Robust adaptive control of linear time-delay systems with point time-varying delays via multiestimation

M. De la Sen

Department of Electricity and Electronics, Faculty of Science and Technology, Leioa (Bizkaia), Aptdo, 644 de Bilbao, Spain

Received 1 March 2005; received in revised form 1 December 2007; accepted 14 December 2007

Available online 31 December 2007

Abstract

This paper presents an adaptive pole-placement based controller for continuous-time linear systems with unknown and eventually time-varying point delays under uncertainties consisting of unmodeled dynamics and eventual bounded disturbances. A multiestimation scheme is designed for improving the identification error performance and then to deal with possibly errors between the true basic delay compared to that used in regressor vector of measurements of the adaptive scheme and also to prevent the closed-loop system against potential instability. Each estimation scheme in the parallel disposal possesses a relative dead-zone which freezes the adaptation process for small sizes of the adaptation error compared to the estimated size of the absolute value of the contribution of the uncertainties to the filtered output versus time. All the estimation schemes run in parallel but only that which is currently in operation parameterizes the adaptive controller to generate the plant input at each time. A supervisor chooses the appropriate estimator in real time which respects a prescribed minimum residence time at each estimation algorithm in operation. That strategy is the main tool used to ensure the closed-loop stability under estimates switching. The relative dead-zone in the adaptation mechanism prevents the closed-loop system against potential instability caused by uncertainties.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Adaptive control; Estimation; Parallel multiestimation scheme; Robust stability; Time delays

1. Introduction

Recent research in adaptive control has been devoted to robustness issues of continuous and discrete adaptive systems against unsuitable unmodeled dynamics and presence noise and to the relaxation of classical assumptions like the stability of the plant inverse and the knowledge of the sign of the high frequency gain (see, for instance, [1–6]). On the other hand, it is well-known that time-delay systems are a natural way for modeling some real processes like population growth, signal and fluid transmission, war and peace models, etc. Such systems have an infinite spectrum and the associate modes cannot be ensured to be close to their delay-free counterparts as the delay size increases what typically might cause instability, [7–9]. Important work has been devoted to the stability and stabilization, [10–15], and robust stability and stabilization ([13,16,17,9])

E-mail address: wepdepam@lg.ehu.es

0307-904X/S - see front matter © 2007 Elsevier Inc. All rights reserved.
of such systems. The design of memoryless stabilizing controllers has been considered in [13–15] while the design of delay-dependent controllers has been considered in [11,12]. In [15], the use of alternative stabilizing control laws with finite or infinite memory for systems subject to bounded point delays has been considered. The adaptive control problem for systems under internal point delays has also been considered (see, for instance, [18,19] and references therein). Adaptive control is based on parametrical estimation of totally or partly unknown physical processes and it is of increasing interest in real life processes [20–24]. The main drawbacks arise when internal delays are present since they typically cause an infinite asymptotic closed-loop spectra, like in the non-adaptive case, unless the adaptive controller compensates for the presence of plant delays. The point of view adopted in [19] has been to consider the choice of either a finite or infinite spectrum in the reference model due to design requirements. The first situation applies to the cases when the presence of delays is parasitic while the second one is useful for those when the presence of the plant delays is suitable in the closed-loop system.

Throughout this paper, the plant is assumed to be linear and possibly subject to unmodeled dynamics and bounded noise and it possibly operates under commensurate point delays. It is not assumed to be inversely stable and an over-bounding function of the contribution of the uncertainties dynamics to the output is not required to be known. Furthermore, the plant parameter vector is unknown and assumed to belong to a known convex set for all time while its time-derivative is not necessarily known and allowed to be impulsive. The main objective of the paper is to present a robustly stable parameter-adaptive scheme for linear time-invariant systems under unknown constant point delays in the presence of unmodeled dynamics and bounded noise. A multiestimation scheme is used to improve the identification error and then to deal with possibly errors between the true basic delay compared to that used in the regressor vector of the adaptive scheme. The various estimation schemes run in parallel but only that which is currently in operation parameterizes the adaptive controller to generate the plant input at each time. Each of those parameterizations is in operation during at least a minimum residence time interval. Each estimation algorithm in the parallel multiestimation scheme is of generalized least-squares type. Furthermore, a relative dead-zone is incorporated to the estimation scheme so that the adaptation is frozen when the identification (or adaptation) errors are small compared to the size of the estimated over bounded function of the contribution of the uncertainties to the output (see, for instance, [2,5,6,11,12] for the delay-free case). This strategy guarantees that the parameter estimation is not disrupted by small identification errors. The adaptive controller is based on pole-placement for the nominal (i.e. disturbance-free) known delay-free plant.

The use of an adaptation relative dead-zone is one of the basic design tools used for adaptive stabilization in the presence of a wide class of unmodeled dynamics and bounded noise, see, for instance, [5,6]. Such dead-zones are implemented in such a way that the estimates are maintained constant when the absolute value of the prediction error is small compared to the size of the contribution of the uncertainties dynamics to the output. In [2,6], the over-bounding function of the contribution of the various uncertainties to the filtered output, which is needed to build the relative adaptation dead-zone is measurable while it has been estimated by extending the estimation scheme in [5]. An alternative technique to achieve robust stability has been the combination of standard estimation procedures with projections of the estimates on known convex sets within of which the stabilization is guaranteed (see [1,2,5,6] and related references for details). Also, additional effort has been devoted in the last years to alleviate some of the cumbersome assumptions usually made on the controlled plant in the classical formulation in adaptive control. In particular, a controllability condition was obtained in [20] by using switching between different tuned controllers while the use of excitation in near-singular cases was proposed in [25] while the plant was not assumed to be controllable for adaptive stabilization in [26]. In [27] a discrete adaptive controller is obtained for non-minimum phase systems, namely, those being non-inversely stable. Also, robustness under switched parameterizations of the controlled system, controller or both have received attention (see, for instance, [28,21]). The research about robustness of adaptive and non-adaptive schemes in theoretical designs and applications has being progressing in the last years including fields, like, for instance, time-delay systems, neural networks of dynamic systems and Robotics (see, for instance, [29–36,22]. The use of a set of simultaneous estimators (multiestimation) has been proposed in order to improve the adaptive performance of the scheme. The basic stabilization and identification performance mechanism usually consists essentially of switching from the current controller to the one associated with a better registered performance at certain times. The performance evaluation is made according to a supervisory evaluation of performance in
terms of an appropriate quality index being the weighted time-integral of the square tracking or tuning errors over some past time interval, see, for instance, [37,38]. A practical reason to proceed in that way is that the use of several simultaneous estimation schemes, perhaps subject to different initial conditions, allows to easily dealing with possible changes in the plant operation points and with possible poor adaptation transients associated with a unique estimation scheme. A general multiestimation framework has been provided in [37] while each estimation algorithm operates for all time by generating a potential plant input but only one of them is injected to the system during appropriate time intervals from each controller switching to the next consecutive one. A related localization-based switching technique proposed in [28] for time-varying discrete systems ensures that the control switching converges rapidly. The closed-loop stability is preserved by appropriate selection of the switching times between controllers in all the above papers. The particular technique proposed [37,38] to evaluate a loss performance of square-integral with forgetting factor type of the identification error evaluated on some past time interval. The control strategy consists basically of switching to the current adaptive controller corresponding to some of the estimators in simultaneous operation to the one leading to the minimum cost according to such a function. That controller is maintained in operation until a new minimum cost is achieved. A prefixed minimum residence time is used as a lower-bound of any residence time in-between each two consecutive switches while it prevents against possible infinitely fast switching and ensures the existence of the problem solution.

As in [37,38], all the estimators operate simultaneously on the plant but the control input is generated as a convex linear combination of the set of potential control signals each associated with the individual adaptive controller associated with each estimator. Each estimator has its own regressor vector components. The plant is allowed to possess stable pole-zero cancellations which are not required to be known and do not influence the adaptive controller parameterization. In this way, the main properties of the adaptive scheme are independent of the plant physical realization being minimal or not and (if it is non minimal) on the number, multiplicity and location within the stability region of the unobservable and/or uncontrollable modes. The multiestimation philosophy might potentially work successfully when the estimation schemes have the same structures but, for instance, different initial conditions and/or different free-design parameters in the adaptive algorithm or when the estimators manipulate distinct structures of the updating algorithms. It has also been proved to be useful when the various estimators run over distinct parameter sets of the parameter space involving projections on the respective boundaries. Robust closed-loop stability is guaranteed for the class of uncertainties dealt with in [5,6]. It is pointed out that the proposed method can be combined with supervision techniques over past measurements to calculate the estimator weights. The extension of the proposed technique to the use of any finite number of estimators while preserving the robust stability of the closed-loop system is also focused on. Compared to the controller synthesis in [19], the parallel multiestimation technique proposed is useful to improve the adaptation transient for arbitrary initial conditions of the controlled process and the estimation algorithms. Another advantage is that it allows closed-loop stabilization if the delays are unknown but close to known nominal values.

