Robust adaptive control of discrete nominally stabilizable plants

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Abstract

This paper presents an indirect pole-placement based adaptive control scheme for discrete linear time-invariant non-necessarily inversely stable and stabilizable systems. The scheme is a pole-placement type without requiring the plant inverse stability assumption. The control objective is that the plant output should asymptotically track in the absence of unmodeled dynamics and bounded disturbances a reference signal, given by an arbitrary stable filter, under a bounded tracking error while ensuring robust closed-loop stability. The adaptive stabilizability and the robustness of this system under the presence of unmodeled dynamics and, possibly, bounded disturbances, are proved without assuming the controllability of the modeled plant. A normalized least-squares algorithm with a relative adaptation dead-zone and ‘a posteriori’ modification of the parameter estimates is used to ensure the controllability of the modified estimation plant model at all sampling instants and at the limit. Such a property is crucial to solve the stable pole-placement problem. On the other hand, the relative adaptation dead-zone is included for robustness purposes under unmodeled dynamics and, possibly, bounded disturbances.

Keywords: Robust control; Unmodeled dynamics; Adaptive stabilizability; Inversely unstable plant; Tracking; Adaptation dead-zone

1. Introduction

The robust adaptive stability of discrete linear time-invariant systems has been successfully developed in the last two decades. In the beginning, as in [1]
and [2], most of the proofs of stability of the adaptive control algorithms proposed were based on a set of standard assumptions. Such assumptions were: (i) the knowledge of an upper bound of the system order, (ii) the knowledge of the system relative degree, (iii) the plant being inversely stable and (iv) the knowledge of the sign of the high frequency gain of the plant as well as an upper bound for its absolute value.

Later, works whose goal was to relax some of these aforementioned assumptions were presented. In this way, the studies in [3] and [4] relaxed the assumptions relative to the high frequency gain by including Nussbaum gain in the adaptation laws. Therefore, in the paper [5] the assumption of inverse stability as well as that of the knowledge of the high frequency plant gain were relaxed. That work focused on the robust adaptative stabilization of discrete linear time-invariant plants. The controllability of the nominal plant model and an overbounding function for the contribution of the unmodeled dynamics and external disturbances were supposed known for purposes of achieving robust adaptive stability. Such a stability result was established without either requiring the injection of persistent excitation probing signals into the system or assuming any prior knowledge of the plant parameters.

The necessary and sufficient condition for the system stabilization via an adaptive pole-placement scheme was established in [6]. Such a condition is simply that the asymptotically reached estimated model of the stabilizable plant be controllable, which is weaker than the controllability of the true plant required in [5].

If the system is not inversely stable then the controllability of the estimated model of the plant has to be ensured when the time tends to infinity to achieve the adaptive stabilization via pole-placement. Basically, two different approaches have been used to circumvent the regions in the parameter space corresponding to uncontrollable models. One of them relies on the use of excitation probing signals, [7–11], while the other one is based on the use of a suitable modification of the plant parameter estimates as in [5,6,12–15]. The estimation modification algorithm used in [5,13,15] includes a hysteresis switching function what it bears a very heavy computation burden before the convergence of the estimates. The modification algorithm presented in [6] involves the use of a hysteresis-free switching function. Then, the convergence of the estimates is usually reached with a light computation burden, but with a high number of switches.

This paper relaxes the controllability condition of the plant nominal model required in [5] for robust adaptive stabilizability of the closed-loop system under the presence of unmodeled dynamics. The control objective is to ensure the stability of the closed-loop system with a bounded tracking error. The proposed scheme is of pole-placement type while it ensures asymptotic reference tracking properties when the reference sequence becomes constant. The scheme is valid to operate in both non-adaptive and adaptive environments.
The only assumption about the true plant is that it is stabilizable. This paper is an extension of the problem solved in [12] to the case of discrete-time plants. A normalized least-squares algorithm with a relative adaptation dead-zone and a parameter modification is used to update the plant parameters at each sampling instant. The above combined technique is the basis to prove the convergence of the estimated parameters and the boundedness of all the signals in the closed-loop system. The knowledge of an upper-bounding function for the contribution of the unmodeled dynamics and disturbances to the output is required to design the relative adaptation dead-zone, [16]. In that way, the adaptation is frozen when an augmented identification error lies below a known upper bound of the absolute value of the contribution of the unmodeled dynamics and bounded disturbances to the output.

Two designs for the parameter modification are presented to fulfill the control objective. Both of them use a hysteresis-free switching function which does not change its value unless it is crucial to ensure the controllability of the estimation model of the plant. In this way, the number of switches and/or the computational cost is reduced with regard to the modification algorithms used in the previous works [5,6,13,15]. It is proved for both modification algorithms that the normalized modified identification error belongs to a residual set defined by an absolute normalized upper bound for the contribution of the unmodeled dynamics and bounded noise to the output.

The paper is organized as follows. Section 2 presents the problem statement with the control objective and the structure of the control law. Section 3 is devoted to formulate the unmodified and modified parameter estimation algorithms. Two alternative algorithms are studied for the parameter modification, both of them providing a controllable plant estimated model for all sampling instants and at the limit. Section 4 establishes the main result of the stability analysis. Section 5 presents a numerical simulation to show the performance of the proposed control scheme. Finally, conclusions end the paper.

2. Problem statement

Consider the following linear discrete-time system

\[ A(q^{-1})y(k) = B(q^{-1})u(k) + \eta(k) \]  

which is not necessarily controllable and can include stable uncontrollable modes as the roots of \( A(q^{-1}) = 0 \), where

\[
A(q^{-1}) = A'(q^{-1})A_0(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_nq^{-n} \\
B(q^{-1}) = B'(q^{-1})A_0(q^{-1}) = b_{n-m}q^{-m-n} + b_{n-m+1}q^{-m-n-1} + \cdots + b_nq^{-n}
\]  

(2)
with \( q^{-1}y(k) \equiv y(k - 1), m \leq n \), \( \eta(k) \) being the contribution of unmodeled dynamics and bounded noise of any order to the output and \( A_0(q^{-1}) \) denoting the possible \( n_0 \geq 0 \) stable common factors of \( A(q^{-1}) \) and \( B(q^{-1}) \). If \( n_0 = 0 \) then \( A_0(q^{-1}) = 1 \). Eq. (1) may be rewritten as

\[
y(k) = \theta^T \phi(k) + \eta(k)
\]

where

\[
\theta = [b_{n-m} \cdots b_n \ a_1 \cdots a_n]^T;
\]

\[
\phi(k) = [u(k - n + m) \cdots u(k - n) - y(k - 1) \cdots -y(k - n)]^T
\]

(4)

are the true plant parameters and measurable regression vectors, respectively.

2.1. Case of known plant and absence of unmodeled dynamics \((\eta(\cdot) \equiv 0)\)

In this case, the control objective is the tracking between the system output sequence and the reference sequence \( y_m(k) \) with a prescribed pole-placement, being zero the tracking error when the reference sequence becomes constant. The control law designed to meet this objective is:

\[
u(k) = Ky_m(k) - R(q^{-1})u(k) - S(q^{-1})y(k) = \frac{K}{1 + R(q^{-1})}y_m(k) - \frac{S(q^{-1})}{1 + R(q^{-1})}y(k)
\]

(5)

where \( y_m(k) \) is given by a stable filter

\[
W_m(q^{-1}) = B_m(q^{-1})/A_m(q^{-1}) = (b'_{n-m} q^{m-n} + \cdots + b'_n q^{-n})/(1 + a'_1 q^{-1} + \cdots + a'_n q^{-n});
\]

\[
R(q^{-1}) = r_{n-m} q^{-(n-m)} + \cdots + r_{n-n_0-1} q^{-(n-n_0-1)};
\]

\[
S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n-n_0-1} q^{-(n-n_0-1)}
\]

(6)

with \( r_i, s_j \in \mathbb{R} \), for \( i \in \{n - m, \ldots, n - n_0 - 1\} \) and \( j \in \{0, \ldots, n - n_0 - 1\} \), being calculated from the following diophantine equation

\[
(1 + R(q^{-1}))A(q^{-1}) + S(q^{-1})B(q^{-1}) = C(q^{-1})
\]

