \(\sigma: \mathcal{R}_{0+} \to \mathbb{P}\) whose associate set of switching time instants \(\mathcal{STI} = \mathcal{STI}(\sigma)\) is subject to the subsequent constraints:

1) The first active parameterization through time after an arbitrary finite time is strictly positive real.
2) All active non positive real parameterization, if any, satisfies the constraint of maximum allowable residence time constraint (10), via (9), and it is preceded by a strictly positive real one.
3) There is a set \(\mathcal{STI}^* \subset \mathcal{STI}\) of marked switching time instants for some of the active strictly positive real parameterizations satisfying the minimum residence time constraint (25) for the time interval \([t_k^*, t_{k+1}^*]\) defined for each two consecutive marked switching instants.
4) The nonlinear feedback device satisfies Popov’s inequality (13) for the set \(\mathcal{STI}\), eventually (13)-(14) if it generates only a finite number of switches.
Then, the closed-loop system is asymptotically hyperstable for such a switching law $\sigma: R_{0+} \to \overline{p}$.

**Proof.** Property (i) follows from Lemma 3 (i) and Theorem 1 [(i), (v)] (since all the matrices in the set $A$ are Hurwitz since the associate transfer function are strictly positive real) guaranteeing global asymptotic stabilization at exponential rate under unconditional switching in the open-loop system since the switched open-loop system possesses a common Lyapunov function for all its active parameterizations under (20). The property is kept for the closed-loop system for any switching law just satisfying Popov’s inequality (13) (eventually (13)-(14), under a finite number of switching actions) in Lemma 2 since from Lemma 3(i), the feed-forward input converges asymptotically to zero under arbitrary switching. Property(ii) follows from the fact that the conditions of Theorem 1[(i)-(iv)] are necessary for unconditional switching of the open-loop forward linear system guaranteeing asymptotic stability. Property (iii) is a result for asymptotic hyperstability of (1)-(2) under conditional switching when a common Lyapunov function either does not exist or it is not guaranteed to exist by combining Lemma 1 (i.2) to be fulfilled for a maximum allowable residence time at non strictly positive real active parameterizations, Theorem 2 with a minimum residence time to be respected at (marked) active strictly positive real parameterizations, and Lemma 2, i.e. Popov’s inequality for the feedback nonlinear device. Then, the input to the feed-forward loop is uniformly bounded for all time and, furthermore, $\lim_{t \to \infty} u(t) = 0$ while the switched linear feed-forward loop is globally exponentially stable from Lemma 3 (ii). Note that there is a wide class of admissible switching laws under the above constraints so that hyperstability of the switched system is achievable since the switching time instants might be generically chosen under such constraints. □

**Remark 4.** Note that, in the general case, the sufficiency-type asymptotic hyperstability conditions of Theorem 3 involve (rather weak) constraints on the switching time instants to two levels, namely, maximum allowable residence time for active non positive real parameterizations and minimum residence time for certain test active strictly positive real parameterizations for any feedback nonlinear devices satisfying Popov’s inequalities. See Theorem 3 (iii). Theorem 3 [(i)-(ii)] refer to unconditional and conditional switching when all the active parameterizations are strictly positive real for the cases of existence or non existence of a common Lyapunov function for the active parameterization. □

**Remark 5.** Note that the stabilization results are applicable to minimal state-space realizations of the transfer functions and also to non-minimal ones with stable zero-pole cancellations of their transfer functions. This is directly concluded since, although non-minimal realizations are either uncontrollable or unobservable or both, the associate modes have asymptotically vanishing contributions with time to the state and output responses for any non-zero initial conditions and for any control input. □

**5. Simulation examples**

This Section contains some simulations illustrating the theoretical results stated in the previous sections. Thus, three different numerical examples will be presented. The first one corresponds to the case when all the linear feed-forward parameterizations are SPR and share a common Lyapunov function, implying that the asymptotic hyperstability is achieved under arbitrary switching. The second one corresponds to the
case when there is at least one SPR parameterization in the feed-forward loop while the remaining ones might not be SPR, being even unstable. In this case, stability is guaranteed if the SPR parameterization is kept active during a sufficiently large time interval. Finally, an application to the boost converter circuit is developed.