Notation
- \( D^i := d/dt \) is the time-derivative operator formally equivalent to the Laplace operator \( s \). Consequently, \( D^i+1 = D \cdot (D^i) = d^{i+1}/dt^{i+1} \) with \( D^0 = 1 \). Also, \( e^{-hD} \) is the base time-delay operator for the base delay \( h \), the commensurate internal delays being \( h^k = kh; k = 1,2,\ldots,g \). For any natural number \( n \), define the finite set \( \bar{n} = \{1,2,\ldots,n\} \) which will allow to simplify the notation.
- \( \delta(\cdot) \) stands for the degree of the \( (\cdot) \)-polynomial and \( \delta_n(\cdot) \) stands for the degree with respect to the variable \( s \) of a quasi-polynomial in \( (s, e^{kh}) \).
- The notation \( v(t) = G(s)[v_0] = g^* v_0 \) is the zero-state response at time \( t \geq 0 \) of the realizable filter \( G(s) = B(s, e^{-h})/A(s, e^{-h}) \) for the signal input \( v_0(t) \) to all \( t \in [0,t] \), where \( B(s, e^{-h}) \) and \( A(s, e^{-h}) \) are quasi-polynomials of degrees satisfying \( \delta_n B \leq \delta_n A - 1 \) and \( g(t) \) is the impulse response of \( G(s) \), i.e., the Laplace inverse transform of \( G(s) = \text{Lap}(g(t)) \), which is the Laplace transform of \( g(t) \) with the Laplace transform argument being denoted by “s-rays”. The set of differential equations whose zero-state solution is \( v(t) \) under input \( u(t) \) is represented by \( A(D, e^{kh}) E(D, e^{kh})v(t) = B(D, e^{kh}) u(t) \).
- Consider the differential equation \( \bar{A}(D)v(t) = \bar{B}(D)u(t) + \eta(t) \) with \( \bar{A}(D) = A(D)E(D) \) and \( \bar{B}(D, e^{-hD}) = B(D, e^{-hD})E(D, e^{-hD}) \) denotes an uncertain linear and time-invariant plant. Thus, the terminology
'nominal plant' applies to the uncertainty-free (i.e. $\eta \equiv 0$) plant modeled by $\tilde{A}(D, e^{-sD})v_0(t) = \tilde{B}(D, e^{-sD})u(t)$. The terminology 'nominal transfer function' applies to $G(s) = B(s, e^{-Ds})/A(s, e^{-Ds})$, i.e. the perfectly modeled cancellation-free transfer function.

- The time argument, as well as the arguments $D$ and $s$, are sometimes omitted in the explicit notation for the sake of notation simplicity when no confusion is expected.

- $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ and trace $(\cdot)$ denote, respectively, the minimum, maximum and trace of the $(\cdot)$-matrix. In particular, $\beta_1 \leq \lambda_{\min}(P(t))$ and $\beta_2 \geq \lambda_{\max}(P(t))$ for $P(t)$ being the covariance matrix of the estimation algorithm.

- The $L_2$-matrix norm is denoted by $\|.(\cdot)\|_2$. If a subscript for norm is not used, it is meant that the kind of norm is irrelevant.

- $L_{\infty}$ is the set of scalar or vector real functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, some $n \geq 1$, such that $|f(t)| < \infty$ for all $t \in \mathbb{R}_0^+$ with $\mathbb{R}_0^+ \equiv \mathbb{R}^+ \cup \{0\} = [0, \infty) \cap \mathbb{R}$.

- $L_p$ is the set of scalar real functions $f : \mathbb{R}_0^+ \to \mathbb{R}$ such that $(\int_0^\infty |f(\tau)|^p d\tau)^{1/p} < \infty (p \geq 1)$.

- Scalar and/or vector functions $f, g : \mathbb{R}_0^+ \to \mathbb{R}^n (n \geq 1)$ are $f = O[g]$ if $|f(t)| \leq K_1(g(t) + K_2$ some real constants $K_{1,2} \geq 0$; and $f = O(g)$ and $\lim_{t \to \infty} |f(t)/g(t)| = 0$.

- The acronyms GS and GES stand for globally stable and globally exponentially stable system.

2. Plant structure and multiestimation scheme

2.1. Uncontrolled dynamic system

Time-delay dynamic systems are very common in nature and therefore of interest in theoretical fields and applications, see, for instance, [7–19] and, more recently, [39,40]. In particular, both academic and practical real life examples of systems with delays are described exhaustively in [7] and in [9]. Time-delay dynamic systems with internal delays (i.e. in the state) are usually infinite-dimensional systems even in the linear time-invariant case (see, for instance, [7,8,19,40]). Therefore, their homogeneous state-trajectory solution is not described by a $C_0$-semigroup generated by an infinitesimal generator as it is the case of their delay-free counterparts. The external description of a single-input single-output time-varying linear system (often referred to as “plant” or “controlled object”) subject to a finite set of time-varying (in general) incommensurate discrete time-varying delays (i.e. they are not constant in general and are not multiple of a base real number) is given by the differential equation

$$A(D, h_1, h_2, \ldots, h_q, t)y(t) = B(D, h_1, h_2, \ldots, h_q, t)u(t) + \eta(t),$$

where $y : \mathbb{R}_0^+ \to \mathbb{R}$ and $u : \mathbb{R}_0^+ \to \mathbb{R}$ are the input and output functions, $\eta : \mathbb{R}_0^+ \to \mathbb{R}$ is a real function including the combined effects of the unmodeled dynamics and external bounded disturbances on the output, and $h_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+ (i \in \bar{q})$ are the time-varying bounded delay functions. $D \equiv d/dt$ is the time-derivative operator. If $h_i = ih$ for some $h \in \mathbb{R}_1$, then the delays are constant, subject to $h_i < h_{i+1}$, and said to be commensurate since they are integer multiple of a real number $h$. Otherwise, the delays are said to be incommensurate, [9]. This is a typical situation in the standard theory of time-delay systems. A typical close practical situation in many real problems is when the delays are time-varying around constant nominal values; i.e. either $h_i(t) \in [i(h - \Delta h_i(t), i(h + \Delta h_i(t))]$ or $h_i(t) \in [h_i^0 - \Delta h_i^0(t), h_i^0 + \Delta h_i^0(t)]$ with $\Delta h_i(t) : \mathbb{R}_0^+ \to \Xi(0), \Xi(0)$ being a neighborhood of zero in $\mathbb{R}$. To set the theoretical framework for such a kind of problems, the time-invariant nominal system subject to time-invariant discrete commensurate delays $h_i = ih$ is first discussed from the point of view of adaptive stabilization. Then, a disturbed system which is subject to local time variations of the delays is investigated from the adaptive stabilization point of view provided that the nominal system is adaptively stabilizable. Consider the single-input single-output $n$th-order continuous-time minimal realization of a linear time-invariant system with $q$ internal point commensurate delays

$$A(D, e^{-hD})y(t) = B(D, e^{-hD})u(t) + \eta(t),$$

where $y(t), u(t)$ and $\eta(t)$ are the scalar output, input and a signal that quantifies the contribution of the unmodeled dynamics and bounded disturbances, respectively, with $D \equiv d/dt$ and $e^{-hD}$ being the time-derivative and delay operators, respectively. Thus, the nominal plant is described by (1) when $\eta \equiv 0$.  

962  

Remark 1. Let “s” be the Laplace argument which is formally identical to the time-derivative operator $D \equiv d/dt$. In the same way, $e^{-hs}$ is formally equivalent to the one-step delay operator $q^{-1}$ defined as $q^{-1}[v(t)] = v(t-h)$ for a given delay $h$. Now, note that the numerator and denominator of a nominal transfer function $G(s,e^{-hs}) = \mathcal{A}(s)e^{-hs}/\mathcal{B}(s)e^{-hs}$ describing equivalently the nominal system (1) are, respectively, quasi-polynomials in the polynomial ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of commensurate delays. By using a formal Laurent series expansion at infinity in the variable $s$ of the form $G(s,e^{-hs}) = \sum_{i=0}^{\infty} H_i(e^{-hs})s^{-i}$ with $H_i(e^{-hs}) \in \mathbb{R}(e^{-hs})$, it follows that there is a ring isomorphism $\mathbb{R}(s,e^{-hs}) \cong \mathbb{R}[s][e^{-hs}]$ namely, the ring of formal Laurent power series with coefficients over $\mathbb{R}$ at infinity. Note that the formal series ring $\mathbb{R}[s][e^{-hs}]$ is the completion of the polynomial matrix ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of noncommensurate delays. Finally, note that the above ideas are extendable to multivariable systems of the form derived in this paper applies also directly to any non minimal realization of (1) described by $G(s,e^{-hs})$, where $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are quasi-polynomials in the polynomial ring $\mathbb{R}(s,e^{-hs})$. In the same way, $e^{-hs}$ becomes lost in the corresponding quasi-polynomials.

Remark 1. Let “s” be the Laplace argument which is formally identical to the time-derivative operator $D \equiv d/dt$. In the same way, $e^{-hs}$ is formally equivalent to the one-step delay operator $q^{-1}$ defined as $q^{-1}[v(t)] = v(t-h)$ for a given delay $h$. Now, note that the numerator and denominator of a nominal transfer function $G(s,e^{-hs}) = \mathcal{A}(s)e^{-hs}/\mathcal{B}(s)e^{-hs}$ describing equivalently the nominal system (1) are, respectively, quasi-polynomials in the polynomial ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of commensurate delays. By using a formal Laurent series expansion at infinity in the variable $s$ of the form $G(s,e^{-hs}) = \sum_{i=0}^{\infty} H_i(e^{-hs})s^{-i}$ with $H_i(e^{-hs}) \in \mathbb{R}(e^{-hs})$, it follows that there is a ring isomorphism $\mathbb{R}(s,e^{-hs}) \cong \mathbb{R}[s][e^{-hs}]$ namely, the ring of formal Laurent power series with coefficients over $\mathbb{R}$ at infinity. Note that the formal series ring $\mathbb{R}[s][e^{-hs}]$ is the completion of the polynomial matrix ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of noncommensurate delays. Finally, note that the above ideas are extendable to multivariable systems of the form derived in this paper applies also directly to any non minimal realization of (1) described by $G(s,e^{-hs})$, where $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are quasi-polynomials in the polynomial ring $\mathbb{R}(s,e^{-hs})$. In the same way, $e^{-hs}$ becomes lost in the corresponding quasi-polynomials.