(7)

where \( C(q^{-1}) = C'(q^{-1})A_0(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{2n-1} q^{-(2n-1)} \) is a Hurwitz polynomial. Eq. (7) is solvable since the \( n_0 \) stable common factors of \( B(q^{-1}) \) and \( A(q^{-1}) \) are included in \( C(q^{-1}) \). If the plant is controllable then all of the factors of the Hurwitz polynomial \( C(q^{-1}) \) can be freely chosen. Note that Eq. (7) is equivalent to that obtained by replacing \( A \rightarrow A', B \rightarrow B' \) and \( C \rightarrow C' \)
after cancelling $A_0$ and that Eq. (1) is equivalent neglecting initial conditions to the equation obtained by replacing $A \rightarrow A'$, $B \rightarrow B'$ and $\eta \rightarrow \eta' = (1/A_0)\eta$. On the other hand, the control parameter $K$ will be calculated such that the tracking of the reference sequence be perfect when it becomes constant. By introducing the control law (5) in the transfer function of the plant $W_0(q^{-1}) = B(q^{-1})/A(q^{-1})$ the transfer function of the closed-loop system is

$$\frac{y(q^{-1})}{r^*(q^{-1})} = \frac{KB_m(q^{-1})B'(q^{-1})A_0(q^{-1})}{A_m(q^{-1})C'(q^{-1})A_0(q^{-1})}$$  \quad (8)

which can include stable cancellations if the degree of $A_0(q^{-1})$ is non-zero and where the relationship (7) has been used. Its dynamics includes the poles of the stable filter $W_m(q^{-1})$. The remaining poles can be fixed according to the arbitrary Hurwitz monic polynomial $C'(q^{-1})$ by the control polynomials $R(q^{-1})$ and $S(q^{-1})$. Then, the dynamics of the closed-loop system is able to practically equate that of the stable filter $W_m(q^{-1})$ if the zeros of the chosen Hurwitz polynomial $C'(q^{-1})$ are sufficiently more stable than the poles of $W_m(q^{-1})$. Note from (8) that the closed-loop characteristic equation is $A_m(q^{-1})C'(q^{-1}) = 0$ and it has $n_m + 2(n - n_0) - 1$ roots. However, note that the computational cost in solving the diophantine equation (7) is fixed irrespective of the degree of $A_m(q^{-1})$. Therefore, the number of closed-loop poles can be increased, if desired, without increasing the computational cost requested to generate the control input. The transfer function from the external input sequence $r^*(k)$ to the tracking error $\varepsilon(k) = y(k) - y_m(k)$ is

$$G_r(q^{-1}) = \frac{\varepsilon(q^{-1})}{r^*(q^{-1})} = \frac{[KB'(q^{-1}) - (1 + R(q^{-1}))A'(q^{-1}) - S(q^{-1})B'(q^{-1])]B_m(q^{-1})}{A_m(q^{-1})C'(q^{-1})}$$ \quad (9)

where the relationship (7) has been used. The control parameter $K$ can be calculated such that the absolute gain of this transfer function be zero when $q^{-1} \equiv e^{-j\omega T} = 1 + j\omega$, with $T$ being the sampling period and $j$ the complex imaginary unity. In this way, the tracking error can be zero if the reference sequence becomes constant. The value of $K$ to meet this objective is from (9):

$$K = \frac{(1 + R(1))A'(1) + S(1)B'(1)}{B'(1)}$$ \quad (10)

If $n = m$, $n_m = m_m$ and $n_0 = 0$, Eqs. (7) and (10) can be written more compactly as

$$[1 \quad q^{-1} \quad \ldots \quad q^{-2n+2} \quad q^{-2n+1}] [M(\theta)v_c - p] = 0$$ \quad (11)
where

\[
v_c = \begin{bmatrix}
1 + r_0 & r_1 & \cdots & r_{n-1} & s_0 & s_1 & \cdots & s_{n-1} & K
\end{bmatrix}^T;
\]

\[
p = \begin{bmatrix}
1 & c_1 & \cdots & c_{2n-1} & 0
\end{bmatrix}^T
\]

and

\[
M(\theta) = \\
\begin{bmatrix}
1 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 & 0 \\
1 & a_1 & \cdots & 0 & b_1 & b_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
an_{n-1} & a_{n-2} & \cdots & 1 & b_{n-1} & b_{n-2} & \cdots & b_0 & 0 \\
an_n & a_{n-1} & \cdots & a_1 & b_n & b_{n-1} & \cdots & b_1 & 0 \\
0 & a_n & \cdots & a_2 & 0 & b_n & \cdots & b_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n & 0 \\
-\left(1 + \sum_{i=1}^{n} a_i \right) & \cdots & -\left(1 + \sum_{i=1}^{n} a_i \right) & -\sum_{i=0}^{n} b_i & -\sum_{i=0}^{n} b_i & \cdots & -\sum_{i=0}^{n} b_i & \sum_{i=0}^{n} b_i
\end{bmatrix}
\]

is the Sylvester matrix of the true plant parameters. Eq. (11) is uniquely solvable if the determinant of \(M(\theta)\) is strictly non-zero, i.e.,

\[
|\text{Det}(M(\theta))| \geq \delta_0 > 0
\]

for some design real small positive constant \(\delta_0\). The relationship (14) will be referred to as the controllability condition of the true plant. Then, if the plant is controllable, i.e., \(A(q^{-1})\) and \(B(q^{-1})\) are relatively prime polynomials, then, the control parameters can be uniquely obtained from the algebraic system (11).

Remark 1. Since the relative degree of the plant and its parameters are unknown, one can choose that the relative degree of the nominal model of the plant be zero \((n = m)\) and \(A_0(q^{-1}) = 1\) \((n_0 = 0)\) to establish the formalism in a simpler way. In the case when the relative degree of the plant is non-zero the estimated parameters, issued from an estimation algorithm, corresponding to those added parameters will tend to zero.

Remark 2. In the case when the polynomials \(A(q^{-1})\) and \(B(q^{-1})\) have stable common factors, the control parameters are obtained from a similar equation to (11) with the Sylvester determinant of \(A'(q^{-1})\) and \(B'(q^{-1})\) being strictly positive.
2.2. Case of unknown plant

The transfer function of the system (1) can be written as 
\[ y(q^{-1})/u(q^{-1}) = W(q^{-1}) = W_0(q^{-1})(1 + v A_1(q^{-1})) + v A_2(q^{-1}), \]
where \( W_0(q^{-1}) = B(q^{-1})/A(q^{-1}) \) is the modeled part of the transfer function of the true plant, \( v A_1(q^{-1}) \) and \( v A_2(q^{-1}) \) are the transfer functions of the multiplicative and additive unmodeled dynamics, respectively, and \( v \) is a positive constant. \( D_1(q^{-1}) \) and \( D_2(q^{-1}) \) must be both exponentially stable and strictly proper so that Assumption 1 below be feasible.

Assumption 1. There exist real constants \( \sigma \in (0, 1), \alpha_0 \geq 0 \) and a constant vector \( a \), which are known, such that

\[
|\eta(k)| \leq \bar{\eta}(k) = a \rho(k) + \alpha_0 \quad \text{for all integer } k \geq 0
\]  

(15)

where

\[
\rho(k) = \sup_{0 \leq k' \leq k} \{|v^T x(k')| \sigma^{k-k'}\}; \quad x(k) = [\varepsilon(k-1) \cdots \varepsilon(k-n) \ u(k-1) \cdots u(k-n)]^T
\]  

(16)

for all integer \( k \geq 0 \), where \( \varepsilon(k) = y(k) - y_m(k) \) is the tracking error with \( y_m(k) = W_m(q^{-1})r^*(k) \) and \( r^*(k) \) being any arbitrary bounded external input sequence.

Remark 3. Assumption 1 is invoked in the design of a relative adaptation dead-zone for robustness purposes in the parameter estimation shown in Section 3. The width of the dead-zone is governed by the overbounding normalized function \( \bar{\eta}_n(k) = \bar{\eta}(k)/(1 + \| \phi(k) \|) \). This assumption is fulfilled by any system in which the signal \( \eta(k) \) is the sum of a bounded term plus a term related to \( u(k) \) by a strictly proper exponentially stable transfer function. [17].

The structure of the control law is maintained as in (5) while replacing the control parameters \( K_i, r_i \) and \( s_i \), for \( i \in \{0, \ldots, n-1\} \), by \( \smooth K_i \), \( \smooth r_i \) and \( \smooth s_i \), respectively, for all integer \( k \geq 0 \). The equations to obtain the control parameters are those of (7) and (10) substituting the plant parameters of the vector \( \theta \) by their corresponding estimates of \( \smooth \theta(k) = [\smooth \theta_0(k) \cdots \smooth \theta_n(k) \smooth \alpha_1(k) \cdots \smooth \alpha_n(k)]^T \) for all integer \( k \geq 0 \). Such estimates are obtained from an estimation and modification algorithm which ensures the controllability of the estimated and modified model at all sampling instants, i.e., \( |\text{Det}(M(\smooth \theta(k)))| \geq \delta_0 > 0 \) for all integer \( k \geq 0 \), as it will be shown in Section 3.
3. Adaptive control

If the plant parameters are unknown, an estimation algorithm has to calculate a controllable estimation plant model. Two alternative algorithms are presented which include a hysteresis-free switching function.

