$H_\infty$ fuzzy adaptive tracking control design for nonlinear systems with output delays

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Abstract

In this study, we develop fuzzy adaptive tracking control via two-layer fuzzy observers, variable structure systems (VSS), and $H_\infty$ control algorithm for nonlinear systems with plant uncertainties, output delays, and external disturbances. The Takagi–Sugeno fuzzy dynamic model with adaptation capability is used to approximate the nonlinear system. When the system states are not available, the states estimated from two-layer fuzzy observers combined with VSS are used to develop the fuzzy adaptive controller. In the first layer, the output delays are partitioned into $m + 1$ equal time intervals to construct the same number of fuzzy observers. The output delayed states in each time interval are used as the premise variables in the IF–THEN rules. The second layer of the fuzzy observers uses output delayed error states as its linguistic variables and it is defuzzified from the first layer. Next, we develop a fuzzy adaptive controller to overcome the nonlinearities, output delayed states, and external disturbances such that $H_\infty$ tracking performance is achieved. The Lyapunov criterion and linear matrix inequalities are used to derive the controller. In the present study, our previous method is extended to handle a class of uncertain nonlinear systems with output delays and external disturbances, which is achieved using robust VSS and $H_\infty$ control techniques. A magnetic levitation system and inverted pendulum system are used as simulation examples to illustrate the validity and confirm the performance of our proposed scheme.

Keywords: Fuzzy adaptive control; $H_\infty$ tracking performance; Inverted pendulum system; Linear matrix inequality; Magnetic levitation system; Output delay; Two-layer fuzzy observer

1. Introduction

Processes that need to be controlled or monitored far from the computing unit in real industrial applications generally have unknown and nonlinear dynamics, where only the output states may be available and measurable. In general, the data measured from the output states are transmitted via a low-rate communication system with long transmission lines, which may lead to time delays in the output states (e.g., rolling mill control systems, chemical processes, traffic networks, and telecommunication systems) and to deterioration in the performance and/or instability in the...
system. Thus, because the dynamics and system nonlinearities are unknown, it is difficult to synthesize stabilizing controllers/observers for systems with output delays, particularly when tracking the control design (e.g., see [1–4] and the references therein). Systems with delays can be stabilized based on delay-independent stability criteria [5] and delay-dependent criteria [6–8]. Furthermore, less conservative delay-dependent stability conditions can be obtained easily in a linear matrix inequality (LMI) form for time-delay nonlinear systems. This is achieved by using the Takagi–Sugeno fuzzy modeling approach when investigating the delay-dependent property [9]. A fuzzy adaptive control scheme that integrates variable structure systems (VSS) has been developed as a robust strategy for handling uncertain control systems, which may include delayed states. This technique usually yields good results in trajectory tracking problems with good data fitting for unknown and nonlinear systems (for example, see [10–12, 18–20]).

Lin et al. [13] proposed a robust fuzzy neural network controller with a nonlinear disturbance observer for a two-axis motion control system using state feedback control, which assumes that all system states are available for measurement. In practice, state feedback control is not always applicable because the system states are not always available. In [14], a high-gain observer was used to estimate the system states for non-affine nonlinear systems. However, the closed-loop system may exhibit a peaking phenomenon during the transient behavior caused by the use of high-gain observers. The design of adaptive observers based on output feedback was proposed in [15], which uses linear dependence conditions. This is achieved after removing unknown parameters and introducing universal function approximators, such as fuzzy systems and neural networks, into the observer structure. In addition to the nonlinearities and uncertainties, the output delay should be included in the dynamic system when considering stability in the control system design (for example, see [1, 2, 16] and the references therein).

However, the uncertainty bounds of the output delay might not be found easily because of the complex structure of the uncertainties. Thus, a previous study [32] proposed fuzzy adaptive observer-based (FAOB) control by partitioning output delays with equal time intervals to form the fuzzy observers, i.e., two-layer fuzzy systems with two forms of linguistic knowledge and one rule, which improve fuzzy inference with linguistic uncertainties. Recently, various techniques have been proposed that use Riccati-like equations and LMIs in the Lyapunov stability theory to find a single positive-definite matrix, a common positive-definite matrix, or a set of positive-definite matrices to guarantee the stability of the closed-loop fuzzy control system [21–25]. Some design methods for fuzzy adaptive systems use $H_\infty$ in the Lyapunov stability theory [26] and VSS schemes [27, 28] to improve the robustness and stability of a system with parameter uncertainties and external disturbances. Conventionally, most of the linguistic input variables in the fuzzy systems used for $H_\infty$ fuzzy adaptive control design are based on measurable system states. However, not all of the states of a controlled system are measurable in the field, which leads to difficulties with the implementation of conventional fuzzy systems. Recently, a chain observer was proposed for estimating the system states of a system with known output time-delays, which assumes that the system parameters are known [29]. The stability of cascade state observers for systems with time-varying output delays in the piecewise area were considered in [30]. Thus, the development of a fuzzy adaptive approach to the control of unknown nonlinear systems based on state observers [31–35] and a two-layer fuzzy observer with output delays [32, 33] could yield successful results.

In the present study, we develop fuzzy adaptive tracking control based on two-layer fuzzy observers, VSS, and an $H_\infty$ control algorithm for nonlinear systems with plant uncertainties, output delays, and external disturbances. Fuzzy adaptive systems are introduced to learn the unknown dynamics with two-layer fuzzy observers for estimating the state variables. In the first layer, the output delays are partitioned into $m + 1$ equal time intervals to construct the same number of fuzzy observers and the output delayed states in each time interval are used as premise variables in the IF–THEN rules. The second layer of the fuzzy observers uses the output delayed error states as its linguistic variables and it is defuzzified from the first layer. The effects of fuzzy approximation errors on the tracking error are eliminated by using a VSS. After characterizing the stabilization problem in terms of a LMI problem within a prescribed attenuation value and transforming the nonlinear matrix inequalities into LMI forms using a decoupling method, the estimation and tracking errors of the overall control system are guaranteed to be uniformly ultimately bounded (UUB). The $H_\infty$ tracking performance in a closed-loop system can be achieved in a Lyapunov sense. Two-layer fuzzy observers can be viewed as an extension of the type-1 fuzzy logic system and interval observer, thus they can efficiently handle nonlinear systems with plant uncertainties, output delays, and external disturbances. The method presented in [32, 33] uses an $H_\infty$ technique without a state variable filter that resulted in undesirable tracking errors due to fuzzy approximation errors. To avoid this problem and to facilitate improved performance, the present study uses a VSS scheme without restrictions on the lumped uncertainty term $(\Delta B_j)$ between 0 and 1. Moreover, a state variable filter is used to obtain all the elements of $x$ and $e$ such that the stability of the estimation error is guaranteed. However, the fuzzy approxima-
For a given reference trajectory \( x_r := [x_r, \dot{x}_r, \cdots, x_r^{(n-1)}]^\top = [x_{r1} x_{r2} \cdots x_{rn}]^\top \), we assume that there exists a compact set \( U_f \) such that \( x_r(t) \in U_f \) for all \( t \geq 0 \). By defining the tracking error vector \( e = x - x_r \), the tracking error dynamic equation can be obtained as

\[
\dot{e} = Ae + B(f(x) + b(x)u + d - x_r^{(n)})
\]