Remark 1. Let “s” be the Laplace argument which is formally identical to the time-derivative operator $D \equiv d/dt$. In the same way, $e^{-hs}$ is formally equivalent to the one-step delay operator $q^{-1}$ defined as $q^{-1}[v(t)] = v(t-h)$ for a given delay $h$. Now, note that the numerator and denominator of a nominal transfer function $G(s,e^{-hs}) = \mathcal{A}(s)e^{-hs}/\mathcal{B}(s)e^{-hs}$ describing equivalently the nominal system (1) are, respectively, quasi-polynomials in the polynomial ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of commensurate delays. By using a formal Laurent series expansion at infinity in the variable $s$ of the form $G(s,e^{-hs}) = \sum_{i=0}^{\infty} H_i(e^{-hs})s^{-i}$ with $H_i(e^{-hs}) \in \mathbb{R}(e^{-hs})$, it follows that there is a ring isomorphism $\mathbb{R}(s,e^{-hs}) \cong \mathbb{R}[s][e^{-hs}]$ namely, the ring of formal Laurent power series with coefficients over $\mathbb{R}$ at infinity. Note that the formal series ring $\mathbb{R}[s][e^{-hs}]$ is the completion of the polynomial matrix ring $\mathbb{R}[s,e^{-hs}]$ defined by the indeterminate complex arguments $(s,e^{-hs})$ which define together with a number of their powers $G(s,e^{-hs})$ in the case of noncommensurate delays. Finally, note that the above ideas are extendable to multivariable systems of the form derived in this paper applies also directly to any non minimal realization of (1) described by $G(s,e^{-hs})$, where $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are quasi-polynomials in the polynomial ring $\mathbb{R}(s,e^{-hs})$. In the same way, $e^{-hs}$ becomes lost in the corresponding quasi-polynomials.

Any possible cancellation quasi-polynomial $E(D,e^{-hD})$ as referred to in the notation is strictly stable and it has been cancelled in (1) and included in $\eta(t)$ although this is not reflected in the notation. In this context, the formalism derived in this paper applies also directly to any non minimal realization of (1) described by

$$\mathcal{A}(D,e^{-hD})y_f(t) = \mathcal{B}(D,e^{-hD})u_f(t) + \eta_f(t),$$

where

$$\mathcal{A}(D,e^{-hD}) = A(D,e^{-hD})E(D,e^{-hD}); \quad \mathcal{B}(D,e^{-hD}) = B(D,e^{-hD})E(D,e^{-hD}),$$

and $\eta_i(t) = E(D,e^{-hD})\eta_j(t)$ with $\partial_E \leq n$ provided that $E(D,e^{-hD})$ has all its zeros in $\mathbb{R}$ and $0 < n$. Cancellation-free $[\text{Assumption A.1(2) below}].$ If $E(D,e^{-hD}) \neq 1$ then $E(D,e^{-hD})$ gives extra poles to the reference model which have to be taken into account to establish its stability abscissa. That extension is direct and no related comments will be further given. The initial conditions of (1) are defined by almost everywhere absolutely continuous functions $\varphi_j; [-nh,0] \to \mathbb{R}$, including possibly bounded discontinuities on a set of zero measure, such that $D(t) = \varphi_f(t)$; $j = 0, 1, \ldots, n - 1$ with $\varphi_f(0) = \varphi_j(0) = x_0$ for $j = 1, 2, \ldots, n$. $A(D,e^{-hD})$ and $B(D,e^{-hD})$ are quasi-polynomials in the time-derivative and delay operator defined by
where $A_{(i)}$ and $B_{(i)}$ are polynomials defined by

$$A_k(D) = \sum_{i=0}^{n} a_{k,i} D^{n-i}; \quad B_k(D) = \sum_{i=0}^{m} b_{k,i} D^{n-i}$$

(5)

where $a_{k,i}, b_{j,i}, k, l = 0, 1, \ldots, n$ and $j, i = 0, 1, \ldots, m$ with $a_{00} = 1$; i.e. $A_0(D)$ is a monic polynomial, $b_{00} \neq 0$ and $m \leq n - 1$. Note that the combined use of the time-derivative and delay operators is easy to deal with. For instance, $D^4 e^{-hD} y_i(t) = e^{-hD} y_i(t - 4D)$ for any $k$th time-differentiable signal $y_i(t)$. In order to improve the filtering properties under the influence of possible disturbances and to accommodate the adaptation transient rates in the adaptive case, define filtered signals from (1)

$$y_f(t) = \frac{1}{F(D)} y(t); \quad \eta_f(t) = \frac{1}{F(D)} \eta(t),$$

(6)

where $F(D) = D^n + \sum_{r=1}^{n} f_r D^{n-i}$ is a n'th order monic Hurwitz polynomial of real constant coefficients. Thus, the filtered plant Eq. (1) becomes

$$A(D, e^{-hD}) y_f(t) = B(D, e^{-hD}) u_f(t) + \eta_f(t) + v(t),$$

(7)

so that the plant equation can be equivalently rewritten in regression form as

$$y(t) = F(D) y_f(t) = (F(D) - A(D, e^{-hD})) y_f(t) + B(D, e^{-hD}) u_f(t) + \eta_f(t) + v(t)$$

$$= \theta^T \varphi(t) + \eta_f(t) + v(t),$$

(8)

where $v(t)$ is an exponentially vanishing signal associated with the initial conditions of the filters, and

$$\theta^T = \left( \varphi_0^T; \theta_1^T; \ldots; \theta_q^T \right) ; \quad \varphi^T(t) = \left( \varphi_0(t); \varphi_1(t); \ldots; \varphi_q(t) \right)$$

$$\theta_0^T = \begin{pmatrix} 0; f_1 - a_{01}, \ldots, f_{n-1} - a_{0,n-1}; b_{00}, b_{01}, \ldots, b_{0m} \end{pmatrix}$$

$$\varphi_0^T(t) = \begin{pmatrix} D^n y_f(t); D^{n-1} y_f(t), \ldots, y_f(t); D^n u_f(t), D^{n-1} u_f(t), \ldots, u_f(t) \end{pmatrix}$$

(9)

$$\theta_i^T = \begin{pmatrix} a_{0i}, a_{1i}, \ldots, a_{mi}; b_{0i}, b_{1i}, \ldots, b_{mi} \end{pmatrix}$$

$$\begin{pmatrix} D^{n-1} y_f(t - ih), \ldots, y_f(t - ih); D^n u_f(t - ih), \ldots, u_f(t - ih) \end{pmatrix} \quad \forall i \in \bar{p}$$

The following assumptions are made on the plant (1):

**Assumption A.1**

1. There exists a known bounded set $\Omega$ which is either a convex compact region or a connected union of a finite number of (disjoint or not) compact sets; i.e. $\Omega = \bigcup_{i=0}^{q} \Omega_i$ such that $\theta \in \Omega_i$; some $i \in \bar{p}$.
2. For all $\theta \in \Omega$, the corresponding polynomials $A_0(D)$ and $B_0(D)$ are relatively prime, i.e. they have no common zeros when considered as complex functions of $D$ so that $(A_0(D), B_0(D)$ is a controllable and observable pair.
3. The base delay $h: [0, \infty) \rightarrow [h^* - \Delta h, h^* + \Delta h] \cap \mathbb{R}$ may be unknown where the nominal $h^*$ and the maximum error $\pm \Delta h (\Delta h \geq 0)$ are both real known constants. For exposition simplicity only, it is assumed through the manuscript that $h = h^*$. is known. The extension to the general case is direct by using distinct regressors for each estimation scheme.
4. $\delta = \sum_{i=1}^{q} (\|A_i(D)\| + \|B_i(D)\|)$ is sufficiently small for all $\theta \in \Omega$. 
Assumption A.2. There exists a nonnegative function of time \( \gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) satisfying
\[
\gamma(t) \leq \epsilon_1 t + \epsilon_2 + \epsilon_2^2 + e_1 \sup_{t \in \mathbb{R}_0^+} (e^{-2\rho_0(t-\tau)}\|\phi(t)\|^2) + e_2 \quad \text{for some known real constant } \rho_0 > 0 \text{ and possibly unknown real constants } e_1, e_2 \geq 0 \text{ such that } |\eta(t)|^2 \leq \gamma(t) \text{ for all time.}
\]

Remark 2. Assumption A.1(1), (2) is standard in pole-placement indirect adaptive control algorithms of time-invariant plants and leads to solvability of the diophantine equation associated with the synthesis of the pole-placement based controller in the delay-free nominal case. It means that any delay-free plant (1) parameterized in \( \Omega \), as well as its associate estimation model, are both controllable and observable. Such a requirement can be easily relaxed and extended to the case when it is stabilizable and detectable since the neglected stable cancellations cause an exponentially decaying neglected term in the control signal which does not modify the properties of the adaptive scheme. The controllability of the estimation model may be guaranteed for all time by using projection of the estimates on the boundary of the \( \Omega \)-domain, if necessary (see, for instance, [5,6], and references therein) what may lead to implementation difficulties. Assumption A.1(3), (4) are used to guarantee closed-loop stability in the presence of uncertainties through the synthesis of a pole-placement based adaptive controller even if the delays are unknown subject to Assumption A.1(3). In addition, global exponential adaptive closed-loop stability is also guaranteed in the absence of disturbances if the delay is unknown while belonging to a prescribed interval of sufficiently small measure [Assumption A.1(3)]. Finally, Assumption A.2 holds if the signal \( \eta(t) \) of the unmodeled dynamics contribution is the sum of a bounded disturbance and a signal related to the input by a strictly proper exponentially stable function. The constants \( e_1 \) and \( e_2 \) are not assumed known but estimated by extending the estimation scheme as proposed in [5] for the delay-free case.