3.1. Estimation and modification algorithms

A least-square estimation algorithm with normalization and a relative adaptation dead-zone is used to obtain an ‘a priori’ estimation of the plant parameters in Step 1. Then, a suitable modification is applied to such ‘a priori’ estimates to obtain a controllable ‘a posteriori’ estimated model of the plant in Step 2.

Step 1 (‘A priori’ estimation)

The ‘a priori’ plant parameter estimates are obtained from the recursive equations

\[
\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{s(k)P(k)\phi_n(k)e_n(k)}{\gamma(k) + s(k)\phi_n^T(k)P(k)\phi_n(k)};
\]

\[
P(k+1) = P(k) - \frac{s(k)P(k)\phi_n(k)\phi_n^T(k)P(k)}{\gamma(k) + s(k)\phi_n^T(k)P(k)\phi_n(k)}
\]

(17a)

where

\[
e_n(k) = \frac{e(k)}{1 + \|\phi(k)\|}; \quad e(k) = y(k) - \phi^T(k)\hat{\theta}(k)
\]

(17b)

are the normalized identification error and the identification error, respectively, \(\hat{\theta}(k) = [\hat{b}_0(k) \ldots \hat{b}_n(k) \hat{a}_1(k) \ldots \hat{a}_n(k)]^T\) and \(\hat{\theta}(k) = \hat{\theta}(k) - \theta\) are the a priori estimation vector of the plant parameters and the parametrical error, respectively, \(\phi_n(k) = \phi(k)/(1 + \|\phi(k)\|), P(0) = P^T(0) > 0\) and \(\gamma(k)\) is a real value sequence chosen such that \(0 < \gamma_1 \leq \gamma(k) \leq \gamma_2 < \infty\) for all integer \(k \geq 0\), with \(\gamma_1\) and \(\gamma_2\) being some design real constants.

The sequence \(s(k)\) is the relative adaptation dead-zone defined as

\[
s(k) = \begin{cases} 
0 & \text{if } w(k) \leq \mu \left(1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)}\right)^{1/2} \bar{\eta}_n(k) \\
\frac{f(k)}{w(k)} & \text{otherwise}
\end{cases}
\]

(17c)

where \(w(k) = (e_n^2(k) + \phi_n^T(k)P^2(k)\phi_n(k))^{1/2}\), and
\[ f(k) = \begin{cases} 
0 & \text{if } w(k) \leq \mu \left( 1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)} \right)^{1/2} \eta_n(k) \\
 w(k) - \mu \left( 1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)} \right)^{1/2} \eta_n(k) & \text{otherwise} 
\end{cases} \] (17d)

with \( \mu > 1 \).

It will be then proved in Lemma 3.1 below that \( \|P^{-1}(k)\hat{\theta}(k)\| \) is bounded for all integer \( k \geq 0 \) and at the limit as \( k \to \infty \). Thus, a bounded vector sequence \( \beta^*(k) \in \mathbb{R}^{2n+1} \) exists such that the true plant parameter vector can be written as \( \theta = \hat{\theta}(k) + P(k)\beta^*(k) \). This fact provides the motivation to obtain the ‘a posteriori’ modified estimates of the plant parameters in Step 2 below.

**Step 2 (‘A posteriori’ modified estimation)**

Two alternative algorithms are designed:

1. **First modification algorithm**

   i. The modified estimates are calculated from the modification rule

   \[ \hat{\theta}(k) = \hat{\theta}(k) + P(k)\beta(k) \] (18)

   where \( \hat{\theta}(k) = [\hat{\theta}_0(k) \ldots \hat{\theta}_n(k) \ \hat{\alpha}_1(k) \ldots \hat{\alpha}_n(k)]^T \) is the ‘a posteriori’ estimation vector of the plant parameters and \( \beta(k) \in \mathbb{R}^{2n+1} \) is a switching sequence which takes values being equal to one of the constant vectors belonging to a finite set as which is computed in (ii) below:

   ii. Let \( e_0 \) be a zero \((2n+1)\)-dimensional vector. Let \( \{e_i^{(1)}\} \) be \((2n+1)\)-dimensional vectors with only one non-zero element which equalizes either +1 or to -1, for \( i \in \{1, \ldots, 2(2n+1)\} \). Similarly, let \( \{e_i^{(j)}\} \) be \((2n+1)\)-dimensional vectors with \( j \) non-zero elements, each of which equalizes either to +1 or to -1, for \( i \in \{1, \ldots, [(2n+1)!/((2n+1)-j)!j!]2\} \). The total number of such vectors is

   \[ L = \sum_{j=0}^{2n+1} \left( \frac{(2n+1)!}{(2n+1-j)!j!2!} \right) \]

   Denote these vectors by \( \{e_j\} \), with \( j \in \mathbb{Z}_L \equiv \{0, 1, \ldots, L - 1\} \), in the increasing order of the number of non-zero elements in \( e_j \). We now precisely define the sequence \( \beta(k) \). Consider a real number \( 0 < \delta \ll 1 \) and let the initial condition of \( \beta(0) \) be \( \delta e_0 \). At each sampling instant, \( \beta(k) \) is defined by

   \[ \beta(k) = \begin{cases} 
\beta(k-1) & \text{if } |\text{Det}(M(\hat{\theta}(k) + P(k)\beta(k-1)))| \geq \delta_0 \\
\delta e_0 & \text{otherwise} 
\end{cases} \] (19)
for all integer \( k \geq 1 \), where \( \delta_0 \) is a small real positive constant and \( e_{j_0} \) is obtained by searching from the vectors \( \{e_j\} \) the smaller \( j_0 \in \mathbb{Z}_e \) such that \(|\text{Det}(M(\hat{\theta}(k) + P(k)\delta e_{j_0}))| \geq \delta_0\).

(2) Second modification algorithm

(i) The modified estimates are calculated from the modification rule (18), where \( \beta(k) = \pi(k)\beta'(k) \), with \( \pi(k) \in \mathbb{R} \) being a scalar switching sequence and \( \beta'(k) \in \mathbb{R}^{2n+1} \) a vector sequence.

(ii) At each sampling instant, the components \( \beta_i'(k) \) for \( i \in \{1, 2, \ldots, 2n + 1\} \), of the vector sequence \( \beta_0(k) \), are calculated from the following equations

\[
\beta_i'(k) = \frac{\text{Det}(P_i'(k))}{\text{Det}(P(k))} \tag{20}
\]

where \( P_i'(k) \) has \( 2n \) columns idented to those of \( P(k) \) while the \( i \)th column is replaced with

\[
v = \begin{bmatrix} 0 & 0 & \ldots & 0 & \frac{n+1}{1} & 0 & \ldots & 0 \end{bmatrix}^T.
\]

(iii) The switching sequence \( \pi(k) \) is zero for \( k = 0 \) and for integers \( k > 0 \) it is defined as follows,

\[
\pi(k) = \begin{cases} 
\pi(k-1) & \text{if } |\text{Det}(M(\hat{\theta}(k) + \pi(k-1)P(k)\beta'(k)))| \geq \delta_0 \\
\pi_0(k) & \text{otherwise}
\end{cases}
\tag{21}
\]

for some small real positive constant \( \delta_0 \). i.e., expression (21) implies that \( \pi(k) \) maintains the value that it had at the previous sampling instant if the estimated and modified plant model obtained with such an aforementioned value is controllable. Otherwise, the function \( \pi(k) \) switches from the value \( \pi(k-1) \) to a new value \( \pi_0(k) \), which depends on the conmutation sampling instant, to avoid non-controllable plant estimated models. Besides, it is convenient that the absolute value of \( \pi_0(\cdot) \) be as small as possible so that parameters modification be smooth. Because of this, \( \pi_0(\cdot) \) is obtained from the following algorithm:

Algorithm to compute \( \pi_0(\cdot) \) at the conmutation sampling instants

This algorithm is splitted into a set of elementary steps:

Step 1: Set \( \pi_0 = 0 \), compute \(|\text{Det}(M(\hat{\theta}(k) + \pi_0P(k)\beta'(k)))|\) and go to Step 2.