Since the functions \( f(x) \) and \( b(x) \) are unknown and nonlinear, the fuzzy logic system can uniformly approximate nonlinear continuous functions to an arbitrary level of accuracy. Thus, the fuzzy systems \( \hat{f}(x) \) and \( \hat{b}(x) \) are used to approximate \( f(x) \) and \( b(x) \), respectively. The fuzzy logic system performs a mapping from \( U_c \subset \mathbb{R}^n \) to \( \mathbb{R} \) and the knowledge base for the fuzzy systems comprises a collection of fuzzy \( 2N_p \) IF–THEN rules with the following forms

\[
R^\ell_f: \text{IF } x_1 \text{ is } F^\ell_{f1} \text{ AND } x_2 \text{ is } F^\ell_{f2} \text{ AND } \cdots \text{ AND } x_n \text{ is } F^\ell_{fn} \text{ THEN } \hat{f}(x) \text{ is } H^\ell_f
\]

and
for $\ell = 1, 2, \ldots, N_p$, where $x = [x_1 x_2 \cdots x_n]^{\top}$ and $\hat{f}(x), \hat{b}(x)$ are the fuzzy system inputs and outputs, respectively; $F_{f_1}^\ell$ and $F_{b_1}^\ell$ are the fuzzy sets associated with the linguistic variables $x$ in the universes of discourses $U_c \subset \mathbb{R}^n$, $H_f^\ell$ and $H_b^\ell$ are the fuzzy sets associated with the linguistic variables $\hat{f}(x)$ and $\hat{b}(x)$ in $W \subset \mathbb{R}$, respectively. The fuzzy inference engine performs a mapping from the fuzzy sets in $U_c$ to the fuzzy sets in $W$. The fuzzifier maps a crisp point $x$ into a fuzzy set in $U_c$. The defuzzifier maps a fuzzy set in $W$ to a crisp point in $W$. Using the singleton fuzzifier, product inference engine, and center average defuzzifier, the fuzzy model outputs can be expressed as:

$$\hat{f}(x) = \sum_{\ell=1}^{N_p} \left( \frac{\prod_{i=1}^{n} \mu_{f_1}^\ell(x_i)}{\sum_{\ell=1}^{N_p} \prod_{i=1}^{n} \mu_{f_1}^\ell(x_i)} \right) h_f^\ell = \sum_{\ell=1}^{N_p} \xi_{f}^{\ell}(x) h_f^\ell$$

$$\hat{b}(x) = \sum_{\ell=1}^{N_p} \left( \frac{\prod_{i=1}^{n} \mu_{b_1}^\ell(x_i)}{\sum_{\ell=1}^{N_p} \prod_{i=1}^{n} \mu_{b_1}^\ell(x_i)} \right) h_b^\ell = \sum_{\ell=1}^{N_p} \xi_{b}^{\ell}(x) h_b^\ell$$

where $h_f^\ell$, $h_b^\ell$ are the centers of the $\ell$th fuzzy set, which are the points in $W$ where $h_f^\ell$, $h_b^\ell$ achieve their maximum values or are equal to 1, and the following nonlinear mappings

$$\xi_{f}^{\ell}(x) = \frac{\prod_{i=1}^{n} \mu_{f_1}^\ell(x_i)}{\sum_{\ell=1}^{N_p} \prod_{i=1}^{n} \mu_{f_1}^\ell(x_i)}$$

and

$$\xi_{b}^{\ell}(x) = \frac{\prod_{i=1}^{n} \mu_{b_1}^\ell(x_i)}{\sum_{\ell=1}^{N_p} \prod_{i=1}^{n} \mu_{b_1}^\ell(x_i)}$$

According to the universal approximation theorem [17], the fuzzy systems (4) and (5) can approximate any nonlinear smooth function $\hat{f}(x)$ and $\hat{b}(x)$ in a compact operating space to any degree of accuracy. In the same manner as most control applications, the appropriate structure of the fuzzy system is specified in advance by the designer in the present study, including the number of rules, number of membership functions for each input, pertinent inputs, and membership function parameters. Moreover, the consequent parameters $h_f^\ell$ and $h_b^\ell$ must be calculated by the learning algorithms.

If we view the $h_f^\ell$’s and $h_b^\ell$’s as adjustable parameters, then (4) and (5) can be rewritten, respectively, as

$$\hat{f}(x, \theta_f) = \theta_f^{\top} \xi_{f}(x)$$

$$\hat{b}(x, \theta_b) = \theta_b^{\top} \xi_{b}(x)$$

where $\theta_f = [h_f^1 h_f^2 \cdots h_f^{N_p}]^{\top} \in \mathbb{R}^{N_p}$ and $\theta_b = [h_b^1 h_b^2 \cdots h_b^{N_p}]^{\top} \in \mathbb{R}^{N_p}$ are free (adjusted) parameter vectors for $\hat{f}(x, \theta_f)$ and $\hat{b}(x, \theta_b)$, respectively, and $\xi_{f}(x) = [\xi_{f}^{1} \xi_{f}^{2} \cdots \xi_{f}^{N_p}]^{\top} \in \mathbb{R}^{N_p}$ and $\xi_{b}(x) = [\xi_{b}^{1} \xi_{b}^{2} \cdots \xi_{b}^{N_p}]^{\top} \in \mathbb{R}^{N_p}$ are two fuzzy basis function vectors.

**Assumption 1.** The fuzzy consequent parameter vectors $\theta_f$ and $\theta_b$ belong to compact sets $\Omega_f$ and $\Omega_b$, respectively, which are defined as

$$\Omega_f = \{ \theta_f \in \mathbb{R}^{N_p} \mid \| \theta_f \| \leq M_{\theta_f} \}$$

$$\Omega_b = \{ \theta_b \in \mathbb{R}^{N_p} \mid 0 < \delta \leq \| \theta_b \| \leq M_{\theta_b} \}$$

where $M_{\theta_f}$, $M_{\theta_b}$, and $\delta$ are designed finite positive constants.
For (6) and (7), there exist optimal parameter vectors $\theta^*_f$ and $\theta^*_b$, which lead to the minimum approximation errors for $f(x)$ and $b(x)$, respectively, as follows:

$$
\begin{align*}
\theta^*_f &\triangleq \arg \min_{M_f \in \mathbb{R}^{N_f}} \left( \sup_{x \in U_c} |f(x) - \hat{f}(x, \theta_f)| \right) \\
\theta^*_b &\triangleq \arg \min_{M_b \in \mathbb{R}^{N_b}} \left( \sup_{x \in U_c} |b(x) - \hat{b}(x, \theta_b)| \right)
\end{align*}
$$

(9)

Define the minimum approximation errors as

$$
\begin{align*}
\delta_f &= f(x) - \hat{f}(x, \theta_f) \\
\delta_b &= b(x) - \hat{b}(x, \theta_b)
\end{align*}
$$

(10)

From (6), (7), and by Assumption 1, it is clear that we can also assume that $|\hat{f}(x, \theta_f)| \leq F$ and $0 < b_{\min} \leq |\hat{b}(x, \theta_b)| \leq b_{\max} < \infty$. Thus, the VSS adaptive fuzzy-based control law can be given by

$$
u = \hat{b}^{-1}(x)(-\hat{f}(x) - K_{\varepsilon}^T \varepsilon + x_n^{(m)} + u_h + u_s)
$$

(11)

where $K_{\varepsilon} = [K_{c_1} \ K_{c_2} \ \cdots \ K_{c_m}]^T$ is selected such that the eigenvalues of the matrix $A - BK_c^T$ all have a negative real part, and $u_h, u_s$ are compensators that are described later for uncertainties and disturbances, respectively.