2.2. Multiestimation algorithm

If the true parameter vector \( \theta \) is unknown and replaced with any estimate \( \hat{\theta}_i(t) \) of any of the estimation algorithms running in parallel \( \forall i \in \bar{p} \) then the identification (or adaptation) error becomes
\[
e_i(t) = y(t) - \hat{y}_i(t) = -\hat{\theta}_i^T(t)\phi(t) + \eta_i(t) + v(t),
\]
\[
\hat{y}_i(t) = (F(D) - \hat{A}_i(D, e^{-hD}, t))y_j(t) + \hat{B}_i(D, e^{-hD}, t)u_j(t)
\]
\[
\forall i \in \bar{p}, \text{where } \hat{y}_i(t) = (F(D) - \hat{A}_i(D, e^{-hD}, t))y_j(t) + \hat{B}_i(D, e^{-hD}, t)u_j(t) \text{ and } \hat{\theta}_i(t) = \hat{\theta}_i(t) - \theta \text{ are the output estimate and the parametrical error, respectively.} \hat{A}_i(D, e^{-hD}, t) \text{ and } \hat{B}_i(D, e^{-hD}, t) \text{ are the estimates of } A(D, e^{-hD}) \text{ and } B(D, e^{-hD}), \text{ respectively, by the } i\text{th estimator } \forall i \in \bar{p} \text{ with respective associate parametrical errors}
\]
\[
\hat{A}_i(D, e^{-hD}, t) = \hat{A}_i(D, e^{-hD}, t) - A(D, e^{-hD}); \quad \hat{B}_i(D, e^{-hD}, t) = \hat{B}_i(D, e^{-hD}, t) - A(D, e^{-hD}).
\]

Assumption A.3. \( \delta_j(t) = \sum_{i=1}^{\bar{p}} (||\hat{A}_j(D, t)|| + ||\hat{B}_j(D, t)||) \) is sufficiently small for all \( \theta \in \Omega_j \forall i \in \bar{p} \) the jth subscript denoting each estimator in the parallel scheme and all \( t \geq 0 \).

A least-squares type multiple estimation algorithm with adaptation relative dead-zone is proposed to prevent against instability caused by uncertainties. The presence of internal commensurate delays with unknown base delay \( h \) assumed
\[
\hat{P}_i(t) = -\beta_i(t)P_i(t)\phi(t)\phi^T(t)P_i(t); \quad P_i(0) = P_i^T(0) \geq k_0 I \quad (k_0 > 0),
\]
\[
(11)
\]
for \( \forall i \in \bar{p} \) with \( \text{Proj} \{ \cdot \} \) being a projection operator, [2,5,6] used to constraint the estimates of the \( i \)-th estimator within the bounded convex region \( \Omega, \forall i \in \bar{p} \) in the light of Assumption A.1 (1), and the relative adaptation dead-zone being

\[
\tilde{v}_i(t) = \begin{cases} 
\frac{z_i \varepsilon_i(t)}{1 + \phi^T(t)P_i(t)\phi(t)} ; & \text{if } |\varepsilon_i(t)| \leq \vartheta \tilde{v}_i(t)^{1/2}, \\
1 - \vartheta \frac{\tilde{v}_i(t)^{1/2}}{|\varepsilon_i(t)|}, & \text{otherwise,}
\end{cases}
\]

for some real constant \( z_1 > 0 \) where \( \vartheta > 1 \) is a design constant, and

\[
\tilde{\varepsilon}_i(t) = \begin{pmatrix} \tilde{v}_i(t) \\ \hat{v}_i(t) \end{pmatrix} \left( \sup_{t \in \mathbb{R}} (e^{-2\vartheta_i(t)}||\varphi(t)||^2, 1) \right),
\]

\[
\tilde{\varepsilon}_1(t) = \frac{z_1 \varepsilon_i(t)}{2(1 + \phi^T(t)P_i(t)\phi(t))} \sup_{t \in \mathbb{R}} (e^{-2\vartheta_i(t)}||\varphi(t)||^2); \quad \tilde{\varepsilon}_i(t) = \frac{z_1 \varepsilon_i(t)}{2(1 + \phi^T(t)P_i(t)\phi(t))}
\]

\( \forall i \in \bar{p} \); with \( \tilde{\varepsilon}_i(0) = \tilde{\varepsilon}_i(0) = 0 \) for any design constant \( z_0 > 0 \) \( \forall i \in \bar{p} \). In the following, the parametrical error is defined as \( \tilde{\theta}_i(t) = \theta_i(t) - \theta_i \forall i \in \bar{p}, j = 1, 2 \) for all \( t \geq 0 \).

2.3. Properties of the estimation algorithm

Note that the estimations of the \( e \)-constants are positive and non decreasing with time until a limit ensured by Theorem 1 is reached. Such a result is proved in Appendix A, is related to the properties of the multistimation algorithm (10)–(13) irrespective of the control law provided that Assumptions A.1, A.2, A.3 hold.

**Theorem 1.** The subsequent two items hold:

(i) \( ||\tilde{\theta}_i|| \in L_{\infty}, \ ||\tilde{\theta}_i|| \in L_{\infty}, \ |\tilde{\varepsilon}_i| = |\tilde{\varepsilon}_i - \varepsilon_i| \in L_{\infty}, |\tilde{\varepsilon}_i| \in L_{\infty}, \ \tilde{\varepsilon}_i(t) \to \varepsilon_i(t) = 0, \ \forall i \in \bar{p} \) as \( \vartheta > 1 \) \( \forall i \in \bar{p} \).

Also, \( b_{ij} = e_{ij} - e_i \in L_{\infty} \) if \( \vartheta > 1 \) and \( ||\tilde{\theta}_i|| \in L_{\infty} \) \( \forall i \in \bar{p} \).

(ii) \( b_{ij} = e_{ij} - e_i \in L_{\infty} \) if \( \vartheta > 1 \) \( \forall i \in \bar{p} \).

3. Adaptive controller design and closed-loop stability properties

3.1. Adaptive controller law

Each estimator in the parallel disposal generates a filtered control law candidate for each time \( t \):

\[
u_{ij}(t) = \frac{S_i(D, e^{-\delta D}, t)}{R_i(D, e^{-\delta D}, t)} (y_j^*(t) - y_j(t)) \quad \forall i \in \bar{p},
\]

the filtered control law for time \( t \in [t_i, t_{i+1}) \) being

\[
u_j(t) = \nu_{ij}(t) \quad \text{some } i \in \bar{p}
\]

for all \( t_i \in S \) (the sequence of switching times), where \( S_i(D, e^{-\delta D}, t) \) and \( R_i(D, e^{-\delta D}, t) = R^0_i(D, e^{-\delta D}, t) + R_i(D, e^{-\delta D}, t) \) are defined by time-varying quasi-polynomials

\[
S_i(D, e^{-\delta D}, t) = \sum_{k=0}^{q} S_{ik}(D, t)e^{-\delta k D}; \quad R^0_i(D, e^{-\delta D}, t) = \sum_{k=0}^{q} R_{ik}(D, t)e^{-\delta k D},
\]
defined through polynomials

\[ S_{ik}(D, t) = \sum_{\ell=0}^{m} s_{ik}^{(\ell)}(t)D^\ell; \quad R_{ik}(D, t) = \sum_{\ell=0}^{n} r_{ik}^{(\ell)}(t)D^\ell \]

(16)

\( \forall i \in \bar{p}; k \in \bar{q} \cup \{0\} \) and a rational (in general transcendent) time-valued complex variable function

\[ \tilde{R}_i(D, e^{-\mu t}) = -\frac{A_{mk}(D)}{A_i(D, e^{-\mu t})} \]

(17)

where the pair \((R_{ik}(D, t), S_{ik}(D, t))\) satisfies uniquely the set of diophantine equations of time-varying polynomials

\[ \tilde{A}_{ik}(D)R_{ik}(D, t) + \tilde{B}_0(D, t)S_{ik}(D, t) = A_{mk}(D) - \sum_{\ell=1}^{k} (\tilde{A}_{ik}(D, t)R_{ik-\ell}(D, t) - \tilde{B}_0(D, t)S_{ik-\ell}(D, t)), \]

(18)

with \( \tilde{A}(S_{ik}) - \tilde{A}(R_{ik}) - 1 = n - 1 \) for \( k = 0, 1, \ldots, n \); provided that \( \tilde{A}_{mk} \leq 2n \) \( (k = 0, 1, \ldots, m + 1) \), since \((A_{ik}, B_{ik})\) are all coprime pairs \((i = 1, 2, \ldots, p)\) from the estimation projection and Assumption A.1(1); and

\[ \tilde{A}_{mk}(D, e^{-\mu t}) = \sum_{k=m+1}^{2\bar{q}} \left[ \sum_{\ell=\text{Max}(0, k-n)}^{n} \tilde{A}_{ik}(D, t)R_{ik-\ell}(D, t) \right] e^{-\mu \ell t}. \]