Step 2: If \(|\text{Det}(M(\hat{\theta}(k) + \pi_0P(k)\beta'(k)))| \geq \delta_0 \) then go out, otherwise go to Step 3.

Step 3: Increase the value of \( \pi_0 \) by the operation \( \pi_0 = \pi_0 + \Delta\pi_0 \), with \( 0 < \Delta\pi_0 \ll 1 \), compute \(|\text{Det}(M(\hat{\theta}(k) + \pi_0P(k)\beta'(k)))|\) and go to Step 4.

Step 4: Set \( \pi_0 = \max(\pi_0, \delta_0) \) and go to Step 1.
Step 4: If \(|\text{Det}(M(\hat{\theta}(k) + \pi_0 P(k)\beta'(k)))| \geq \delta_0\) then set \(\pi_1 = \pi_0\) and go to Step 5, otherwise go to Step 3.

Step 5: Set \(\pi_0 = 0\) and go to Step 6.

Step 6: Decrease the value of \(\pi_0\) via \(\pi_0 = \pi_0 - \Delta \pi_0\), compute \(|\text{Det}(M(\hat{\theta}(k) + \pi_0 P(k)\beta'(k)))|\) and go to Step 7.

Step 7: If \(|\text{Det}(M(\hat{\theta}(k) + \pi_0 P(k)\beta'(k)))| \geq \delta_0\) then go to Step 8, otherwise go to Step 6.

Step 8: If \(|\pi_1| \leq |\pi_0|\) then set \(\pi_0 = \pi_1\), and go out.

Remark 4. From (17b) and (18),

\[ y(k) = \phi^T(k)\bar{\Theta}(k) + \bar{e}(k) - \beta^T(k)P(k)\phi(k) = \phi^T(k)\bar{\Theta}(k) + e_a(k) \quad (22) \]

is obtained where \(e_a(k) = e(k) - \beta^T(k)P(k)\phi(k)\) is referred to as ‘a posteriori’ identification error. Such a model has \(u(k)\) and \(y(k)\) as input and output, respectively, and \(e_a(k)\) as an external disturbance. The controllability condition of the plant estimated model is \(|\text{Det}(M(\bar{\Theta}(k))| > 0\) where \(M(\bar{\Theta}(k))\) has the same structure as \(M(\theta)\), in Eq. (13) of Section 2, by replacing the components of the true parameters vector \(\theta\) with the corresponding one of their modified estimates \(\bar{\Theta}(k)\) through (18).

3.2. Adaptive control law

The control law is based in (5) and becomes:

\[ u(k) = K(k) y_m(k) - R(q^{-1}, k) u(k) - S(q^{-1}, k) y(k) \]

\[ = \frac{K(k)}{1 + R(q^{-1}, k)} y_m(k) - \frac{S(q^{-1}, k)}{1 + R(q^{-1}, k)} y(k) \quad (23) \]

where the control parameters \(K(k), R_i(k)\) and \(S_i(k)\) (coefficients of the time-varying polynomials \(R(q^{-1}, k)\) and \(S(q^{-1}, k)\)), for \(i \in \{0, \ldots, n-1\}\), are calculated by similar equations to (7) and (10). In those equations, the parameters of the polynomials \(A(q^{-1})\) and \(B(q^{-1})\) must be substituted by their estimated and modified ones of \(\bar{A}(q^{-1})\) and \(\bar{B}(q^{-1})\), respectively. The adaptive controller parameters can be obtained through a similar equation to (11).

3.3. Properties of the adaptive control system

The convergence and stability scheme’s properties are given in the following results. Step 1 of the estimation algorithm possesses the properties given in the following lemma:
Lemma 3.1

(i) $P(k)$ is symmetrical for all integer $k \geq 0$, uniformly bounded and it asymptotically converges to a finite (at least semidefinite positive) limit as $k$ tends to infinity.

(ii) $\|\hat{\theta}(k)\| < \infty$, $\eta_n(k) < \infty$, $|\eta_n(k)| < \infty$, $|e_n(k)| < \infty$ and $w(k) < \infty$ for all integer $k \geq 0$.

(iii) $f(k) < \infty$, $s(k)w^2(k) < \infty$, $\sum_{i=0}^k f^2(i) < \infty$, $\sum_{i=0}^k f(i) < \infty$ and $\sum_{i=0}^k s(i) \times w^2(i) < \infty$ for all integer $k \geq 0$, $\lim_{k \to \infty} f^2(k) \to 0$, $\lim_{k \to \infty} f(k) \to 0$ and $\lim_{k \to \infty} s(k)w^2(k) \to 0$.

(iv) $\|\hat{\theta}(k) - \hat{\theta}(k-1)\| < \infty$ for all integer $k \geq 1$ and $\lim_{k \to \infty} \|\hat{\theta}(k) - \hat{\theta}(k-1)\| \to 0$.

(v) $0 \leq s(k) < 1$, $\|P(k)\| > 0$ and $s(k)e_n^2(k) < \infty$ for all integer $k \geq 0$, $\lim_{k \to \infty} s(k) \to 0$ and $\lim_{k \to \infty} s(k)e_n^2(k) \to 0$.

(vi) $\|P^{-1}(k)\hat{\theta}(k)\| < \infty$ for all integer $k \geq 0$.

The proof is given in Appendix A. Note that Property (v) ensures the nonsingularity of the matrix $P(k)$, for all integer $k \geq 0$. In this way, the plant parameters are estimated from (17a)–(17d) until $w(k)$ be smaller than $\mu(1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)})^{1/2}\eta_n(k)$, provided that $\phi(k)$ does not become or it is closed to a constant vector prior to the convergence of the plant estimates to their limits.

Step 2 applies, independent of the computation method of $\beta(k)$:

Lemma 3.2

(i) There exists a finite number of switches in $\beta(k)$, it is bounded, for all integer $k \geq 0$, and asymptotically converges.

(ii) $\theta(k)$ is a bounded vector sequence, for all integer $k \geq 0$, and asymptotically converges to a finite limit.

(iii) The control parameters $K(k)$, $\tau_i(k)$ and $\bar{\tau}_i(k)$, for all integer $k \geq 0$ and $i \in \{0, 1, \ldots, n-1\}$, are bounded sequences and asymptotically converge to finite limits.

(iv) $e_{an}(k) = e_a(k)/(1 + \|\phi(k)\|)$ verifies

$$\lim_{k \to \infty} \left\{ e_{an}^2(k) - 2\beta_{\text{max}}(k)\mu^2 \left(1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)}\right) \eta_n^2(k) \right\} \leq 0$$

where $\beta_{\text{max}}(k) = \max\{1, \|\beta(k)\|^2\}$, for all integer $k \geq 0$.

The proof is given in Appendix B.

Remark 5. In view of Property (iv) of Lemma 3.2, the normalized ‘a posteriori’ identification error $e_{an}(k)$ belongs to the residual set defined below
If \( \eta_n(k) \) converges to zero, which can occur with \( z_0 = 0 \) in (15) if \( \rho(k) \) converges to zero, then, the residual set \( D_e \) converges to the zero equilibrium.

4. Stability analysis

Eqs. (22) and (23) lead to the following time-varying 2n-auxiliary system

\[
x(k) = \mathcal{A}_e(k - 1)x(k - 1) + B_1 \vartheta_1(k) + B_2 \vartheta_2(k)
\]  

(24a)

with

\[
x(k - 1) = [\varepsilon(k - 1) \ldots \varepsilon(k - n) \ u(k - 1) \ldots \ u(k - n)]^T
\]  

(24b)

\[
\vartheta_1(k) = [\mathcal{B}_0(k - 1)(\mathcal{K}(k - 1) - s_0(k - 1)) - 1 - r_0(k - 1)]y_m(k) + \sum_{i=1}^{n-1}(\mathcal{B}_0(k - 1)s_i(k - 1) + (1 + r_0(k - 1))s_i(k - 1))y_m(k - i) - \mathcal{A}_u(k - 1)(1 + r_0(k - 1))y_m(k - n) + (1 + r_0(k - 1))e_u(k)
\]  

(24c)

\[
\vartheta_2(k) = \mathcal{K}(k - 1)y_m(k) + \sum_{i=1}^{n-1}(\mathcal{B}_0(k - 1)\mathcal{A}_i(k - 1) - s_i(k - 1))y_m(k - i) - s_0(k - 1)\mathcal{A}_n(k - 1)y_m(k - n) + s_0(k - 1)e_u(k)
\]  

(24d)
First, note that $\overline{A}_e(k - 1)$ is uniformly bounded from Lemma 3.2 (Properties (ii) and (iii)). On the other hand, the number of switches of the sequence $\beta(\cdot)$ is finite. This fact implies that the time interval between two consecutive switches is finite. Besides, the control parameters are bounded at all sampling instants since the controllability of the modified estimation plant is ensured and the closed-loop modified estimation model possesses a stable dynamics. Then, there is no finite escape between the initial time instant and the sampling instant $k_1 T$ at which the last switch in $\beta(\cdot)$ occurs. Thus, one can redefine the time origin as $k_1 T$ and study the system stability for $k \geq k_1$.