3. $H_\infty$ fuzzy adaptive tracking control design using a two-layer delayed output feedback observer

The states of the system are not fully available for measurement, thus not all of the derivative signals are available for measurement. Therefore, a T–S fuzzy linear dynamic model is proposed that represents the local linear input/output relations as well as the state observers of the nonlinear system with delayed output, as described in (2). Let $\hat{x}$ be the estimate of the state $x$, $\hat{x}_j(t)$ be the estimate of the delayed state $x_j(t)$, $\hat{x}(t) = x(t - \Delta + \frac{m}{\delta}) = [x_1 \ x_2 \ \cdots \ x_n]^T$, $\hat{e} = \hat{x} - x$, the estimate of the error state $e$, and $\hat{e}_j(t) = \hat{x}_j(t) - x$ is the estimate of the delayed error state $e_j(t)$, $e_j(t) = e(t - \Delta + \frac{m}{\delta}) = [e_1 \ e_2 \ \cdots \ e_n]^T$. From (3) and (11), we have

$$
\dot{\hat{e}} = A\hat{e} + B(\hat{f}(x) - \hat{\hat{f}}(\hat{x}) - K_{\varepsilon}^T \hat{e}_j + (b(x) - \hat{\hat{b}}(\hat{x}))u + u_h + u_s + d)
$$

(12)

where $K_{\varepsilon} = [K_{c_1} \ K_{c_2} \ \cdots \ K_{c_m}]^T$, $j = 0, 1, \ldots, m$, is chosen such that the eigenvalues of the matrix $A - BK_c^T$ all have a negative real part. Define $[\eta_1 = [|\eta_1| \ |\eta_1| \ \cdots \ |\eta_1|]^T$, where $\eta_j = y(t) - \hat{\hat{C}}x_j(t - \frac{d}{\delta}j)$ denotes the delayed output error of the $j$th sub-observer. We use a two-layer fuzzy observer to reconstruct the states for systems with delayed output, where the output delayed states in $m + 1$ time slices partitioned with an equal time interval, $t - \Delta + \frac{m}{\delta}j$, $j = 0, 1, \ldots, m$, are used as premise variables in the IF–THEN rules of the first layer fuzzy observer. The second layer fuzzy observer that uses the delayed output error states as its linguistic variables is defuzzified from the first layer. Next, a fuzzy observer with $N_{\varepsilon_0} + N_{\varepsilon_1} + \cdots + N_{\varepsilon_m}$ IF–THEN rules is proposed to handle state estimation for the system with delayed output in (2), as follows:

$$
\begin{align*}
R_j^f: \text{IF } \hat{e}_{j_1} \text{ is } F_{j_1}^f \ \text{AND } \hat{e}_{j_2} \text{ is } F_{j_2}^f \ \text{AND } \cdots \ \text{AND } \hat{e}_{j_n} \text{ is } F_{j_n}^f \\
\text{THEN } \begin{cases} 
\hat{e}_j = \hat{A}\hat{e}_j - BK_c^T \hat{e}_j + K_{\varepsilon j} (y(t) - \hat{\hat{y}}(t)) \\
\hat{e}_{j_1}(t) = C\hat{e}_j
\end{cases}
\end{align*}
$$

(13)

where $\hat{e}_j = [\hat{e}_{j_1} \ \hat{e}_{j_2} \ \cdots \ \hat{e}_{j_n}]^T$ is the premise input variable of the $j$th fuzzy observer, $j = 0, 1, \ldots, m$, and $K_{\varepsilon j_1} = [K_{\varepsilon_{j_1}} \ K_{\varepsilon_{j_2}} \ \cdots \ K_{\varepsilon_{j_{N_{\varepsilon_j}}}}]^T$ are the constant matrices for the $j$th fuzzy observer that needs to be designed, $\hat{\hat{y}}(t) = \hat{\hat{C}}x_j(t) \in \mathbb{R}$, and $\hat{\hat{e}}_j(t) \in \mathbb{R}$ is the output of the $j$th fuzzy observer, $j = 0, 1, \ldots, m$. Using the singleton fuzzifier, product inference engine, and the center average defuzzifier, the fuzzy observer in the first layer is given by
\[
\dot{e}_j = \frac{1}{\sum_{\ell_j=1}^{N_{\ell_j}} \prod_{i=1}^n F^{\ell_j}_{hi}(\hat{e}_j)} \left( \sum_{\ell_j=1}^{N_{\ell_j}} \prod_{i=1}^n F^{\ell_j}_{hi}(\hat{e}_j) (A\hat{e}_j - BK^T_{\ell} \hat{e}_j + K_{o_{\ell j}} (y(t) - \hat{y}_j(t))) \right) \\
= \sum_{\ell_j=1}^{N_{\ell_j}} v^{\ell_j}(\hat{e}_j) (A\hat{e}_j - BK^T_{\ell} \hat{e}_j + K_{o_{\ell j}} (y(t) - \hat{y}_j(t))) 
\]

where

\[
v^{\ell_j}(\hat{e}_j) = \frac{\prod_{i=1}^n F^{\ell_j}_{hi}(\hat{e}_j)}{\sum_{\ell_j=1}^{N_{\ell_j}} \prod_{i=1}^n F^{\ell_j}_{hi}(\hat{e}_j)}
\]

for which \(F^{\ell_j}_{hi}(\hat{e}_j)\) is the grade of membership of \(\hat{e}_j\) in \(F^{\ell_j}_{hi}\). Without any loss of generality, we can assume that \(\sum_{\ell_j=1}^{N_{\ell_j}} v^{\ell_j}(\hat{e}_j) = 1, j = 0, 1, \ldots, m \).

For the second layer, a fuzzy observer with \(m + 1\) IF–THEN rules is proposed to handle state estimation for the system with delayed output in (2), which uses the absolute value of the delayed output error state as its linguistic variable, as follows:

\[
R_j: \text{IF } |\eta_j| \text{ is } G_j \text{ THEN} \\
\hat{e} = \sum_{\ell_j=1}^{N_{\ell_j}} v^{\ell_j}(\hat{e}_j) (A\hat{e}_j - BK^T_{\ell} \hat{e}_j + K_{o_{\ell j}} (y(t) - \hat{y}_j(t))) 
\]

for \(j = 0, 1, \ldots, m \). Using the singleton fuzzifier, product inference engine, and the center average defuzzifier, the fuzzy observer in the second layer can be expressed as follows:

\[
\dot{\hat{e}} = \frac{1}{\sum_{j=0}^{m} G_j(|\eta_j|)} \left( \sum_{j=0}^{m} G_j(|\eta_j|) \left( \sum_{\ell_j=1}^{N_{\ell_j}} v^{\ell_j}(\hat{e}_j) (A\hat{e}_j - BK^T_{\ell} \hat{e}_j + K_{o_{\ell j}} (y(t) - \hat{y}_j(t))) \right) \right) \\
= \sum_{j=0}^{m} w_j(|\eta_j|) v^{\ell_j}(\hat{e}_j) (A\hat{e}_j - BK^T_{\ell} \hat{e}_j + K_{o_{\ell j}} (y(t) - \hat{y}_j(t))) 
\]

where

\[
w_j(|\eta_j|) = \frac{G_j(|\eta_j|)}{\sum_{j=0}^{m} G_j(|\eta_j|)}
\]

where \(G_j(|\eta_j|)\) is the grade of membership of \(|\eta_j|\) in \(G_j\). Without any loss of generality, we can assume that \(\sum_{j=0}^{m} w_j(|\eta_j|) = 1 \).