(19)

3.2. Model multiestimation and control switching rule

The choice of the current filtered control input \((10. b)\) from those ones generated by the overall parallel multiestimation scheme is made by the subsequent switching rule. Define \( S = \{t_i; i \geq 1\} \) as the finite (or infinite) set of switching instants between estimation models \( i \in \bar{p} \) which satisfy the following: Let \( t_i \in S \). Then, \( t_{i+1} \in S \) if

1. \( T_t = t_{i+1} - t_i \geq T \) (\( T \) being the so-called minimum residence time) for all time in the switching sequence \( S \).
2. \( J(t_{i+1}) = \text{Min} (\ell \in P; J_\ell(t_{i+1})) \) \( \forall k \in P \) then \( J(t_i) = J_{k_1}(t_i) \Rightarrow J(t_{i+1}) = J_{k_2}(t_{i+1}) \) with \( k_1 \neq k_2 \) for any estimators \( k_1, k_2 \in P \), where \( J_{k}(t) = \int_{t-\tau}^{t} e^{-\mu(\ell-\tau)}(\lambda \xi^2(t) + (1-\lambda)u_{i1}^2(t))d\tau \), all \( i \in P \) for some prescribed forgetting factor \( \zeta > 0 \) and weighting factor \( \lambda \in (0, 1) \) which is a loss function which is a measure of a combined quality index for the identification and control effort. Note that each estimator is running for all time. However, the adaptive controller is parameterized by each estimator scheme during a time interval, subject to the above minimum residence time, before potential switching for re-parameterization of the adaptive controller.

3.3. Robust stability

The combined Eqs. (10) and (14) may be described through the auxiliary extended system

\[ \dot{x}(t) = \sum_{j=0}^{\bar{q}} A_j(t)x(t - j\mu) + be(t) + g(t) + v(t), \]

(20)

where the state vector, forcing signals and parameterization are defined by

\[ x^T(t) = \begin{pmatrix} D^{\bar{n} - 1}y_f(t), \ldots, Dy_f(t), y_f(t); & D^{\bar{n} - 1}u_f(t), \ldots, Du_f(t), u_f(t) \end{pmatrix}, \]
Theorem 2. The subsequent items hold:

(i) Assume that the plant (1) is uncertainty-free and perfectly known and that a stable reference model of transfer function \( 1/A_m(s) \) is set with \( \delta A_{mk} \leq 2n \) \((k = 0, 1, \ldots, m + 1)\). Thus, if Assumption A.1(1)–(3) holds then \( y_f(t), uf(t), y(t) \) and \( u(t) \) are bounded for all time provided that \( y^* \in L_1 \). Furthermore, if \( y^* = 0 \) then \( y_f(t) \to 0 \) as \( t \to \infty \); \( u_f(t) \to 0 \) and \( y(t) \to 0 \) as \( t \to \infty \); \( u(t) \to 0 \) exponentially as \( t \to \infty \) or any bounded initial conditions. If the delay is unknown but the error between the true and measured delay is sufficiently small then the closed-loop system is still GES.

(ii) Assume that the plant is uncertainty-free with unknown parameters and subject to Assumption A.1(1)–(3), and A.3. Assume also that the delay-free auxiliary system is GES. Thus, the adaptive controller based on the estimation algorithm (10)–(12) with a single estimation scheme leads to a GES closed-loop system provided for some \( i \)th estimator \( i \in \bar{p} \); all \( j \in \bar{q} \cup \{0\} \), running and generating the plant control input at any time \( t \). The vector signal \( v(t) \) is generated by the initial conditions of the filters. It may be directly obtained from \( v(t) \) in (7) and it is exponentially vanishing.

Assumption A.4. \( A(t) \equiv A_0(t) \) is almost everywhere time-differentiable in any open interval \((t, t + T)\) except possibly at a finite set \( \mathcal{C}_t \) of isolated instants \( t_i \in (t, t + T) \); \( i = 1, 2, \ldots, c_t \), where it is impulsive taking values \( K_0 \delta (t - t_i) \), so that \( \| \sum_{i=1}^{c_t} K_i \| \leq \alpha \) (i.e. \( A(t_i) \) is discontinuous and then \( \dot{A}(t_i) \) is impulsive at \( t_i \in (t, t + T) \)). Furthermore, there exist real constants \( \mu \in [0, \mu^*), \alpha \geq 0 \), some \( \mu^* > 0 \) such that for some real \( T > 0 \) and all \( t \geq 0 \),

\[
\int_t^{t+T} \| \dot{A}(\tau) \|^2 \, d\tau \leq \mu^2 T + \alpha,
\]

(or, alternatively, \( \sup_{t \geq 0} (\| \dot{A}(t) \|) \leq \mu \), some \( \mu \in [0, \mu^*]) \). Thus, the system is globally exponentially stable (GES) if \( \mu^* \) is sufficiently small.

Furthermore, note that \( A(t) \) is uniformly bounded and almost everywhere time-differentiable in any open interval \((t, t + T)\) and it has bounded entries and eigenvalues in \( \text{Re} s \leq -\rho_0 < 0 \) for all \( t \geq 0 \). This follows from Theorem 1 and the fact that the reference model is strictly stable with stability abscissa not exceeding \(-\rho_0 < 0\) which accounts for possible stable plant cancellations included as poles of the reference model. The time instants where it is not differentiable are those where the estimation scheme switches between two estimators. There is a finite number of switching instants within any finite interval since each estimator parameterizes the adaptive controller subject to a minimum residence time. The intuitive implications of Assumption A.4 are that the adaptation rate is sufficiently slow and that the residence time for each estimator to parameterize the controller, i.e., the interval between any two consecutive estimator switches where \( \dot{A}(t) \) is impulsive, is sufficiently large. The following main stability result is proved in Appendix B by first obtaining appropriate global stability results related to the stability of the auxiliary system (20)–(22):

Theorem 2. The subsequent items hold:

\[
A_j(t) = A_j^{(i)}(t) = \begin{bmatrix}
-a_{j1}^{(i)}(t), \ldots, -a_{jm}^{(i)}(t) \\
\vdots \\
I_{n-1}^T ; 0 \\
-b_{j0}^{(i)}(t), \ldots, -b_{jn-1}^{(i)}(t) \\
0 \\
I_{n-1}^T ; 0
\end{bmatrix}
\]

\[
G_i(t) = \begin{bmatrix}
0^T \\
s_i_{n-1}(t) \ldots s_i(t) \\
0^T
\end{bmatrix}
\]

\[
b^T = (1, 0, \ldots, 0) ; \quad g^T(t) = \sum_{j=0}^{n-1} G_j(t) y_{ef}^j(t - jh) ; \quad y_{ef}^T(t) = (D^{n-1}y_f^j(t), \ldots, Dy_f^j(t), y_f^j(t)),
\]

for some \( i \)th estimator \( i \in \bar{p} \); all \( j \in \bar{q} \cup \{0\} \), running and generating the plant control input at any time \( t \). The vector signal \( v(t) \) is generated by the initial conditions of the filters. It may be directly obtained from \( v(t) \) in (7) and it is exponentially vanishing.
that the error between the true and measured delay in Assumption A.1(4) and \( \delta_j = \text{Sup}_{t \geq 0}(\delta_j(t)) \forall j \in \bar{p} \) in Assumption A.3 are both sufficiently small compared to the stability abscissa of the delay-free auxiliary system. The precise condition becomes alleviated if the above parallel multiestimation model is used instead the single estimation one. The results still hold if the auxiliary delay-free system is GES.

Remarks

1. Assume that the plant is subject to uncertainties satisfying Assumption A.2 and estimated from (13). Thus, the closed-loop system is GS under similar conditions as in (ii).

4. Numerical example

A system of order \( n = 4 \) with unmodeled dynamics defined by the parasitic output versus input transfer function \( 0.1/(s + 21) \) and nominal delay-free parameter vector \( (1.68, -1.02, 0.15, 0.25, -0.28, 0.05) \) is considered where “s” is the Laplace transform argument. The delay is assumed to be 0.02\( s \) while unknown for controller synthesis purposes. The controller scheme assumes that there is a small unknown delay belonging to the interval \([0, 0.02]\). A number of real physical systems are described approximately by such a model. For instance, a cascade of four resistor-capacitor filters whose zeros are not very relevant in the dynamics for the range of frequencies of interest. The whole system possesses linear time-invariant parasitic coupling dynamics which is approximately described by the above transfer function. The linear reference model transfer function is given by a fixed third-order Hurwitz denominator polynomial with a real pole and two complex conjugates ones which is defined by \( A_m(s) = s^3 + 21s^2 + 199s + 660 \). Two estimation schemes are in parallel bi-estimation disposal are used which have identical linear filters for the input and output all defined by transfer functions \( R_{ik}, S_{ik} \) for \( k = 0, 1, \ldots, m \) and any \( \delta \)-estimator involve powers \( k = 0, 1, \ldots, 2m \) of \( e^{-\delta t} \) for \( k = 2m + 1 \) to \( 2q \).