The following theorem establishes the main result of the convergence analysis.

**Theorem 4.1** (Main result). The adaptive control law stabilizes the plant (1) in the sense that $u(k), \varepsilon(k)$ and $y(k)$ are bounded for all finite initial states and any bounded reference sequence $r^*(k)$, subject to Assumption 1, provided that $z$ in (15) is sufficiently small so that

$$c_a \sup_{k_1 \leq k' \leq k} \left\{ \sqrt{2\beta_{\max}(k')} \left( 1 + \frac{\dot{\lambda}_{\max}(P(0))}{\gamma_1} \right)^{1/2} \right\} \mu \|v\| < 1$$

for all integer $k \geq k_1$ and some finite sufficient large $k_1$, where $c_a$ is a bounded constant which depends on an upper bound of the control parameters vector and an upper bound of the norm of the state transition matrix associated with $\overline{A}_e(k)$, $\sigma_0 \in (0, 1)$ is a lower bound of the absolute value of the eigenvalues of $\overline{A}_e(k)$ for all integer $k \geq k_1$, and $\dot{\lambda}_{\max}(P(0))$ denotes the maximum eigenvalue of $P(0)$.

The proof is given in Appendix C.

5. Simulations results

In this section, some simulation results are presented to illustrate the effectiveness of the robust adaptive control scheme described in Sections 2 and 3. The nominal plant to be controlled is given by $W_0(q^{-1}) = (1 - 1.4q^{-1} + 0.33q^{-2})/(1 - 1.6q^{-1} + 0.39q^{-2})$ and it is subject to a multiplicative and an additive unmodeled dynamics given, respectively, by $\nu \Delta_1(q^{-1}) = (0.02q^{-1})/(1 + 0.2q^{-1})$ and $\nu \Delta_2(q^{-1}) = (0.02q^{-1})/(1 + 0.25q^{-1})$. The stable filter $W_s(q^{-1}) = (q^{-1}(1 + q^{-1}))/((1 + 0.5q^{-1} + 0.22q^{-2})$ and the external input signal

$$r^*(kT) = \begin{cases} \sin 3kT + \sin 7kT + \sin 10kT & \text{for } 0 \leq kT < 40 \\ 1 & \text{for } kT > 40 \end{cases}$$
for all integer $k \geq 0$ where $T = 0.01s$, are considered. The following parameters are chosen in (15) for the computation of the overbounding function for the unmodeled dynamics: $\alpha = 10^{-5}$, $\alpha_0 = 0$ and $\sigma = 0.5$. Also, $v = [0.1 \ 0.2 \ -0.3 \ 0.1]^T$ in (16), the sequence $\gamma(k) = 4$ for all integer $k \geq 0$ in (17a) and $\mu = 1.0001$ in (17c) and (17d), and

$$P(0) = 55 \times \begin{bmatrix}
1 & 0.4 & 0.4 & 0.4 & 0.4 \\
0.4 & 1 & 0.4 & 0.4 & 0.4 \\
0.4 & 0.4 & 1 & 0.4 & 0.4 \\
0.4 & 0.4 & 0.4 & 1 & 0.4 \\
0.4 & 0.4 & 0.4 & 0.4 & 1
\end{bmatrix}$$

and $\hat{\theta}(0) = [-1.3 \ -0.7 \ -0.5 \ 1.2 \ -0.9]^T$ are the initial conditions of the estimation algorithm with $\delta_0 = 0.001$. The parameter $\delta = 10^{-5}$ for the first modification algorithm and $\Delta \pi_0 = 10^{-4}$ for the second modification algorithm are chosen. The control objective is defined by the Hurwitz polynomial $C(q^{-1}) = 1 + 0.4q^{-1} - 0.25q^{-2} - 0.1q^{-3}$. The simulation results are shown in Figs. 1–8.

(i) Results with the first modification algorithm

![Fig. 1. Tracking error signal in the interval [0,50].](image-url)
Fig. 2. Control signal in the interval [0,50].

Fig. 3. Parameters of the plant estimated and modified model in the interval [0,10].
(ii) Results with the second modification algorithm

Fig. 4. Components of the vector $e_h$ in the interval [0,0.5].

Fig. 5. Tracking error signal in the interval [0,50].
Fig. 6. Control signal in the interval [0,50].

Fig. 7. Parameters of the plant estimated and modified model in the interval [0,10].
Remark 6. Note that Figs. 1, 2, 5 and 6 exhibit bounded jumps in the tracking error and control signals. These jumps occur at the sampling instants at which the sequence $\beta(k)$ changes its value. The modified estimated parameters present also bounded jumps at the same sampling instants at which the sequence $\beta(k)$ changes its value. The presence of such jumps is crucial to ensure the controllability of the estimated and modified model of the plant when the unmodified one loses (or it is close to lose) its controllability. Note also that the numerator and denominator polynomials of the modeled plant present the common factor $(1 - 0.3q^{-1})$. I.e., the controllability condition for the nominal plant is not fulfilled. However, the adaptive system becomes stabilizable since the estimated model of the plant is controllable thanks to the modification algorithm. Note that $\beta(k)$ switches a finite number of times before converging for both modification algorithms.

6. Conclusions

An adaptive pole-placement based control algorithm has been presented that stabilizes an, in general, inversely unstable discrete-time system in the presence of unmodeled dynamics and, eventually, bounded noise. The algorithm includes the use of a relative adaptation dead-zone for the closed-loop
robust adaptive stabilization. The controllability of the estimated model is ensured by incorporating an appropriate modification in the standard least-squares estimation algorithm which is implemented as the first estimation level. The boundedness of the tracking error and all the remaining signals in the closed-loop system is ensured without providing the controllability of the nominal plant.

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Appendix A. Proof of Lemma 3.1

(i) The matrix $P(0)$ is chosen symmetrical and $P(k + 1) - P(k)$ is also symmetrical from (17a) for all integer $k \geqslant 0$. Then, $P(k)$ is a monotonic non-increasing symmetrical matrix sequence since $P(k + 1) - P(k) \geqslant 0$ for all integer $k \geqslant 0$ from Eq. (17a). Thus, if $P(k_1) = 0$ for any integer $k_1 \geqslant 0$, then, $P(k_1 + 1) - P(k_1) = 0$ from (17a) and then, $P(k) = 0$ for all integer $k \geqslant k_1$. Consequently, $P(k)$ is upper bounded by $P(0)$ and lower bounded by zero, i.e., $0 \leqslant P(k) \leqslant P(0)$, and it also converges asymptotically to a finite limit as $k$ tends to infinity because of it is monotonic non-increasing matrix sequence.

(ii) Consider the non-negative sequence $V(k) = \Omega^T(k)P^{-1}(k)\tilde{\Theta}(k) + \text{tr}(P(k))$ for all integer $k \geqslant 0$. The matrix inversion lemma, [18], applied to the second equation of (17a) yields

$$P^{-1}(k + 1) = P^{-1}(k) + \frac{s(k)}{\gamma(k)}\phi_n(k)\phi_n^T(k) \quad (A.1)$$

From Eqs. (17a)–(17d), (15) and (A.1), it follows that

$$V(k + 1) - V(k) = -\frac{s(k)w^2(k)}{\gamma(k) + s(k)\phi_n^T(k)P(k)\phi_n(k)} + \frac{s(k)}{\gamma(k)}\eta_n^2(k)$$

$$\leq -\left(\frac{\mu^2 - 1}{\mu^2}\right)\frac{f^2(k)}{\gamma(k) + s(k)\phi_n^T(k)P(k)\phi_n(k)} \quad (A.2)$$

where $s(k) \in [0, 1)$ and $s(k)w^2(k) = f(k)w(k) \geq f^2(k)$ have been used. From (A.1) it follows that

$$\lambda_{\min}(P^{-1}(k)) \geq \lambda_{\min}(P^{-1}(0)) \quad (A.3)$$
for all integer $k \geq 0$. Eq. (A.2) leads to

$$V(k) \leq V(0) \iff \bar{\theta}^T(k)P^{-1}(k)\bar{\theta}(k) + \text{tr}P(k) \leq \bar{\theta}^T(0)P^{-1}(0)\bar{\theta}(0) + \text{tr}P(0)$$