The control law for the \(j\)th fuzzy rule is given by:

\[
u_j = (-\theta^T_{f j} \xi_j(\hat{x}_j) - K^T_{c j} \hat{e}_j + x^{(n)}_c + u_{bj} + u_{sj}) / (\theta^T_{b j} \xi_b(\hat{x}_j)) \quad (15)
\]

Thus, the fuzzy rules for the controller can be given by

\[
R_j: \text{IF } |\eta_j| \text{ is } G_j \text{ THEN } u = u_j
\]

and the certainty equivalent fuzzy controller (11) can be rewritten as follows

\[
u = \sum_{j=0}^{m} w_j(|\eta_j|) \frac{-\theta^T_{f j} \xi_j(\hat{x}_j) - K^T_{c j} \hat{e}_j + x^{(n)}_c + u_{bj} + u_{sj}}{\theta^T_{b j} \xi_b(\hat{x}_j)} \quad (16)
\]

Therefore, the closed-loop dynamic system (12) can be rewritten as follows:
\[
\dot{e} = \sum_{j=0}^{m} w_j(|\eta|)(Ae + B(f(x) - f(\hat{x}_j)) - K_j^T \dot{e}_j + (b(x) - \hat{b}(\hat{x}_j))u_j + u_{hj} + u_{sj} + d))
\]  

(17)

Let the observation error be defined as \( \tilde{e} \triangleq e - \hat{e} = x - x_r - (\hat{x} - \hat{x}_j) = x - \hat{x} \), and \( \dot{e}_j \triangleq e - \hat{e}_j = x - \hat{x}_j \). Then, from (14) and (17), we have

\[
\hat{e} = \sum_{j=0}^{m} \sum_{i=1}^{N_{ej}} w_j(|\eta|)v^{\ell}(\hat{e}_j)(A\hat{e}_j + B(f(x) - f_j(\hat{x}_j) + (b(x) - \hat{b}_j(\hat{x}_j))u_j + u_{hj} + u_{sj} + d)
\]

where \( \hat{e} \equiv e - \hat{e} \), and \( \hat{e}_j \equiv e - \hat{e}_j \). Then, the filtered augmented system vector denoted by \( \hat{e} \) is given by \[39\]

\[
\hat{e} = \sum_{j=0}^{m} \sum_{i=1}^{N_{ej}} w_j(|\eta|)v^{\ell}(\hat{e}_j)((A - K_{o(j)}C)\hat{e}_j + B(\theta_j^T \xi_j(\hat{x}_j) + \theta_b^T \xi_b(\hat{x}_j)u_j) + B(\theta_j^T \xi_j + \theta_b^T \xi_b)u_j + \delta_j + \delta_{\hat{b}} u_j + u_{hj} + u_{sj} + d) + K_{o(j)}C\bar{x} - K_{o(j)}C(t - \Delta))
\]

where \( \theta_j^T = \theta_j^T - \theta_j^T, \delta_j = \theta_j^T - \theta_j^T, \xi_j = \xi_j(x) - \xi_j(\hat{x}_j), \) and \( \xi_b = \xi_b(x) - \xi_b(\hat{x}_j). \)

Hence, from (14) and (18), we have the following augmented system:

\[
\begin{align*}
\hat{e}_a &= \sum_{j=0}^{m} N_{ej} w_j(|\eta|)v^{\ell}(\hat{e}_j)(A_{a(j)} \hat{e}_a + \bar{A}_{a(j)} \bar{x}_{a\Delta} + B_a(\theta_j^T \xi_j \hat{x}_j) \\
\bar{y}_a &= C_a \hat{e}_a
\end{align*}
\]

where

\[
\begin{align*}
\hat{e}_a &= \begin{bmatrix} \hat{e} \\ \hat{e}_j \end{bmatrix}, \quad \hat{e}_a &= \begin{bmatrix} \hat{e}_j \\ \hat{e}_j \end{bmatrix}, \quad \bar{x}_{a\Delta} &= \begin{bmatrix} x(t - \Delta) \\ x \end{bmatrix}, \quad A_{a(j)} = \begin{bmatrix} A - BK_j^T & K_{o(j)}C \\ 0_{n \times n} & A + K_{o(j)}C \end{bmatrix}, \\
B_a &= \begin{bmatrix} 0_{n \times 1} \\ B \end{bmatrix}, \quad \bar{A}_{a(j)} = \begin{bmatrix} K_{o(j)}C & -K_{o(j)}C \\ 0_{n \times n} & K_{o(j)}C \end{bmatrix}, \\
d_a &= \Delta F_j + \Delta B_j(-\bar{f}_j(\hat{x}_j) - K_j^T \hat{e}_j + x_{(n)} + u_{hj} + u_{sj}) + d,
\end{align*}
\]

\( \bar{y}_a \) is the output of the augmented system, and \( C_a = \text{diag}(C \ C) \), where \( \Delta F_j = \theta_j^T \xi_j + \delta_j \), and \( \Delta B_j = (\theta_b^T \xi_b + \delta_{\hat{b}}) \hat{\delta}_j^{-1}(\hat{x}_j) = \omega_b \hat{\delta}_j^{-1}(\hat{x}_j) \) are lumped uncertainties. It is assumed that the lumped uncertainties are bounded, that their bounds need to be known in advance, and that they satisfy \(|\Delta F_j| \leq \kappa_f \) and \(|\Delta B_j| \leq \kappa_b = b^{-1} \sum_{i \geq 0} |\omega_{bj}| \), where \( \kappa_f \) and \( \kappa_b \) are given positive constants. The system has delayed output and not all of the states are available for measurement, thus it is not possible to obtain all of the elements of \( x \) and \( e \). Hence, the augmented system should be filtered by stable filters and the filtered signals can then be used to update the fuzzy parameters. Let the filter denoted by \( L(s) \) be given by [39]

\[
L(s) = (\frac{\omega_s}{s^2 + \omega_s^2})^{2n}
\]

(20)

where \( s \) denotes the complex Laplace transfer variable, \( \omega > 0 \) is a constant that needs to be determined, and \( 2n \) is the dimension of the augmented system vector. Define a set of state variable filters \( \Omega_i(s) \) by

\[
\Omega_i(s) = L(s)x_i, \quad i = 0, 1, 2, \ldots, 2n - 1
\]

(21)

Then, the filtered augmented system vector denoted by \( e_a = \Omega_i(s)[\hat{e}_a] = [\hat{e}_F \ \hat{e}_F]^T, \quad i = 0, 1, 2, \ldots, 2n - 1, \) and \( e_a = [e_{a_0} \ e_{a_1} \ \cdots \ e_{a_{2n-1}}]^T, \) is given by
Assumption 2. The external disturbance \( d(t) \) is assumed to satisfy
\[
|d(t)| \leq \zeta_d
\]
where \( \zeta_d \) is a finite positive constant.