5. Concluding remarks

The paper has been devoted to multistation in systems subject to a finite number of point delays. The delays are allowed to be time-varying and the case when they evolve locally around fixed nominal values has
been studied in detail from a point of view of adaptive stabilization. Firstly, the investigation has been concerned with the adaptive controller synthesis for the case of known controlled objects. Then, the adaptive control scheme is provided based on a multistimation scheme. This is useful for the case when the delays are not known exactly. The robust adaptive stabilization properties have been also established and discussed.

**Acknowledgements**

The author is very grateful to the Spanish Ministry of Education and to the Basque Government by their respective partial support of this work via Project DPI 2006-00714, and Research Grants: Research Group Nos. IT-269-07 and SAIOTEK 2006/S-PED06UN10. He is also grateful to the referees by their useful comments.

---

Fig. 1. Accumulated time-integral of the squared tracking errors versus time.
Appendix A. Properties of the single estimates

Proof of Theorem 1. (i) Consider Lyapunov’s-like candidate functions

\[ V_i(t) = \frac{1}{2} (\tilde{\theta}_i^T(t)P_i^{-1}(t)\tilde{\theta}_i(t) + (\bar{v}_i^2(t) + \bar{v}_{i2}(t))) \]  

(A.1)

\( \forall i \in p. \) One gets by using the estimation algorithm (10)–(13), \( \dot{P}_i^{-1}(t) = -P_i^{-1}(t)\dot{P}_i(t)P_i^{-1}(t), \) and (9) and neglecting the asymptotically vanishing signal \( \vartheta(t) \)

\[ \dot{V}_i(t) = b_i(t)\tilde{\theta}_i^T(t)\varphi(t)e_i + \frac{1}{2} b_i(t)(\tilde{\theta}_i^T(t)\varphi(t))^2 + \frac{1}{2} b_i(t)(\bar{v}_i^2(t) + \bar{v}_{i2}(t))(\sup_{0 \leq \tau \leq t}(e^{-2p_0(\tau-t)}\|\varphi(\tau)\|^2, 1)) \]

\[ \leq -\frac{1}{2} b_i(t)(\gamma_i(t) + \bar{\gamma}_i(t)) \]  

(A.2)

with

\[ \gamma_i^T(t) = (\bar{v}_i(t), \bar{v}_{i2}(t)) \left( \sup_{0 \leq \tau \leq t}(e^{-2p_0(\tau-t)}\|\varphi(\tau)\|^2, 1) \right), \]

(A.3)

\[ \bar{\gamma}_i(t) \geq (\vartheta^2 - 1)\gamma_i^T(t) \geq \frac{1}{2} b_i(t) \left( \frac{\vartheta^2 - 1}{\vartheta^2} \right) e_i^2(t) \geq 0. \]

(A.4)

Since the time-derivatives of the estimates of the \( \bar{v}_i \)-constants are always nonnegative and their initial conditions are zero, \( \gamma_i(t) \leq 0 \) for all \( t \geq t_0 \) with \( t_0 = \max(t_{i0}, j = 1, 2) \) being such that \( \bar{v}_{ij}(t) \geq \bar{v}_j \) for all \( t \geq \bar{v}_{ij}(t) \geq \bar{v}_j; j = 1, 2; \forall i \in p. \) It is obvious that such a finite time \( t_{i0} \) exists \( \forall i \in p. \) Thus, \( \dot{V}_i(t) \leq 0 \) for all \( t \geq t_0 \) (some finite \( t_{i0} \)) so that \( V_i \leq L_{\infty}, \) if \( \dot{\theta}(0) \) is bounded, guarantees \( \|\dot{\theta}_i\| \leq L_{\infty}, \|\dot{\theta}_i\| \leq L_{\infty} \) and \( \|\bar{e}_{ij}\| \leq L_{\infty}, \) \( \|\bar{e}_{ij}\| \leq L_{\infty}, \) and have nonnegative finite limits. It also exists a finite \( t_0 = \max(t_{i0}, \forall i \in p \forall i \in p) \) such that all the \( \dot{V}_i(t) \leq 0 \) are non positive for all \( t \geq t_0 \) \( \forall i \in p. \) Since \( \|\dot{\theta}_i\| \leq L_{\infty} \) and \( \gamma_i^0 \in L_{\infty} \) from (A.8), so that \( \{b_i\gamma_i\} \in L_{\infty} \) and \( b_i^2/\vartheta_i \in L_{\infty} \) since \( \gamma_i^T(t) \geq |e_i(t)| \) for \( t \geq t_0 \) (finite) if \( b_i \neq 0 \) for \( \forall i \in p. \) Also, \( b_i|\dot{\theta}_i| \in L_{\infty}, \) and, furthermore, \( \theta_i \leq L_i \cap L_{\infty} \) since \( \gamma_i \in L_i \cap L_{\infty} \) and \( \|b_i|P_i\varphi\varphi^T\| \in L_{\infty}. \) Also, \( b_i|\eta_i - e_i^2| \in L_{\infty} \) if \( \vartheta > 1 \forall i \in p. \) Item (i) has been fully proved.

(ii) Since \( \gamma_i(t) \geq 0 \) and \( \bar{\gamma}_i(t) \geq 0 \) if \( \bar{e}_{ij} \leq \bar{v}_j \) \( \forall i \in p; j = 1, 2 \) with possible switching-off the estimation of those constants, both terms that upper-bound for all time the time-derivatives of the Lyapunov functions in (A.1) are no positive. Thus, \( b_i(\gamma_i - e_i^2) \in L_1 \cap L_{\infty}, b_i^1/2(\gamma_i^{1/2} + |e_i|) \in L_1 \cap L_{\infty} \) and \( b_i^1/2(\gamma_i^{1/2} - |e_i|) \in L_1 \cap L_{\infty}, b_i^1/2(\gamma_i^{1/2}) \in L_1 \cap L_{\infty} \) and \( b_i^1/2|e_i| \in L_1 \cap L_{\infty} \). Also, all the above signals converge asymptotically to zero as time tends to infinity. Also, \( b_i(\eta_i - e_i)^2 \in L_1 \cap L_{\infty} \) and converges asymptotically to zero as time tends to infinity if \( \vartheta > 1 \). The proof of (ii) is complete.

Appendix B. Preparatory auxiliary results for Theorem 2

Proof of Theorem 2. (i) The diophantine closed-loop equation for the time-invariant case of known parameters becomes from (17), (18)

\[ A(D, e^{-HD})R(D, e^{-HD}) + B(D, e^{-HD})S(D, e^{-HD}) = A_m(D, e^{-HD}), \]

(B.1)

where \( A_m(D, e^{-HD}) = \sum_{k=0}^{2q} A_mk(D)e^{-HD} \) with \( \partial A_mk \leq 2m \) \( (k = 0, 1, \ldots, m), \) since \( \sim R(D, e^{-HD}) = \frac{-A_m(D, e^{-4HD})}{A(D, e^{-2HD})}, \)

and

\[ A(D, e^{-HD})R^0(D, e^{-HD}) = \sum_{k=0}^{2q} \sum_{\ell=\max(0,k-n)}^{\min(n,k)} A_{\ell}(D)R_{k-\ell}(D)e^{-H\ell D}, \]

(B.2a)

\[ B(D, e^{-HD})S(D, e^{-HD}) = \sum_{k=0}^{2q} \sum_{\ell=\max(0,k-m)}^{\min(m,k)} B_{\ell}(D)S_{k-\ell}(D)e^{-H\ell D}, \]

(B.2b)
Note that \( \frac{S(s, e^{-ht})}{R(s, e^{-ht})} \) is a rational realizable transfer function since the denominator quasi-polynomial has a larger degree in \( s \) than the numerator quasi-polynomial. Now, combining the filtered output and control equations, one gets directly the closed-loop relations

\[
y_j(t) = \frac{B(D, e^{-hD})S(D, e^{-hD})}{A_m(D, e^{-hD})} y_j^*(t), \quad u_j(t) = \frac{A(D, e^{-hD})S(D, e^{-hD})}{A_m(D, e^{-hD})} y_j^*(t),
\]

which guarantees the global internal closed-loop stability and yields directly the result. \( \square \)

**Theorem 2(i)** has been proved. **Theorem 2(ii)–(iii)** are proved through the subsequent set of stability results.

### B.1. Auxiliary delay-free homogeneous system

The subsequent result guarantees that the auxiliary system (20)–(22) is GES under Assumptions A.1, A.2–A.4.

**Lemma B.1.** Consider the \( n \)th time-varying homogeneous system \( \dot{z}(t) = A(t)z(t) \), where

1. \( A(t) \) has bounded entries and eigenvalues in \( \Re s \leq -\rho_0 < 0 \) for all \( t \geq 0 \).
2. \( A(t) \) satisfies **Assumption A.4**. then the auxiliary system (16)–(18) is GES.