(A.4)

for all integer $k \geq 0$, and then

$$\lambda_{\text{max}}(P^{-1}(0))\|\bar{\theta}(0)\|^2 + \text{tr}P(0)$$

$$\geq \lambda_{\text{min}}(P^{-1}(0))\|\bar{\theta}(0)\|^2 + \text{tr}P(k) \iff \|\bar{\theta}(0)\|^2$$

$$\leq \frac{\lambda_{\text{max}}(P(0))}{\lambda_{\text{min}}(P(0))}\|\bar{\theta}(0)\|^2 + \frac{\text{tr}P(0) - \text{tr}P(k)}{\lambda_{\text{min}}(P^{-1}(0))} < \infty$$

(A.5)

for all integer $k \geq 0$, where (A.3) and the boundedness of $\text{tr}P(k)$ have been used. Hence $\bar{\theta}(k)$ and thus $\theta(k)$, for all integer $k \geq 0$, are both bounded. From Assumption 1 and (15), (16):

$$|\eta(k)| \leq \bar{\eta}(k) = \alpha \rho(k) + \alpha_0 = \alpha \sup_{0 \leq k' \leq k} \{\|v^T x(k')\|\sigma^{k-k'}\} + \alpha_0$$

$$\leq c_2 \sup_{0 \leq k' \leq k} \{\|v\|\|\phi(k')\|\sigma^{k-k'}\} + c_1$$

(A.6)

for some real positive constants $c_1$ and $c_2$, where it has been used the fact that the absolute value of the scalar product of two vectors is lower or equal than the product of their norms, the definition of $\phi(k)$ in (4) and $e(j) = y(j) - y_m(j)$, for any integer $j \geq 0$, since $y_m(j)$ is bounded. From (A.6) and (15), (16), it follows that

$$\bar{\eta}_n(k) = \frac{\bar{\eta}(k)}{1 + \|\phi(k)\|} \leq \frac{c_2 \sup_{0 \leq k' \leq k} \{\|v\|\|\phi(k')\|\sigma^{k-k'}\} + c_1}{1 + \|\phi(k)\|} < \infty$$

(A.7)

so that $|\eta_n(k)| < \infty$ for all integer $k \geq 0$. Finally, since $e(k) = -\phi^T(k)\bar{\theta}(k) + \eta(k)$ from (17b) and the definition of $e_n(k, \phi_n(k)$ and $\eta_n(k)$ one obtains that

$$e_n(k) = -\phi_n^T(k)\bar{\theta}(k) + \eta_n(k)$$

(A.8)

and then $|e_n(k)| < \infty$, for all integer $k \geq 0$, since $\|\phi_n(k)\| < \infty$, $\|\bar{\theta}(k)\| < \infty$ and $|\eta_n(k)| < \infty$. Then, $w(k) < \infty$ for all integer $k \geq 0$ since $P(k)$ is bounded.

(iii) From Eq. (17d), $f(k) < \infty$ for all integer $k \geq 0$ since $w(k)$, $P(k)$ and $\bar{\eta}_n(k)$ are bounded from (i)–(ii) and $\phi_n(k)$ is also bounded and $\gamma(k) > 0$ from their definitions. From (A.2) it follows that

$$\left(\frac{\mu^2 - 1}{\mu^2}\right) \sum_{i=0}^k \frac{f^2(i)}{\gamma(i) + s(i)\phi_n^T(i)P(i)\phi_n(i)} \leq V(0) - V(k+1) \leq V(0) < \infty$$

(A.9)
and therefore \( \sum_{i=0}^{k} f^2(i) < \infty \) and \( \sum_{i=0}^{k} f(i) < \infty \) for all integer \( k \geq 0 \). Thus, \( \lim_{k \to \infty} f^2(k) = 0 \) and \( \lim_{k \to \infty} f(k) = 0 \). In a similar way, \( \sum_{i=0}^{k} s(i)w^2(i) < \infty \), \( s(k)w^2(k) < \infty \) for all integer \( k \geq 0 \), and \( \lim_{k \to \infty} s(k)w^2(k) = 0 \) from (A.2) since \( s(k) \in [0, 1] \), for all integer \( k \geq 0 \), from (17c).

(iv) From (17a) and (17c), one gets

\[
\|\hat{\theta}(k+1) - \hat{\theta}(k)\|^2 \leq \frac{s(k)w^2(k)A^2 \{P(k)\} \|\phi_n(k)\|^2}{\gamma^2(k)} \leq k_0 s(k)w^2(k)
\]

(A.10)

for some positive real constant \( k_0 \), where the facts that \( s(k) \in [0, 1] \) and \( P(k) \) and \( \|\phi_n(k)\| \) are bounded for all integer \( k \geq 0 \) have been used. From (A.10), it follows that \( \|\hat{\theta}(k+1) - \hat{\theta}(k)\| < \infty \) for all integer \( k \geq 0 \) and \( \lim_{k \to \infty} \|\hat{\theta}(k+1) - \hat{\theta}(k)\| = 0 \) since \( s(k)w^2(k) < \infty \) and \( \lim_{k \to \infty} s(k)w^2(k) = 0 \) from (iii).

(v) From (A.2), it follows that

\[
M_1(k) = \sum_{i=0}^{k} s(i) \left( w^2(i) - \left[ 1 + \frac{\phi_n^T(i)P(i)\phi_n(i)}{\gamma(i)} \right] \eta^2_n(i) \right) \leq V(0) - V(k+1) \leq V(0) < \infty
\]

(A.11)

Let \( I_1 \) and \( I_2 \) be \( I_1 = \{ k \in \{0, 1, 2, \ldots \} \mid s(k) = 0 \} \) and \( I_2 = \{ k \in \{0, 1, 2, \ldots \} \mid s(k) \neq 0 \} \), respectively. At the sampling instants \( kT \) such that \( k \in I_2 \), \( w^2(k) > \mu^2(1 + (\phi_n^T(k)P(k)\phi_n(k))/\gamma(k))\eta^2_n(k) \) is fulfilled. Then, (A.11) implies since \( s(i) \in [0, 1] \):

\[
M_2(k) \sum_{i=0}^{k} s(i) \left( w^2(i) - \left[ 1 + \frac{\phi_n^T(i)P(i)\phi_n(i)}{\gamma(i)} \right] \eta^2_n(i) \right) \leq M_1(k) < \infty
\]

(A.12)

with

\[
M_2(k) = \inf_{i \in I_2} \left\{ \frac{w^2(i) - \left[ 1 + \frac{\phi_n^T(i)P(i)\phi_n(i)}{\gamma(i)} \right] \eta^2_n(i)}{\gamma(i) + s(i)\phi_n^T(i)P(i)\phi_n(i)} \right\} > 0
\]

for all integer \( k \geq 0 \). Then, \( \lim_{k \to \infty} s(k) = 0 \). Thus, it follows from (A.1) that

\[
P^{-1}(k+1) = P^{-1}(0) + \sum_{i=0}^{k} s(i)\phi_n(i)\phi_n^T(i)
\]

(A.13)

From (A.13) and the facts that \( \|P(0)\| > 0 \), \( \phi_n(k) \) is bounded, \( \gamma(k) > 0 \) for all integer \( k \geq 0 \) and \( \lim_{k \to \infty} s(k) = 0 \), it follows that \( \|P^{-1}(k)\| < \infty \), \( \|P(k)\| > 0 \).
for all finite \( k \) and as \( k \to \infty \). Finally, it follows, from the definition of \( w(k) \) in Eqs. (17), that

\[
\sum_{i=0}^{k} s(i)e_n^2(i) \leq \sum_{i=0}^{k} s(i)w^2(i) - \inf_{\substack{i \in \{0, 1, \ldots, k\}}} \{ \phi_n^T(i)P^2(i)\phi_n(i) \} \sum_{i=0}^{k} s(i) < \infty
\]

(A.14)

where the boundedness of \( \sum_{i=0}^{k} s(i)w^2(i) \), \( \sum_{i=0}^{k} s(i) \), \( \phi_n(k) \) and \( P(k) \) for all integer \( k \geq 0 \) have been used. Thus, \( 0 \leq s(k)e_n^2(k) < \infty \) for all integer \( k \geq 0 \) and \( \lim_{k \to \infty} s(k)e_n^2(k) = 0 \).