**Theorem 1.** Consider system (2) with uncertain nonlinear functions \( f(x) \), \( b(x) \), the two-layer fuzzy observers (13), and (14), and let Assumptions 1 and 2 be true. If the variable structure adaptive fuzzy-based control law is given as in (16) with
\[
\begin{align*}
    u_{hj} &= -\frac{1}{r(1+\kappa_{bj})} \mathbf{e}^T_{aj} \mathbf{PB}_a \\
    u_{sj} &= -\frac{2\alpha_j}{1+\kappa_{bj}} \text{sgn}(\mathbf{B}_a^T \mathbf{Pe}_{aj})
\end{align*}
\]
and the adaptive control laws:
\[
\begin{align*}
    \dot{\theta}_{fj} &= -\gamma_f \xi_j(x_j) \mathbf{B}_a^T \mathbf{Pe}_{aj} \\
    \dot{\theta}_{bj} &= -\gamma_b \xi_b(x_j) \mathbf{B}_a^T \mathbf{Pe}_{aj} u_j
\end{align*}
\]
where \( x_{a,j} = [x_F(t-\Delta) x_F(t)]^T \) and \( y_a \) are the filtered outputs of \( \tilde{x}_{a,j} \) and \( \tilde{y}_a \), respectively, and
\[
\begin{align*}
    z_j &= \Omega_{0}(s) \Delta G_j - \Delta G_j \\
    &= \Omega_{0}(s) \left\{ \theta_{fj}^T \xi_j(x_j) + \tilde{\theta}_{bj}^T \xi_b(x_j) u_j + \theta_{fj}^T \xi_f u_j + \delta_j^f + \theta_{bj}^T \xi_b u_j + \delta_j^b u_j + d \\
    &\quad + u_{hj} + u_{sj} \right\} - \left\{ \theta_{fj}^T \xi_j(x_j) + \tilde{\theta}_{bj}^T \xi_b(x_j) u_j + \theta_{fj}^T \xi_f u_j + \delta_j^f + \theta_{bj}^T \xi_b u_j + \delta_j^b u_j + d + u_{hj} + u_{sj} \right\}
\end{align*}
\]
Note that although the right-hand side of (23) contains unknown parameters, we can obtain the vector \( \mathbf{e}_a \) based on the available filtered output \( y_a \). It is assumed that the filtered signals are bounded and that they satisfy \( |z_j| \leq \kappa_{zj} \), where \( \kappa_{zj} \) is a given positive constant. Therefore, \( \mathbf{e}_a \) is available and it can be used for further parameter estimation and control design.

Control objectives: A fuzzy adaptive control scheme is used to eliminate the uncertainties \( f(x) \) and \( b(x) \) as fully as possible, and a two-layer fuzzy observer with the injection of the delayed outputs of the system is utilized to reconstruct the states of the system such that the following conditions are satisfied:

(i) For any \( t \geq t_0, \tilde{e}_{aj}, \mathbf{e}_{aj}, \tilde{\theta}_{fj}(t), \tilde{\theta}_{bj}(t), \forall j \), are bounded and \( \mathbf{e}_{aj} \) is UUB.

(ii) The \( H_{\infty} \) tracking performance of the overall system satisfies the following relationship:
\[
J = \sum_{j=0}^{m} w_{j}(|\eta|) \int_{0}^{T} \mathbf{e}_{aj}^T \mathbf{Q} \mathbf{e}_{aj} dt \leq \mathbf{e}_{a}^T(0) \mathbf{P} \mathbf{e}_{a}(0) + \frac{1}{\gamma_f} \int_{0}^{T} \tilde{\theta}_{fj}^T(0) \tilde{\theta}_{fj}(0) dt + \frac{1}{\gamma_b} \int_{0}^{T} \tilde{\theta}_{bj}^T(0) \tilde{\theta}_{bj}(0) dt + \rho^2 \int_{-\Delta}^{0} \mathbf{e}_{a}^T(\varsigma) \mathbf{e}_{a}(\varsigma) d\varsigma \\
+ \rho^2 \int_{0}^{T} (\mathbf{x}_{f,\Delta}^T \mathbf{x}_{f,\Delta}) dt
\]
\]
To derive the control laws, the following assumptions must hold for all \( \dot{x} \in U_c, \theta_{fj} \in \mathbb{R}^{N_F} \) and \( \theta_{bj} \in \mathbb{R}^{N_B} \).

**Assumption 2.** The external disturbance \( d(t) \) is assumed to satisfy
\[
|d(t)| \leq \zeta_d
\]
where \( \zeta_d \) is a finite positive constant.
where \( \alpha_j \triangleq \kappa_{e_j} + \kappa_{f_j} + \kappa_{\theta_j} (| - \dot{f}_j(\hat{x}_j) - K_{\ell_j}^T \hat{e}_j + x_{j(n)}|) \), \( r \) is a positive constant that is determined by the designer and \( \gamma_f, \gamma_{\theta} \) are the adaptation rates. Furthermore, there exists a symmetric positive-definite matrix \( P = P^T > 0 \), which is the solution of the following LMI

\[
\begin{align*}
A_{\alpha_{jl}}^T P + PA_{\alpha_{jl}} &= \left(1 + \frac{2}{\phi^2} + \frac{1}{\rho^2}\right) P A_{\alpha_{jl}} \hat{A}_{\alpha_{jl}}^T P + \left(1 - \frac{2}{\rho^2} - \frac{2}{r}\right) P B_d B_d^T P + \tilde{Q} < 0
\end{align*}
\]

(28)

where \( \phi \) is a positive constant that is determined by the designer, \( \rho \) is a prescribed attenuation level, and \( \tilde{Q} = 2\phi^2 I + Q, Q > 0 \), where \( I \) is the identity matrix. Then, the \( H_\infty \) tracking performance can be achieved for a prescribed attenuation level.

**Proof.** See Appendix A. \( \square \)

Based on the analysis given above, determining a common solution \( P = P^T > 0 \) for (28) is the most important issue for the fuzzy observer-based tracking control problem. To simplify the design problem, a decoupling technique is used to solve \( P \), as follows:

\[
P = \begin{bmatrix}
P_1 & 0 \\
0 & P_2
\end{bmatrix}
\]

(29)

where \( P_1, P_2 > 0 \), and \( \tilde{Q} \) in (28) is given by

\[
\tilde{Q} = \begin{bmatrix}
\tilde{Q}_1 & 0 \\
0 & \tilde{Q}_2
\end{bmatrix}
\]

(30)

where \( \tilde{Q}_1, \tilde{Q}_2 > 0 \). This choice is suitable for the separate designs of a fuzzy controller and observer, as described later. By substituting (29) and (30) into (28), we obtain

\[
\begin{align*}
R_{11j} &= (A - BK_{\ell_j})^T P_1 + P_1 (A - BK_{\ell_j}) + \left(2 + \frac{4}{\phi^2} + \frac{2}{\rho^2}\right) P_1 K_{\alpha_{jl}} C (K_{\alpha_{jl}} C)^T P_1 + \tilde{Q}_1 \\
R_{22j} &= (A - K_{\alpha_{jl}} C)^T P_2 + P_2 (A - K_{\alpha_{jl}} C) + \left(2 + \frac{4}{\phi^2} + \frac{2}{\rho^2}\right) P_2 K_{\alpha_{jl}} C (K_{\alpha_{jl}} C)^T P_2 \\
&+ \left(1 - \frac{2}{\rho^2} - \frac{2}{r}\right) P_2 B_d B_d^T P_2 + \tilde{Q}_2
\end{align*}
\]

Let \( T_1 = P_1^{-1} \) and multiply it into (31) with \( Y_{1j} = K_{\ell_j}^T T_1 \), then by using the Schur complements [25] and \( Y_{2\alpha_{jl}} = P_2 K_{\alpha_{jl}} \), the matrix inequalities (31) can be represented in the following form

\[
\begin{bmatrix}
\Pi_{11j} & \Pi_{12j} & 0 & 0 \\
* & \Pi_{22j} & Y_{2\alpha_{jl}} & 0 \\
* & * & \Pi_{33} & P_1 K_{\alpha_{jl}} C \\
* & * & * & \Pi_{44}
\end{bmatrix} < 0
\]