**Proof.** Since \( A(t) \) is a stability matrix for all \( t \geq 0 \), there is a unique symmetric positive definite matrix \( Q(t) \) that satisfies the Lyapunov’s matrix equation

\[
A^T(t)Q(t) + Q(t)A(t) = -I,
\]

with \( Q(t) = \int_0^\infty e^{A(t)\tau} e^{A^T(t)\tau} \) being everywhere time-differentiable by construction and satisfying furthermore

\[
A^T(t)\dot{Q}(t) + \dot{Q}(t)A(t) = -(\dot{A}(t)\dot{Q}(t) + \dot{Q}(t)\dot{A}(t)),
\]

with \( \dot{Q}(t) = \int_0^\infty e^{A(t)\tau} (\dot{A}(t)\dot{Q}(t) + \dot{Q}(t)\dot{A}(t)) e^{A^T(t)\tau} \) so that

\[
\lambda_{\max}(Q(t)) := \|Q(t)\|_2 \leq \frac{K_0^2}{2\rho_0}; \quad \|\dot{Q}(t)\|_2 \leq \frac{K_0^2}{\rho_0} \lambda_{\max}(Q(t))\|\dot{A}(t)\|_2 \leq \frac{K_4^2}{2\rho_0^2} \|\dot{A}(t)\|_2,
\]

since \( Q(t) \) is symmetric positive definite and \( \|e^{A(t)\tau}\|_2 \leq K_0 e^{-\rho_0 \tau} \leq K_0 e^{-\rho_0 \tau} \) for some \( K_0 : [0, \infty) \rightarrow \mathbb{R}^+ \) and \( \rho_0 : [0, \infty) \rightarrow \mathbb{R}^+ \) upper and lower-bounded by \( K_0 \) and \( \rho_0 \), respectively, since \( A(t) \) is bounded with negative stability abscissa and \( Q(t) \) is symmetric positive definite for all \( t \geq 0 \) so that

\[
\beta_1 \leq \lambda_{\min}(Q(t)) \leq \lambda_{\max}(Q(t)) \leq \beta_2.
\]

Consider the Lyapunov function candidate \( V(t) = z^T(t)Q(t)z(t) \) of time-derivative

\[
\dot{V}(t) \leq ||z(t)||_2^2 + |z^T(t)\dot{Q}(t)z(t)| \leq -(1 - c||\dot{A}(t)||_2)||z(t)||_2^2,
\]

\[
\beta_1^{-1} V(t) \geq \lambda_{\min}(Q(t)) V(t) \geq ||z(t)||_2^2 \geq \lambda_{\max}(Q(t)) V(t) \geq \beta_2^{-1} V(t),
\]

where \( c = \frac{K_4^2}{\rho_0} \). Thus, from (B.8) and (B.9),

\[
\dot{V}(t) \leq -(1 - c||\dot{A}(t)||_2) \beta_2^{-1} V(t) \quad \text{if} \quad 1 > c \sup_{t \geq 0}(||\dot{A}(t)||_2),
\]

\[
\dot{V}(t) \leq -(\beta_1^{-1} - c||\dot{A}(t)||_2 \beta_1^{-1}) V(t),
\]

otherwise. Thus, from (B.10), one gets

\[
||z(t)||_2^2 \leq \beta_1^{-1} V(t) \leq \beta_1^{-1} \exp \left\{- \int_0^t (\beta_2^{-1} - c\beta_1^{-1}) ||\dot{A}(\tau)||_2^2 d\tau \right\} V(0),
\]

\[
\leq \beta_1^{-1} \beta_2 \exp(-\beta_2^{-1} t) \exp\{c\beta_1^{-1} [\mu^2 t^2 + \alpha t^3]||z(0)||_2^2, \}
\]

since
\[
\int_t^{t+T} \|\dot{A}(\tau)\|_2 d\tau \leq (\mu^2 T + z)^{\frac{3}{2}} \sqrt{T} \leq (\mu \sqrt{T} + \sqrt{z}) \sqrt{T} \quad \text{(B.13)}
\]

Now, assume that \(\mu^* \leq \frac{\varepsilon}{\lambda} = \frac{2\varepsilon}{K^2} \) then \(\|z(t)\|_2^2 \to 0 \) exponentially as \(t \to \infty\) provided that \(t \to \infty\). Similarly, if \(\|\dot{A}(\tau)\|_2\) is bounded with \(1 > c \sup_{t \geq 0} (\|\dot{A}(t)\|_2) = c\mu = \frac{K^2}{2\varepsilon^2} \mu\).

Thus, \(\dot{V}(t) \leq -\delta V(t)\) with \(\delta = \beta_2^{-1} \mu = \beta_2^{-1}(1 - c \sup_{t \geq 0} (\|\dot{A}(t)\|_2)) \) so that \(V(t) \leq -e^{-\delta t} V(0)\) and \(\|z(t)\|_2^2 \leq \beta_1^{-1} \beta_2 \exp(\beta_2^{-1} \mu) \|z(0)\|_2^2 \to 0 \) exponentially as \(t \to \infty\) with \(\varepsilon = 1 - \frac{K^2}{2\varepsilon^2} \mu\) and the result has been fully proved. \(\square\)

### B.2. Unforced auxiliary time-delay system

The stability of the inhomogeneous auxiliary system (20)–(22) is discussed provided that the homogeneous delay-free system is subject to Lemma B.1 and Assumptions A.1, A.2 hold. In other words, if the error between the true and estimated delays is sufficiently small and the true and parameter estimates lie within a region where the matrices associated with the delayed dynamics have a sufficiently small norms. It is assumed in addition that the forcing signal grows non faster than linearly with \(O[\sup_{t} (\|x(t)\|)] \) with sufficiently small slope.

**Lemma B.2.** The following two items hold:

(i) Consider the time-varying inhomogeneous system:

\[
\dot{x}(t) = \sum_{k=0}^{q} A_k(t)x(t - kh) + f(t),
\]

with initial conditions defined by the absolutely continuous vector function \(\varphi: [-qh, 0] \to \mathbb{R}^n\) except possibly at set of zero measure of bounded discontinuities. Assume that:

1. \(A(t) = \sum_{k=0}^{q} A_k(t)\) satisfies all the assumptions of Lemma B.1; i.e., Assumption A.4 and it is strictly stable of constant eigenvalues for all time; and \(A_k(t) (k = 1, 2, \ldots, q)\) has bounded entries on \([0, \infty)\).
2. The base true delay \(h = h^*\) is known with a maximum error of \(\pm \Delta h (\Delta h \geq 0)\).
3. \(\|f(t)\| = o[\sup_{t - q(h+\Delta h) \in \mathbb{R}} (\|x(t)\|)]\).

Thus, the system (B.14) is GES if

\[
\rho > K\exp[2a(1 + q\Delta h)] + M_0,
\]

where \(M_0 \geq 0\) is such that \(\|f(t)\| \leq M_0 \sup_{t - q(h+\Delta h) \in \mathbb{R}} (\|x(t)\|)\) with \(h = h^* + \Delta h\), and

\[
a = \sup_{t \geq 0} \left( \sum_{k=1}^{q} ||A_k(t)|| \right); \quad b = \sup_{t \geq 0} \left( \sum_{k=0}^{q} \|A_k(t)\| \right) \leq a + \sup_{t \geq 0} (\|A_0(t)\|).
\]

Also, the system (B.14) is GES if

\[
\rho' > K' \exp[a(1 + 2qb\Delta h) + M_0],
\]

provided that \(z_0(t) = A_0(t)z_0(t)\) is GES with its fundamental matrix satisfying \(\rho' > K' \exp[a(1 + 2qb\Delta h) + M_0]\) for some real constants \(K' \geq 1\) and \(\rho' > 0\).

(ii) Now consider \(f(t) = O[\sup_{t \in \mathbb{R}} (\|x(t)\|)]\). Thus, the system (B.14) is GS if \(\rho > K\] 2a(1 + q\Delta h) + M_0\) provided that the homogeneous \(\dot{z}(t) = A(t)z(t)\) is GS. Also, (B.14) is GS if the homogeneous system \(\dot{z}_0(t) = A_0(t)z_0(t)\) is GES and \(\rho' > K' \] a(2 + q\Delta h) + M_0\).

**Proof.** (i) Eq. (B.14) may be rewritten equivalently as

\[
\dot{x}(t) = A(t)x(t) + \sum_{k=1}^{q} A_k(t)(x(t - kh) - x(t)) + \sum_{k=1}^{q} A_k(t)(x(t - kh) - x(t - kh')) + f(t),
\]

and since \(\|y(t, \tau)\| \leq Ke^{-\mu(t-\tau)}\) [Lemma B.1] one gets by taking \(\ell_2\)-matrix and vector norms:
\[ \|x(t)\| \leq Ke^{-\rho t}\left(\|x(0)\| + \int_0^{q\bar{h}} e^{\rho \tau} \left[ \sum_{k=1}^{q} A_k(\tau)(x(t-\bar{h}k)-x(\tau)) + M_0 \right] d\tau \right) + \int_{q\bar{h}}^{t} e^{\rho \tau} \left( 2a(1+qb\Delta h) + M_0 \right) \sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x'\|) \right) d\tau, \]

since

\[ \|x(t-\bar{h})-x(t-\bar{h}')\| \leq \int_0^{q\bar{h}} \sup_{\tau-q\bar{h} \leq \tau \leq t-h+\Delta h} (\|x(\tau)\|) d\tau \leq q\Delta h(b+M_0 C), \]

and

\[ \|x(t-\bar{h})-x(t-\bar{h}')\| \leq 2qb\Delta h \sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x(\tau)\|) \leq 2q\Delta h(b+M_0 C), \]

some \( \zeta_k(\tau) \in (t-k\bar{h}, t-k(h-\Delta h)) \) where La Rolle mean value theorem for integrals has been applied in the above first inequality. Thus, (B.19) leads to

\[ \|x(t)\| \leq Ke^{-\rho t}\left( m + M \int_{q\bar{h}}^{t} e^{\rho \tau} \sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x'\|) d\tau \right), \]

for \( t \geq q\bar{h} \), where

\[ m = \|x(0)\| + \left\| \int_0^{q\bar{h}} e^{\rho \tau} \sum_{k=1}^{q} A_k(\tau)(x(t-\bar{h}k)-x(\tau)) + M_0 \right\|, \quad M = 2a(1+qb\Delta h) + M_0. \]