(vi) From (17a) and (A.1), one gets

\[
\|P^{-1}(k)\tilde{\theta}(k)\| \\
\leq \|P^{-1}(k-1)\tilde{\theta}(k-1)\| + \frac{s(k-1)}{\gamma(k-1)} \|\phi_n(k-1)\| \|\phi_n^T(k-1)\tilde{\theta}(k-1)\| \\
+ \frac{1 + \gamma(k-1)}{\gamma(k-1)} s(k-1)\|\phi_n(k-1)\|\|e_n(k-1)\|
\]

(A.15)

From \( e_n(i) = -\phi_n^T(i)\tilde{\theta}(i) + \eta_n(i) \) for all integer \( i \geq 0 \), it follows that \( |\phi_n^T(i)\tilde{\theta}(i)| \leq |e_n(i)| + |\eta_n(i)| \). By introducing this expression in (A.15), one obtains

\[
\|P^{-1}(k)\tilde{\theta}(k)\| \\
\leq \|P^{-1}(k-1)\tilde{\theta}(k-1)\| + \frac{s(k-1)}{\gamma(k-1)} \|\phi_n(k-1)\|\|\eta_n(k-1)\| \\
+ \|2 + \gamma(k-1)\|e_n(k-1)\| \\
\leq \|P^{-1}(0)\tilde{\theta}(0)\| + \sup_{0 \leq i \leq k-1} \left\{ \frac{1}{\gamma(i)} \|\phi_n(i)\|\|\eta_n(i)\| + \|2 + \gamma(i)\|e_n(i)\| \right\} \\
\times \sum_{i=0}^{k-1} s(i) < \infty
\]

(A.16)

for all integer \( k \geq 0 \) since \( \sum_{i=0}^{k-1} s(i) < \infty \), \( \|\phi_n(k)\| < \infty \), \( |\eta_n(k)| < \infty \), \( |e_n(k)| < \infty \) and \( 0 < \gamma(k) < \infty \).

Appendix B. Proof of Lemma 3.2

(i) These properties have to be proved for both the first modification algorithm and the second modification algorithm.

For the first modification algorithm, it has been proved in [6] that there exits a positive sufficient small real constant \( \delta_0 \) such that \( |\text{Det}(M(\tilde{\theta}(k) + P(k)\delta e_i))| \geq \delta_0 \) for at least one of the vectors \( e_i \) at each sampling instant, for
Thus, from the fact that \( P(k) \) and \( \hat{\theta}(k) \) are bounded and converge (see Properties (i), (ii) and (iv) of Lemma 3.1) and from (19), it follows that the number of switches in \( \beta(k) \) is finite and then \( \beta(k) \) converges asymptotically. Besides, \( \beta(k) \) is uniformly bounded since it equates to the product of \( d \) with one of the aforementioned bounded constant vectors \( e_i \) at each sampling instant.

For the second modification algorithm, one has to prove that there exists a bounded sequence \( p(k) \) such that \( |\text{Det}(M(\tilde{\theta}(k)))| \geq \delta_0 > 0 \), where the matrix defined as \( M(\tilde{\theta}(k)) \) is:

\[
M(\tilde{\theta}(k)) = \begin{bmatrix}
1 & 0 & \cdots & 0 & \tilde{\theta}_0 & 0 & \cdots & 0 & 0 \\
\tilde{a}_1 & 1 & \cdots & 0 & \tilde{\theta}_1 & \tilde{\theta}_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{a}_{n-1} & \tilde{a}_{n-2} & \cdots & 1 & \tilde{\theta}_{n-1} & \tilde{\theta}_{n-2} & \cdots & \tilde{\theta}_0 & 0 \\
\tilde{a}_n & \tilde{a}_{n-1} & \cdots & \tilde{a}_1 & \tilde{\theta}_n & \tilde{\theta}_{n-1} & \cdots & \tilde{\theta}_1 & 0 \\
0 & \tilde{a}_n & \cdots & \tilde{a}_2 & 0 & \tilde{\theta}_n & \cdots & \tilde{\theta}_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tilde{a}_n & 0 & 0 & \cdots & \tilde{\theta}_n & 0 \\
-\left(1 + \sum_{i=1}^{n} \tilde{a}_i\right) & -\left(1 + \sum_{i=1}^{n} \tilde{a}_i\right) & \cdots & -\left(1 + \sum_{i=1}^{n} \tilde{a}_i\right) & -\sum_{i=0}^{n} \tilde{\theta}_i & -\sum_{i=0}^{n} \tilde{\theta}_i & \cdots & -\sum_{i=0}^{n} \tilde{\theta}_i & \sum_{i=0}^{n} \tilde{\theta}_i \\
\end{bmatrix}
\]  

(B.1)

From (18) and the expression \( \beta(k) = \pi(k)\beta'(k) \), the vector \( \tilde{\theta}(k) \) can be written as:

\[
\tilde{\theta}(k) = \hat{\theta}(k) + \pi(k)P(k)\beta'(k)  
\]  

where the components of the vector sequence \( \beta'(k) \) are calculated by means of (20). i.e., \( \beta'(k) \) is the solution of the matrix equation:

\[
P(k)\beta'(k) = v  
\]  

where

\[
v = \begin{bmatrix}
0 & 0 & \cdots & 0 & n+1 \\
0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}^T \in \mathbb{R}^{2n+1}.  
\]  

That is, the components of \( \beta'(k) \) are such that, at each sampling instant:

\[
p_i^T(k)\beta'_i(k) = 0 \quad \text{for} \quad i \in \{1, \ldots, n\} \cap \{n+2, \ldots, 2n+1\};
\]

\[
p_{n+1}^T(k)\beta'_{n+1}(k) = 1  
\]  

where \( p_i(k) \) denotes the \( i \)-column of the matrix \( P(k) \). By introducing (B.2) and (B.4) into (B.1), it follows that:
of determinant

$$\text{Det}(M(\tilde{\theta}(k))) = (\hat{b}_n(k) + \pi(k))^{n+1} + g(\hat{\theta}(k), \pi(k))$$

where both $g(\hat{\theta}(k), \pi(k))$ and $g'(\hat{\theta}(k), \pi(k))$ are $n$-order polynomials in $\pi(k)$.

From (B.6), the controllability condition of modified estimation plant model is:

$$|\pi(k)^{n+1} + g'(\hat{\theta}(k), \pi(k))| \geq \delta_0$$  \hspace{1cm} (B.7)

Note that function $g'(\hat{\theta}(k), \pi(k))$ is a sum of a finite number of terms of the form $\pi(k)^{c_0} \hat{\theta}_1(k)^{c_1} \cdots \hat{\theta}_j(k)^{c_j}$ for some non-negative integers $c_0, c_1, \ldots, c_j$, and where $\hat{\theta}_j(k)$ and $\hat{\theta}_j(k)$ denote some components of the vector $\hat{\theta}(k)$. Then, $g'(\hat{\theta}(k), \pi(k))$ is analytic for all $\pi(k) \in \mathbb{R}$ and for all $\hat{\theta}(k) \in \mathbb{R}^{2n+1}$. Then, the expression $\text{Det}(M(\tilde{\theta}(k)))$ is a $(n+1)$-order polynomial in $\pi$, with all coefficients well defined at all sampling instants. Thus, there exists some absolute bounded value for $\pi(k)$ which verifies the controllability condition (B.7). Besides, the vector sequence $\beta'(k)$ is uniformly bounded and it asymptotically converges from (20) and the boundedness and asymptotic convergence of $P(k)$ [Lemma 3.1, (i) ]. Then, $\pi(k)$ asymptotically converges to a bounded constant from (21) and the uniform boundedness and asymptotic convergence of $\hat{\theta}(k)$ [Lemma 3.1, (ii) and (iv)]. Thus, $\beta(k)$ is uniformly bounded, has a finite number of switches and asymptotically converges.

(ii) It follows directly from (i) since $P(k)$ and $\hat{\theta}(k)$ are bounded.

(iii) It follows directly since $\|\tilde{\theta}(k)\| < \infty$ for all integer $k \geq 0$, $\tilde{\theta}(k)$ converges to a limit as $k \to \infty$ and $|\text{Det}(M(\tilde{\theta}(k)))| \geq \delta_0 > 0$ for all integer $k \geq 0$.