(32)

for \( j = 0, 1, \ldots, m \), where

\[
\begin{align*}
\Pi_{11j} &= T_1 A^T + A T_1 - Y_{1j}^T B^T - BY_{1j} + T_1 \tilde{Q}_j T_1 \\
\Pi_{22j} &= A^T P_2 + P_2 A - C^T Y_{2\alpha_{jl}}^T - Y_{2\alpha_{jl}} C + \left(1 - \frac{2}{\rho^2} - \frac{2}{r}\right) P_2 B_d B_d^T P_2 + \tilde{Q}_2 \\
\Pi_{33} &= \Pi_{44} = \left(-\left(2 + \frac{4}{\phi^2} + \frac{2}{\rho^2}\right) I\right)^{-1} \\
\Pi_{12j} &= K_{\alpha_{jl}} C - \left(2 + \frac{4}{\phi^2} + \frac{2}{\rho^2}\right) P_1 K_{\alpha_{jl}} C C^T Y_{2\alpha_{jl}}^T
\end{align*}
\]
Since the four parameters $P_1$, $P_2$, $K_a$, and $K_{o_f}$ determined from (32) cannot be solved simultaneously, the control and observer problems should be decoupled and solved separately using the following two-step method by selecting (29) and (30). In the first step, (32) implies that

$$
\Pi_{22j} = A^T P_2 + P_2 A - C^T Y_{2o_{j1}} - Y_{2o_{j1}} C + \left( \frac{1}{\rho^2} - \frac{2}{r} \right) P_2 B B^T P_2 + \tilde{Q}_2 < 0
$$

(33)

By solving (33), the parameters $P_2$ and $Y_{2o_{j1}}$ can be obtained for a prescribed attenuation level $\rho$. In the second step, after substituting these known parameters $P_2$ and $Y_{2o_{j1}}$ (thus $K_{o_{f1}} = P_2^{-1} Y_{2o_{j1}}$) into (32), (32) becomes a convex optimization problem. Similarly, the parameters $T_1$ and $Y_{1j}$ (thus $P_1 = T_1^{-1}$ and $K^j = Y_{1j} T_1^{-1}$) can be solved from (32). By minimizing the attenuation level $\rho^2$ such that the $H_\infty$ tracking performance in (28) is reduced as far as possible,

$$
\min_{\{P_1, P_2\}} \rho^2
$$

subject to $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$ and (32)

(34)

Remark 1. To constrain the estimations $\theta_{f_j}, \theta_{b_j}$ within the sets $\mathbb{R}^N_\rho$ and $\Omega_{\rho}$ for the true parameters $\theta^*_f$ and $\theta^*_b$, respectively, we need to modify the adaptive laws in (27) using the parameters projection algorithm [37], as follows:

$$
\dot{\theta}_{f_j} = \begin{cases} 
-\gamma_f \xi^*_f (\hat{x}_j) B^T_a P e_{a_j}, & \text{if } (\| \theta_{f_j} \| < M_{\theta_{f_j}}) \text{ or } (\| \theta_{f_j} \| = M_{\theta_{f_j}} \text{ and } e_{a_j} P B_{a} \xi^*_f (\hat{x}_j) \theta_{f_j} \geq 0) \\
\text{Proj}[-\gamma_f \xi^*_f (\hat{x}_j) B^T_a P e_{a_j}], & \text{if } (\| \theta_{f_j} \| = M_{\theta_{f_j}} \text{ and } e_{a_j} P B_{a} \xi^*_f (\hat{x}_j) \theta_{f_j} < 0) 
\end{cases}
$$

and

$$
\dot{\theta}_{b_j} = \begin{cases} 
-\gamma_b \xi^*_b (\hat{x}_j) B^T_a P e_{a_j} u_j, & \text{if } (\| \theta_{b_j} \| < M_{\theta_{b_j}}) \text{ or } (\| \theta_{b_j} \| = M_{\theta_{b_j}} \text{ and } e_{a_j} P B_{a} \xi^*_b (\hat{x}_j) \theta_{b_j} u_j \geq 0) \\
\text{Proj}[-\gamma_b \xi^*_b (\hat{x}_j) B^T_a P e_{a_j} u_j], & \text{if } (\| \theta_{b_j} \| = M_{\theta_{b_j}} \text{ and } e_{a_j} P B_{a} \xi^*_b (\hat{x}_j) \theta_{b_j} u_j < 0) 
\end{cases}
$$

where the projection operator $\text{Proj}[-\gamma_f \xi^*_f (\hat{x}_j) B^T_a P e_{a_j}]$ is defined as

$$
\text{Proj}[-\gamma_f \xi^*_f (\hat{x}_j) B^T_a P e_{a_j}] = -\gamma_f \xi^*_f (\hat{x}_j) B^T_a P e_{a_j} + \gamma_f e_{a_j} P B_{a} \frac{\theta_{f_j}^T \xi^*_f (\hat{x}_j)}{\| \theta_{f_j} \|^2}
$$

and

$$
\text{Proj}[-\gamma_b \xi^*_b (\hat{x}_j) B^T_a P e_{a_j} u_j] = -\gamma_b \xi^*_b (\hat{x}_j) B^T_a P e_{a_j} u_j + \gamma_b e_{a_j} P B_{a} \frac{\theta_{b_j}^T \xi^*_b (\hat{x}_j) u_j}{\| \theta_{b_j} \|^2}
$$

Remark 2. The weighting matrices $2\rho^2 I + Q$ are replaced by $\tilde{Q}$ for the LMI in (28). The positive definite solution of $P$ can be found easily using MATLAB’s LMI toolbox. However, this reduces the capacity to overcome uncertainties in the control scheme.

Remark 3. To obtain the positive definite solution of $P$ in (28), first the matrix inequalities (32) can be obtained by decoupling $P = \text{diag}(P_1, P_2)$. Next, if a prescribed attenuation level $\rho$ and the weighting matrices $\tilde{Q}$ are given in advance, the parametric matrices $P_2$ and $Y_{2o_{j1}}$ can be solved from (33). By substituting $P_2$ and $Y_{2o_{j1}}$ into (32) to obtain $P_1$ and $Y_{1j}$, we can then solve for the positive definite matrix $P$.

Remark 4. Since the attenuation level $\rho^2$ can be obtained from (34) in the LMI procedure and it satisfies

$$
\frac{1}{\rho^2} - \frac{2}{r} \leq 0 \quad \text{or} \quad \rho^2 \geq \frac{r}{2}
$$

the errors, including the estimation and tracking errors of the closed-loop system with delayed output and disturbances, can be decreased in an effective manner to achieve $H_\infty$ tracking performance.
Remark 5. In the design of the two-layer fuzzy observer, the traditional T–S fuzzy observers are used for each delayed time slice in the first layer such that the system states can be reconstructed. In the second fuzzy observer layer, the delayed output error states are selected as the linguistic variables and its output is defuzzified from the \( m + 1 \) decentralized fuzzy sub-observers in the first layer.

Remark 6. The lumped uncertainty \( \delta b_j \hat{b}_j^{-1}(\hat{x}_j) \) is assumed to be bounded and it satisfies \( 0 < |\delta b_j \hat{b}_j^{-1}(\hat{x}_j)| < 1 \), where \( \delta b_j \hat{b}_j^{-1}(\hat{x}_j) \) can be selected independently as \( \delta b_j \hat{b}_j^{-1}(\hat{x}_j) \leq \epsilon b_j / M \theta b_j \), and where \( \epsilon b_j = \max |\Delta \theta b_j| \) is the maximal perturbation of \( \theta b_j \) since \( \xi b(\hat{x}_j) \in [0, 1] \) and \( \sum^n i \xi b(\hat{x}_j) = 1 \) [38]. It can be seen that \( \epsilon b_j \) is also the maximal perturbation of \( \hat{b}(\hat{x}_j) \), i.e., \( \epsilon b_j = \max |\Delta \hat{b}(\hat{x}_j)| \), and it should be less than max \( |\hat{b}(\hat{x}_j)| \).