5.1 All the system feed-forward parameterizations are SPR and share a common Lyapunov function

In the first simulation we consider a set of three SPR active parameterizations of the feed-forward loop of the closed-loop system sharing a common Lyapunov function. The SPR linear parameterizations are given by:

\[ G_1(s) = \frac{s + 3}{s^2 + 7s + 10}, \quad G_2(s) = \frac{s + 2}{s^2 + 6s + 9}, \quad G_3(s) = \frac{s + 1}{s^2 + 12s + 12} \]

The strict positive realness of these transfer functions can be easily verified since they satisfy \( \text{Re}(G_i(s)) > 0 \) for \( \text{Re}(s) \geq 0 \). Furthermore, these parameterizations share a common Lyapunov function. In fact, if the dynamics matrices of these parameterizations are given by:

\[
A_1 = \begin{pmatrix} -7 & -10 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -6 & -9 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -12 & -12 \\ 1 & 0 \end{pmatrix}
\]

then, the function \( V(x) = x^T P x \) with \( P = P^T = \begin{pmatrix} 0.0451 & 0.0417 \\ 0.0417 & 1.0417 \end{pmatrix} > 0 \), which is the solution of the Lyapunov equation \( A_3^T P + P A_3 = -I \), is a common Lyapunov function for the complete set since \( \text{spec}(A_1^T P + P A_1) = \{-1.02, -0.36\} \), \( \text{spec}(A_2^T P + P A_2) = \{-1.01, -0.19\} \) and \( \text{spec}(A_3^T P + P A_3) = \{-1, -1\} \). In addition, the family of nonlinear feedback functions is given by \( \phi_i(y, t) = \tanh(\gamma_i y) \) where \( \gamma_i \in \{0.2, 0.3, 0.4\} \). Thus, the most general case defined by different nonlinear functions for each linear system parameterization is considered here. These nonlinear functions satisfy the integral-type Popov’s constraint specified by Eq. (3) since \( \tanh(\gamma_i y) > 0 \) for \( y \neq 0 \) and \( \tanh(\gamma_i y) y = 0 \) for \( y = 0 \) and \( i = 1, 2, 3 \), implying that \( \int_0^t \tanh(\gamma_i y) y dt > 0 \) for any \( t > 0 \) and all \( i = 1, 2, 3 \). Under these circumstances, Theorem 4(i) guarantees the closed-loop asymptotic hyperstability under arbitrary switching. This result is illustrated in the simulation showed in Figures 2 and 3 with initial conditions \( x_1(0) = x_2(0) = 1 \).
Figure 2. Switching function between the three SPR feed-forward parameterizations

Figure 3. State variables evolution

Figure 2 shows an arbitrary switching function with different residence times at each activated parameterization. The time evolution of the state components under this switching signal is showed in Figure 3. It can be appreciated that both states converge asymptotically to zero regardless of the switching between system feed-forward parameterizations, being then the closed-loop system asymptotically hyperstable, as Theorem 3(i) states. Furthermore, Figure 4 shows the control effort. The control action is not continuous since switching in the nonlinear feedback function introduces abrupt changes in it, some of which are marked in Figure 4. However, it can be appreciated in Figure 4 that the control signal vanishes asymptotically, as Lemma 3(i) states.
The next example is concerned with non-SPR parameterizations of the system feed-forward loop.

5.2. Not all the parameterizations are SPR in the system feed-forward loop

This second example is devoted to the study of hyperstability of switched systems when not all the parameterizations in the feed-forward loop are SPR. For this purpose, the following two transfer functions of parameterizations will be considered:

\[ G_1(s) = \frac{s + 3}{s^2 + 7s + 10}, \quad G_2(s) = \frac{s + 2}{s^2 - 6s + 9} \]

Notice that \( G_1(s) \) is the same SPR parameterization introduced in the previous example and that \( G_2(s) \) is unstable. Thus, we are now considering the most critical case. The family of nonlinear feedback functions is again given by \( \phi_i(y,t) = \tanh(\gamma_i y) \) where \( \gamma_i \in [0.2, 0.3] \). The following simulation showed in Figures 6 and 7 is obtained for the switching law depicted in Figure 5:

\[ \gamma_i \in [0.2, 0.3] \]
The switching function starts activating the stable parameterization and switches to the unstable one at certain time instants, showed in Figure 5. When the switching law activates the stable system, this is maintained active during a sufficiently large time interval in order to guarantee the boundedness of all signals of the system. Figure 6 shows the effect of switching in the state-space variables. When the SPR parameterization is active, the closed-loop system is asymptotically hyperstable and both state variables converge to zero. However, in the time intervals when the unstable parameterization is active, the state variables tend to diverge as Figure 6 shows. The effect of a sufficiently large activation time period for
the SPR system is to force the state variables to converge again to zero, maintaining them globally bounded. Finally, Figure 7 depicts the control signal. The control signal tends asymptotically to zero when the SPR parameterization is active but when the unstable one is activated, its value increases since the state variables and output of the system diverge. When the SPR parameterization is activated, then the control signal converges to zero again. In conclusion, if the SPR system is activated during a sufficiently large time interval before switching to a non-SPR parameterization, the state variables, input and output of the system are bounded, as Theorem 3(iii) establishes.