(iv) Since $e_{an}(k) = e_{n}(k) - \beta^T(k)P(k)\phi_a(k)$, one obtains from the relation $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$:

$$e_{an}(k) \leq 2(e_{n}^2(k) + \|\beta(k)\|^2 \phi_a^T(k)P^2(k)\phi_a(k))$$

$$\Rightarrow \lim_{t \to \infty} \left\{2\beta_{\max}(k) \left[ \nu^2(k) - \mu^2 \left( 1 + \frac{\phi_a^T(k)P(k)\phi_a(k)}{\gamma(k)} \right) \pi^2_{an}(k) \right] \right\} \leq 0$$  \hspace{1cm} (B.8)
since \( \lim_{k \to \infty} s(k) \to 0 \) so that \( w^2(k) \leq \mu^2 \left( 1 + (\phi_n^1(k)P(k)\phi_n(k))/\gamma(k) \right) \overline{m}_n^2(k) \) as \( k \to \infty \). □

**Appendix C. Proof of Theorem 4.1**

\( \overline{A}_c(k) \) is bounded because of \( \overline{b}(k) \), and the controller parameters \( \overline{K}(k) \) and the coefficients of \( \overline{K}(q^{-1}, k) \) and \( \overline{s}(q^{-1}, k) \) are bounded, from Lemma 3.2(ii–(iii)). The eigenvalues of \( \overline{A}_c(k) \) are in \( |z| < 1 \) for all integer \( k \geq 0 \). Besides,

\[
\sum_{k'=k_0+1}^{k} \| \overline{A}_c(k') - \overline{A}_c(k' - 1) \|^2 \leq \beta_0 + \beta_1 (k - k_0) \tag{C.1}
\]

for all integers \( k \) and \( k_0 \) such that \( k > k_0 \geq 0 \), some positive constants \( \beta_0 \) and \( \beta_1 \) with \( \beta_1 \) being sufficiently small. Note that (C.1) is fulfilled with a suitable constant \( \beta_0 \), which takes into account the changes of value of the entries of \( \overline{A}_c(\cdot) \) when a switch in the sequence \( \overline{b}(k) \) occurs, and a slow enough estimation rate via suitable \( P(0) \) and \( \gamma(k) \) in (17a) so that \( \beta_1 \) is sufficiently small. Thus, the time-varying homogeneous system \( x(k + 1) = \overline{A}_c(k)x(k) \) is exponentially stable and its transition matrix \( \psi(k, k') = \prod_{j=k}^{k'-1} \overline{A}_c(j) \) satisfies \( \|\psi(k, k')\| \leq c_1 \sigma_0^{k-k'} \) for all \( k \geq k' \) where \( \sigma_0 \in (0, 1) \) and \( c_1 \) is a norm-dependent constant, [19,20].

We now redefine the time origin \( k_1 T \) as the sampling instant such that for \( kT > k_1 T \) there are not changes of value in \( \beta(k) \). From (24a), if follows that

\[
x(k) = \psi(k, k_1)x(k_1) + \sum_{k'=k_1}^{k} \psi(k, k')[B_1 \overline{b}_1(k') + B_2 \overline{b}_2(k')]
\]

for all integer \( k \geq k_1 \geq 0 \). From (C.2), one obtains

\[
\|x(k)\| \leq c_1 \sigma_0^{k-k_1} \|x(k_1)\| + \sum_{k'=k_1}^{k} c_1 \sigma_0^{k-k'} (c_2 + c_3 |e_a(k')|) \\
\leq c_4 + \sum_{k'=k_1}^{k} c_5 \sigma_0^{k-k'} |e_a(k')| \tag{C.3}
\]

since \( \| \sum_{i=1}^{2} B_i \overline{b}_i(k) \| \leq c_2 + c_3 |e_a(k)| \), for some positive real constants \( c_2, c_3, c_4 \) and \( c_5 \), from (24c) and (24e), Schwarz’s inequality, \( \sigma_0 \in (0, 1) \), and the boundedness of \( \overline{K}(\cdot), \overline{b}_0(\cdot), \overline{a}_i(\cdot) \), for \( i \in \{1, \ldots, n\} \), the sequence \( y_m(\cdot) \) and \( \|x(k_1)\| \). From (B.8), \( e_{\inf}^2(k) \leq 2\beta_{\text{max}}(k)w^2(k) \) so that
where \( w(k) \) has been split into the two additive terms \( f(k) \) and

\[
w(k) - f(k) \leq \mu \left( 1 + \frac{\phi_n^T(k)P(k)\phi_n(k)}{\gamma(k)} \right)^{1/2} \cdot \eta_n(k)
\]

from (17d). It follows from (15), (16) and (C.4) that

\[
\|x(k)\| \leq c_6 + c_5 \mu \|x\| \sup_{k_1 \leq k' \leq k} \left\{ \sqrt{2\beta_{\max}(k')} \left( 1 + \frac{\phi_n^T(k')P(k')\phi_n(k')}{\gamma(k')} \right)^{1/2} \right\}
\times \sup_{k_1 \leq k' \leq k} \left\{ \frac{1 - \sigma_0^{-k+k+1}}{1 - \sigma_0} \right\}
\times \sum_{k' = k_1}^k c_5 \sigma_0^{k-k'} \sqrt{2\beta_{\max}(k')} f(k') (1 + \|\phi(k')\|)
\]  

\[(C.5)\]

for some positive constant \( c_6 \), where \( \|P(k)\| < \infty \) and \( \|v\| \sup_{k_1 \leq k' \leq k} \{\|x(k')\|\} \geq \sup_{k_1 \leq k' \leq k} \{v^T x(k')|a^{k-k'}\} \) have been used. Besides, the right-hand side of (C.5) is monotonic non-decreasing in \( k \). Then, for \( k \gg k_1 \) and provided that

\[
x < \frac{1 - \sigma_0}{\mu c_5 \|v\|} \sup_{k_1 \leq k' \leq k} \left\{ \sqrt{2\beta_{\max}(k')} \left( 1 + \frac{\phi_n^T(k')P(k')\phi_n(k')}{\gamma(k')} \right)^{1/2} \right\}
\]

one deduces from (C.5) that

\[
\|x(k)\| \leq c_6 + \sum_{k' = k_1}^k c_5 \sigma_0^{k-k'} \sqrt{2\beta_{\max}(k')} f(k') (1 + \|\phi(k')\|)
\]  

\[(C.6)\]

From (3) and (4), with \( n = m \), (15), (16) and (23) and the boundedness of \( \overline{K}(\cdot), \overline{\eta}(\cdot) \) and \( \overline{\sigma}(\cdot) \), one gets
\[ \| \phi(k) \| \leq |u(k)| + \sum_{i=1}^{n} (|u(k-i)| + |y(k-i)|) \]

\[ \leq |u(k)| + \sum_{i=1}^{n} (|u(k-i)| + |e(k-i)|) + c_7 \leq c_8 + c_9 \sup_{k_1 \leq k' \leq k} \{ \| x(k') \| \} \]

(C.7)

for some positive constants \( c_i \), for \( i \in \{7, 8, 9\} \). From (C.6) and (C.7), it follows directly that

\[ \| x(k) \| \leq c_6 + \sum_{k' = k_1}^{k} c_5 \sigma_0^{k-k'} \sqrt{2\beta_{\text{max}}(k') f(k')} \left( 1 + c_{10} + c_{11} \sup_{k_1 \leq k'' \leq k'} \{ \| x(k'') \| \} \right) \]

(C.8)

Since the right-hand side of (C.8) is monotonic non-decreasing, it follows that

\[ \sup_{k_1 \leq k' \leq k} \{ \| x(k') \| \} \leq c_{12} + \sum_{k' = k_1}^{k} f(k') \left( c_{13} + c_{14} \sup_{k_1 \leq k'' \leq k'} \{ \| x(k'') \| \} \right) \]

\[ \leq c_{15} + c_{16} \sum_{k' = k_1}^{k} f(k') \sup_{k_1 \leq k'' \leq k'} \{ \| x(k'') \| \} \]

(C.9)

for some positive constants \( c_i \), for \( i \in \{12, \ldots, 16\} \), where the boundedness of \( f(k) \) and \( \beta_{\text{max}}(k) \) has been used. The use of the discrete Gronwall’s lemma, [21], in (C.9) leads to

\[ \sup_{k_1 \leq k' \leq k} \{ \| x(k') \|^2 \} \leq c_{15} + \sum_{k_1 \leq k \leq k} \left[ \prod_{i \leq j < k} (1 + c_{16} f^2(j)) c_{15} c_{16} f^2(i) \right] < \infty \]

(C.10)

Thus, \( \| x(k) \| \) is bounded and then, \( u(k-i), e(k-i) \) and \( y(k-i) \) are bounded for \( i \in \{0, \ldots, n-1\} \). From (C.10), (4) and (15), \( \phi(k) \), \( \rho(k) \) and \( \overline{\eta}(k) \) are also bounded. The proof has been completed. \( \Box \)

References