From the above analysis, the block diagram of the overall two-layer fuzzy adaptive observer control system is shown in Fig. 1. Hence, to summarize the above analysis, the design procedure of the proposed control scheme is delineated as follows:

**Step 1:** Give the delay time \( \Delta \) and partition it into \( m + 1 \) time slices with equal intervals.

**Step 2:** Compute \( \kappa b_j \) and select \( \delta b_j, f_j, g_j, \kappa f_j, \kappa c_j \), and the initial conditions \( x_0, x_1, \hat{e}_j, \hat{\theta} f_j, \hat{\theta} b_j, j = 0, 1, \ldots, m \).

**Step 3:** Give fuzzy membership functions for the unknown nonlinear functions of the plant and select appropriate matrices \( Q_1, Q_2 \), and the values \( \phi, r, \rho \).

**Step 4:** Solve (33) to obtain \( P_2 \) and \( Y_{2aj} = P_2 Y_{2a} \).
We considered a system with output delay times \( \Delta \) and \( g \) to control the nonlinear dynamic system. The magnetic levitation system [36] shown in Fig. 2 is used to illustrate the performance and efficiency of the proposed approach. Let the reference signals be \( x_r \) and \( y_r \). The magnetic levitation system is given by:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= f(x) + b(x)u + d(t) \\
y(t) &= x_1(t - \Delta)
\end{align*}
\]

where \( x_1(t) \) is the height of the magnet above the electromagnet, \( x_2(t) \) is the velocity of the magnet, \( f(x) = -g - \frac{\beta}{M}x_2 \), \( b(x) = -\frac{d}{M} \), and the external disturbance \( d(t) = 0.1 \sin(t) \), where \( M = 3 \) kg is the mass of the magnet, \( g = 9.81 \) m/s\(^2 \) is the gravitational constant, \( \alpha = 15 \) is the viscous friction coefficient, which is determined by the medium where the magnet moves, \( \beta = 12 \) is the field strength constant, which is determined by the number of turns of the wire on the electromagnet and the strength of the magnet, and the variable ranges are \( x_1 \in [0, 5] \) and \( x_2 \in [-5, 10] \).

We selected the same filter, \( L(s) = 25/(s + 5)^2 \) [28], for both \( \hat{e}_j \) and \( \hat{\hat{e}}_j \).

Let the reference signals be \( x_{r1} = 2 + 0.5 \sin(0.5t) \) and take \( \dot{x} \) as the premise variable since \( x \) is not available. For comparison, we considered a system with output delay times \( \Delta = 0.2 \) s and \( 0.3 \) s to analyze the trajectory tracking performance of the closed-loop system.

**Step 1:** The number of sub-observers was set as 2 with \( m = 1 \) and as 3 with \( m = 2 \) for a system with output delay times of \( \Delta = 0.2 \) s and \( 0.3 \) s, respectively. The computational complexity as a function of \( m \) is shown in Table 1.

**Step 2:** If we require that \( 0 \leq x_1 \leq 5 \), then the bound \( b_{\text{min}}^{-1} \) can be computed \( b_{\text{min}}^{-1} = 1 \) and we selected

\[
(\sup_{t \geq 0} |\omega_{b_j}|) = 2, \quad \delta = 0.01, \quad \gamma_f = 100, \quad \gamma_b = 0.01, \quad \kappa_f = 0.1, \quad \kappa_z = 0.05, \quad \text{and the initial conditions } x_1(0) = 1, \quad x_2(0) = 0.1, \quad \hat{e}_{j1}(0) = \hat{e}_{j2}(0) = 0.2, \quad \hat{\theta}_j(0) = 3I_{25 \times 1}, \quad \dot{\theta}_b(0) = 3I_{25 \times 1}, \quad j = 0, 1, 2.
\]
Table 1
Computational complexity as a function of 
m
for nonlinear systems.

<table>
<thead>
<tr>
<th>Value of ( m )</th>
<th>( m = 1 )</th>
<th>( m = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of ( j )</td>
<td>0 and 1</td>
<td>0, 1 and 2</td>
</tr>
<tr>
<td>Number of sub-observers</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Number of T–S sub-observers</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Time instants of observer</td>
<td>( t - \Delta, t - \Delta ), and ( t )</td>
<td>( t - \Delta, t - \Delta ), and ( t )</td>
</tr>
<tr>
<td>Fuzzy controller rule</td>
<td>( R_0 ) and ( R_1 )</td>
<td>( R_0, R_1, ) and ( R_2 )</td>
</tr>
<tr>
<td>Number of controller gains needed from LMI inequality</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Number of T–S sub-observer gains needed from LMI inequality</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Fig. 3. Membership function for \( \hat{x}_{j1}, j = 0, 1, 2 \).

Fig. 4. Membership function for \( \hat{x}_{j2}, j = 0, 1, 2 \).

**Step 3:** To approximate the unknown nonlinear functions \( f(x) \) and \( b(x) \), we used the fuzzy membership functions for \( \hat{x}_{ji}, j = 0, 1, 2, i = 1, 2 \), as shown in Figs. 3 and 4 and the fuzzy membership functions for system states \( \hat{e}_{j1}, j = 0, 1 (m = 1) \) and \( j = 0, 1, 2 (m = 2) \), are given in Fig. 5. The two-layer T–S fuzzy observers and the fuzzy set \( G_j \) as shown in Fig. 6 used to reconstruct the system states are given by:

**Observer Rule \( R_j \):**

IF \( |\eta_j(t)| \) is \( G_j \) THEN \( \dot{\hat{e}}(t) = \hat{e}(t) \)

\[ R_j^{\ell_j} : \text{IF } \hat{e}_{ji}(t) \text{ is } F_{ji}^{\ell_j} \text{ THEN } \dot{\hat{e}}_j(t) = A\hat{e}_j - BK_j^\ell_j \hat{e}_j(t) + K_{ji}\ell_j(y(t) - \hat{y}_j(t)) \]

\( j = 0, 1, 2, \ell_j = 1, 2, i = 1, 2 \). Moreover, the defuzzification of the switching observer is described as follows:

\[
\dot{\hat{e}}(t) = \begin{cases} 
\hat{e}_0(t), & G_0(|\eta_0|) \geq G_1(|\eta_1|) \land G_0(|\eta_0|) \geq G_2(|\eta_2|), \\
\hat{e}_1(t), & G_1(|\eta_1|) \geq G_0(|\eta_0|) \land G_1(|\eta_1|) \geq G_2(|\eta_2|), \\
\hat{e}_2(t), & \text{otherwise}
\end{cases}
\]
Fig. 5. Membership function for $\hat{e}_{j1}$ for the first layer of the two-layer fuzzy observer.

Fig. 6. Membership function for $\eta_j$, $j = 0, 1, 2$, for the second layer of the two-layer fuzzy observer.

Let $\tilde{Q}_1 = 0.0001 \times I_{2 \times 2}$, $\tilde{Q}_2 = 0.0001 \times I_{2 \times 2}$, $j = 0, 1, 2$, for $m = 1$ and $m = 2$, and select $\phi = 1$, $r = 0.01$, and $\rho^2 = 0.9$.