5.3 Boost converter switching circuit

This final example is concerned with the PWM-driven boost converter circuit represented in Figure 8.

Figure 8. The boost converter circuit

Here, \( L \) is the inductance, \( C \) the capacitance, \( R \) the load resistance and \( v_s(t) \) the source voltage. With this converter it is possible to convert the source voltage \( v_s(t) \) into a higher voltage \( v_C(t) \) at the load \( R \). This is possible by means of a switching device that can be active for two different physical positions. In the following, switch positions will be denoted by \( \sigma \in \{0,1\} \) as indicated in Figure 8. Hence, the boost converter is naturally a switched linear system with two different parameterizations. The application of basic circuit theory to the schematics of Figure 8 leads to the dynamics equations:

\[
\begin{bmatrix}
0 & -1 \sigma \\
1 & -\frac{L}{RC}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\dot{x}(t)
\end{bmatrix}
= \begin{bmatrix}
1 \\
L
\end{bmatrix} v_s(t)
\]

with state vector \( x(t) = [i_L \ v_C]^T \). If the output is given by \( y(t) = i_L = [1 \ 0]^T x(t) \) and \( R = 10^7 \ \Omega \), \( L = 10mH \) and \( C = 50nF \), then the two parameterizations are represented by the transfer functions:

\[
G_{\sigma=0}(s) = G_0(s) = \frac{100s + 200}{s(s+2)}, \quad G_{\sigma=1}(s) = G_1(s) = \frac{100s + 200}{s^2 + 2s + 210^7}
\]

Notice that \( G_1(s) \) is SPR while \( G_0(s) \) is not. Note also that \( G_1(s) \) has a stable zero-pole cancellation at \( s = -2 \) which is inherent to the boost converter device to work in the described way. This implies that such a mode is either not controllable or not observable or both depending on the state-space realization. However, since the cancellation is stable, it contribution to the output response vanishes asymptotically as time tends to infinity. Also, there is no error in the computation of the transient response associated with this cancellation issue if the state-space realization is directly used to calculate such a response. The
problem we will face consists in turning off the circuit, i.e. we want all the electric variables converge to zero from an initial steady-state value $x_0$. If the source voltage is directly set to zero, and the switching rule is given by Figure 9, the evolution of the intensity $i_L$ (output) is given by Figure 10 for $x_0 = (1, 1)^T$. A similar behavior is obtained for $v_C(t)$.

As it is appreciated in Figure 10, the output is very oscillating. Thus, in order to avoid these oscillations, a nonlinear static feedback controller is to be applied to this situation according to the diagram showed in Figure 1. This controller could be implemented, for instance, by nonlinear resistors. The objective of this controller is to make the oscillations vanish in a faster way, regardless of the potential switching between parameterizations. In the simulation, the nonlinear controller is given by the same function as in
Examples 1 and 2, $\varphi(y) = \tanh(y)$ with $\gamma = 0.8$. In this case, there is only one nonlinear static function, i.e. both feed-forward parameterizations are controlled by the same nonlinear controller. The evolution of the output is showed in Figure 11 while the control signal is depicted in Figure 12.

![Evolution of the intensity with the nonlinear static feedback controller](image1)

Figure 11. Evolution of the intensity with the nonlinear static feedback controller

![Control signal](image2)

Figure 12. Control signal

As Figure 11 shows, the nonlinear controller reduces significantly the oscillation amplitudes in the output regardless the switching process while the control signal also converges to zero, as Figure 12 reveals. The general frame introduced in this paper allows guaranteeing the stability of the closed-loop for any nonlinear device used as controller if it satisfies the Popov's integral-type constraint given by Eq. (3). Thus, the nonlinear controller could be implemented in this application in a variety of forms, guaranteeing the stability for all of them without needing to perform ad-hoc proofs for each one. Figures 11 and 12 also
show that when switching occurs, the output and the control signal change their behaviors as corresponds to the different system’s dynamics.

6. Concluding remarks
This paper has been devoted to the investigation of the asymptotic hyperstability of switched time-varying regulated dynamic systems subject to switching among linear time-invariant parameterizations in the feed-forward loop for any given feedback nonlinear time-varying regulator potentially also being subject to switching through time. The nonlinear control device is assumed to satisfy a Popov’s-type integral inequality. Asymptotic hyperstability is achieved in the most general case if at least one of the feed-forward parameterization possesses a strictly positive real transfer function, a minimum residence time interval is respected for each activation time interval of such a parameterization and a maximum allowable residence time interval is kept for all active parameterization which is not positive real, if any. When all the feed-forward activated parameterizations have a common Lyapunov function, asymptotic hyperstability of the switched dynamic system holds for any arbitrary switching law.

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