Define \( \sigma(t) = e^{\rho t}\sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x(\tau)\|) \) for all \( t \geq 0 \) with \( x(t) = \varphi(t) \) for \( t \in [-q\bar{h}, 0] \). Then, define \( t'(t) = \max\{t-q\bar{h} \leq \tau \leq t : \|x(\tau)\| \leq \|x(t')\| \} \) so that one gets from (B.21) \( \sigma(t) \leq Ke^{\rho(t-t')} (m+M \int_0^{t'} \sigma(\tau) d\tau) \) and Gronwall’s Lemma yields directly since \( t-t' \leq q\bar{h} \)

\[ \sigma(t) \leq Ke^{\rho t}\exp(Ke^{\rho M_0 t}), \]

so that

\[ \|x(t)\| \leq \sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x(\tau)\|) \leq Ke^{\rho t}\exp(-\rho - Ke^{\rho M}t), \]

which converges to zero exponentially as \( t \to \infty \) if \( \rho > Ke^{\rho M} \) from (B.23) and thus the system is GES if (B.19) holds. If \( z_0(t) = A_0z_0(t) \) is GES with its fundamental matrix satisfying \( \|\Psi(t, \tau)\| \leq K'e^{-\rho(t-s)} \) for constants \( K' \geq 1 \) and \( \rho > 0 \). Thus, (B.14) becomes

\[ \dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^{q} A_k(t)x(t-\bar{h}k) + \sum_{k=1}^{q} A_k(t)(x(t-\bar{h}k)-x(t-\bar{h})) + f(t), \]

so that

\[ \|x(t)\| \leq K'e^{-\rho t}\left[ m + \int_0^{q\bar{h}} e^{\rho \tau} \left[ \sum_{k=1}^{q} \|A_k(\tau)\| + 2qb\Delta h + M_0 \right] \right] \sup_{\tau-q\bar{h} \leq \tau \leq t} (\|x(\tau)\|) d\tau, \]

which has the same structure as (B.15) by replacing \( K \to K', \rho \to \rho ', M \to M' = a(1+2qb\Delta h) + M_0 \) so that the system is GES if (B.17) holds.

(ii) Note from (B.14) that there exist real constants \( 0 \leq M_0 < \infty \) and \( 0 \leq M_0' < \infty \) such that

\[ f(t) = O\left( \sup_{s-t \leq s \leq t} (\|x(\tau)\|) \right) = M_0 \sup_{0 \leq s \leq t} (\|x(\tau)\|) + M_0', \]

\[ \|x(t)\| \leq Ke^{-\rho t}m + K\left( \frac{2a(1+qb\Delta h) + M_0}{\rho} \right) \sup_{0 \leq s \leq t} (\|x(\tau)\|) + \frac{KM_0'}{\rho}. \]
For each $t$, there exists $t'(t) = \text{Max}(\tau \leq t; \|x(t')\| = \text{Sup}_{0 \leq \tau \leq (t)}(\|x(\tau)\|))$. From (B.26), one gets
\[
\text{Sup}_{0 \leq \tau \leq t}(\|x(\tau)\|) \leq K(e^{-\lambda t}m_{\rho} + M_{0}^2) \leq K\left(\frac{m_{\rho}}{e^{\lambda t} + M_{0}}\right) < \infty,
\] (B.27)
with $e_0 = 1 - 2\alpha K(1 + qh\Delta h) + M_{0} > 0$ and (B.14) is GES if $1 > \frac{K(2\alpha(1 + qh\Delta h) + M_{0}^2)}{\rho}$. Alternatively, if the equivalent system description (B.24) is used then a similar reasoning establishes that the system (B.14) is GES if $1 > \frac{K(2\alpha(1 + qh\Delta h) + M_{0})}{\rho}$. \hfill \square

### B.3. Particular cases of Lemma B.2

1. If $h = h'$ (i.e. $\Delta h = 0$) then (B.14) is GES if $h < \frac{1}{q} \ln \frac{\rho}{K(2\alpha + M_{0})}$ provided that $\rho > K(2\alpha + M_{0})$.

2. If $h$ satisfies the above inequality then $\Delta h < \Delta h$ if $\Delta h > 0$ and $\Delta h = \Delta h = 0$, otherwise, where
\[
\Delta h = \text{Inf}_{z \in [0,1]} \left(\frac{1}{q} \left| \ln z \right|, \frac{2\alpha h - M_{0} - 2a}{2ab} \right),
\] (B.28)
provided that $\rho > \frac{K(2\alpha + M_{0})}{h}$.

This holds since $\rho e^{-\alpha c e^{-qh\Delta h}} > M = M_{0} + 2a(1 + qh\Delta h)$. Very close results apply to the case when $z_{0}(t) = A_{0}(t)z_{0}(t)$ is GES.

### B.4. Multiestimation schemes

**Lemma B.3.** Assume that a multiestimation model is used and that all assumptions of Lemmas B.1, B.2 hold. Thus, Lemmas B.1, B.2 still hold if there is a minimum residence time for each estimator (i.e. if both the homogeneous and inhomogeneous systems are GES).

**Proof.** First note from Lemma B.1 that $\|\Psi(t,0)\|_{2} = \text{Max}_{t \geq 0}(\|\Psi(t,0)\|_{2}) \leq \frac{1}{2} e^{\frac{\rho}{\rho_{2}} - \frac{\rho}{\rho_{2}}} = \ln K$ for the $\ell_{2}$-matrix norm. Let $t_{i+1}$ and $t_{i}$ two consecutive switches between estimation models with $t_{i+1} = t_{i}$. Any switching of the estimator at time $t = t_{i}$ causes the modification $\int_{t_{i+1}}^{t_{i+1}}\|A(\tau)\|_{2} d\tau \leq \mu_{2} T_{i} + \alpha + x_{i}$, to be considered in the developments of Lemma B.1, provided that no switches take place in $(t_{i}, t_{i} + T_{i})$ while only the referred one occurs in $(t_{i}, t_{i} + T_{i})$ with $A(t) = \sum_{k=0}^{N} A_{k}(t)$ and
\[
\beta = \text{Max}_{i \geq 0}(\beta_{i}); \quad \beta_{i} = \left| \sum_{k=0}^{N} A_{k}(t_{i}) - A_{k}(t_{i}) \right| \leq 2qa,
\] (B.29)
Thus, the fundamental matrix $\Psi(t, \tau)$ of $A(t)$ satisfies $\|\Psi(t, \tau)\|_{2} \leq Ke^{\frac{\rho}{\rho_{2}}(1 - t)}$ for $t \in [0, T_{i}]$ with $K_{2} = \frac{1}{2} c^{\rho_{2}}(\alpha + \beta)$. Thus, $\|\beta_{i+1}\|_{2} < \lambda \|\beta_{i}\|_{2}$ for any given real constant $\lambda \in (0,1)$ and all consecutive switching times $t_{i}$ between estimation models belonging to the sequence $S$ satisfy $t_{i+1} = t_{i} + T_{i}$ if $\|\Psi(t, \tau)\| \leq Ke^{\frac{\rho}{\rho_{2}}(1 - t)} < \lambda$ what holds if the interval between consecutive switches satisfies:
\[
T_{i} \geq 1 \frac{1}{\rho} \left[ \frac{1}{2} \ln(\alpha + \beta_{i}) + \ln \left( \frac{c\beta_{2}}{\sqrt{2}\beta_{1}^{3/2}} \right) + \ln |\lambda| \right]
\geq 1 \frac{1}{\rho} \left[ \frac{1}{2} \ln(\alpha + 2qa) + \ln c + \ln \beta_{2} - \frac{3}{2} \ln \beta_{1} - \frac{1}{2} \ln 2 + |\ln \lambda| \right],
\] (B.30)
Thus, the inhomogeneous system (B.24) with switches between estimation models is GES if, in addition to the assumptions of Lemma B.1, there is a minimum residence time between any two consecutive switches subject to (B.29), (B.30). Now, take real constants $\lambda_{i} \in [0, \rho]$, and $\rho'$ satisfying $K_{3}e^{\rho_{2}T_{i}} < \lambda_{i} = e^{\rho_{2}T_{i}} < 1 \forall i \in \rho_{2}$ so that $\rho' = \frac{\ln |\lambda_{i}|}{T_{i}} \leq \rho$. Thus, the inhomogeneous system (B.24) is GES if Lemma B.2 holds under similar conditions by replacing $\rho \to \rho'$, $K_{2} \to K_{2}^{2} = \frac{1}{2}(\alpha + \beta_{2})^{2}$ (or $K_{2} \to \frac{1}{2}(\alpha + 2qa)e^{\frac{\rho}{\rho_{2}}}$) $\forall i \in \rho$. \hfill \square

**Proof of Theorem 2 (ii–iii).** Note that $A(t)$ is bounded from Theorem 1 and it is almost everywhere time-differentiable except possibly at isolated switching times and with constant time-invariant eigenvalues of negative
real parts. Thus, Assumption A.4 and all the assumptions of Lemma B.1 hold so that the delay-free auxiliary system (20)–(22) is GES and Theorem 2(ii) follows from Lemma B.2, related to the single estimation scheme and Lemma B.3 related to the multiestimation scheme. Theorem 2(iii) follows from Lemma B.2(ii) for single estimation and (B.4) for multiestimation.

**References**