**Step 4:** From (32), we obtained $P_2$, for $m = 1$ and $m = 2$, $x_{r1} = 2 + 0.5 \sin(0.5t)$ as

$$P_2 = \begin{bmatrix} 5.6913 & 0.0186 \\ 0.0186 & 5.1762 \end{bmatrix}$$

and the observer gain vectors $K_{o j \ell j}$, $j = 0, 1$, $\ell j = 1, 2$, for $m = 1$ and $m = 2$, were obtained using

$$K_{o01} = [0.6186 \ 0.0166]^T, \ K_{o02} = [0.6805 \ 0.0182]^T,$$
$$K_{o11} = [0.7427 \ 0.0199]^T, \ K_{o12} = [0.8042 \ 0.0216]^T,$$
$$K_{o21} = [0.8661 \ 0.0232]^T, \ K_{o22} = [0.9279 \ 0.0249]^T.$$

**Step 5:** From (32), we obtain $P_1$, for $m = 1$ and $m = 2$, $x_{r1} = 2 + 0.5 \sin(0.5t)$ as

$$P_1 = \begin{bmatrix} 0.0040 & 0.0026 \\ 0.0026 & 0.0040 \end{bmatrix}$$

and the feedback control gain vectors $K_j^i$, $j = 0, 1, 2$, for $m = 1$ and $m = 2$, were obtained using

$$K_0^1 = [2.0630 \ 2.1395]^T, \ K_1^1 = [2.2693 \ 2.3534]^T, \ K_2^1 = [2.2464 \ 2.9782]^T$$

**Step 6:** The minimal attenuation level $\rho^2$ was determined as 0.65 using MATLAB’s LMI toolbox.

**Step 7:** By substituting $P = \text{diag}(P_1, P_2)$ and $\rho^2$ back into (28), the matrix inequality holds.

**Step 8:** Obtain the observer-based VSS indirect adaptive fuzzy $H\infty$ tracking control law from (16). Then, the controller rules are as follows:
Then, compute the adaptive laws (27) to adjust the parameter vectors \( \Delta \) for

\[
\Delta = \frac{-\hat{f}_j(\hat{x}_j) - K_j^T \hat{e}_j + u_{b,j} + u_{s,j}}{b_j(\hat{x}_j)},
\]

where the fuzzy rules \( R_j \) are used for \( m = 1 \) and \( R_0, R_1, R_2 \) for \( m = 2 \), respectively, and for (25) and (26), \( u_{b,j} = -\frac{1}{0.01(1+2)} \epsilon_{a,j}^T \mathbf{P} \epsilon_{a,j} \), and \( u_{s,j} = -\frac{2\alpha_j}{1+2} \mathbf{P} \mathbf{e}_{a,j} \mathbf{e}_{a,j}^T \), where \( \alpha_j = 0.1 + 0.05 + 2(|\hat{f}_j(\hat{x}) + \hat{e}_j| - K_j^T \hat{e}_j) \). Then, compute the adaptive laws (27) to adjust the parameter vectors \( \theta_j \), \( \theta_{b,j} \), \( j = 0, 1, 2 \).

The responses of the tracking performance using two fuzzy controllers with two fuzzy sub-observers, i.e., two fuzzy observer rules \( R_0 \) and \( R_1 \) and two fuzzy controller rules \( R_0 \) and \( R_1 \) for \( m = 1 \), to estimate the true states when the system output had a delay time of \( \Delta = 0.2 \) s are shown in Figs. 7–11. The responses of the actual state \( x_1 \), estimated state \( \hat{x}_1 \), and desired state \( \hat{x}_2 \) are shown in Fig. 7. It can be seen that \( \hat{x}_1 \) approached \( \hat{x}_1 \) and \( \hat{x}_1 \) reached \( x_1 \) in less than 7 s. Fig. 8 shows the trajectories of the actual state \( x_2 \), estimated state \( \hat{x}_2 \), and desired state \( \hat{x}_2 \). Based on Fig. 8, it is clear that \( \hat{x}_2 \) approached \( \hat{x}_2 \) and \( \hat{x}_2 \) approached \( x_2 \) in less than 8 s. The responses of the error in the distance and the velocity states were both less than 0.1 after 6 s, as shown in Figs. 9 and 10, respectively. Fig. 11 shows the trajectories of the control input. It should be mentioned that the response of the estimated state \( \hat{x}_1 \) did not reach the true state \( x_1 \) for \( \Delta = 0.3 \) s and \( m = 1 \) (see Fig. 12). However, when \( m = 2 \), i.e., three fuzzy observer rules \( R_0, R_1, R_2 \) and three fuzzy controller rules \( R_0, R_1, R_2 \) were used for \( \Delta = 0.3 \) s, the output response of the estimated state \( \hat{x}_1 \) converged to the actual state \( x_1 \) satisfactorily and the maximum steady state estimation error was around 2.8%, as shown in Fig. 13.
Fig. 9. Trajectories of the error state of the distance for a delay time of 0.2 s with $m = 1$.

Fig. 10. Trajectories of the error state of the velocity for a delay time of 0.2 s with $m = 1$.

Fig. 11. Trajectory of the control input $u(t)$ for a delay time of 0.2 s with $m = 1$.

Fig. 12. Trajectories of $x_1(t)$, $\hat{x}_1(t)$, and $\tilde{x}_1(t)$ for a delay time of 0.3 s with $m = 1$. 

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Figs. 13 and 14 show that the true states $x_1$ and $x_2$ approach the desired reference signal at about 6 s for $\Delta = 0.3$ s with $m = 2$. The responses of the error state of the distance and the velocity were both in the range $[-0.04, +0.04]$, as shown in Figs. 15 and 16, respectively. The control input is shown in Fig. 17 for $\Delta = 0.3$ s with $m = 2$. It can be seen that the errors in the system performance, including estimation and tracking errors, can be decreased effectively when there are disturbances and uncertainties in the systems. Thus, the proposed control scheme achieved good $H_\infty$ tracking performance with a satisfactory robustness level when controlling a system with delayed outputs and disturbances. To demonstrate the effectiveness of our proposed control scheme, we performed a comparison with the study presented in [36] based on the magnetic levitation system using the same reference signal $x_{r1} = 2 + 0.5 \sin(0.5t)$. The responses of the simulation results are shown in Figs. 18 and 19. Fig. 18 shows that the responses of the distance states tracked the reference signal in about 5 s with both control schemes and the maximum steady state tracking errors were 3.23% and 3.86% after 10 s with the DAFC scheme [36] and the proposed control scheme, respectively. However, the settling time for the velocity states, as shown in Fig. 19, was about 7 s. The maximum steady state tracking error was 3.45% after 10 s using the DAFC scheme [36] whereas the maximum steady state tracking error was 2.92% after 10 s using the proposed scheme. These results clearly demonstrate the superior performance of our proposed control scheme when it was used to control the magnetic levitation system with output delays and external disturbances.

Example 2. As shown in Fig. 20, an inverted pendulum system was used to illustrate the performance and efficiency of the proposed scheme compared with FAOB [33]. We consider the inverted pendulum system with the dynamics described in [33], as follows:

$$
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= f(x) + b(x)u + d(t) \\
y(t) &= x_1(t - \Delta),
\end{align*}
$$

(39)

where $x_1(t) = \theta(t)$ represents the angle of the pendulum, $x_2(t) = \dot{\theta}(t)$ represents the angular velocity, $g = 9.8 \text{ m}/\text{s}^2$ is the acceleration due to gravity, $m_c$ is the mass of the cart, $m$ is the mass of the pole, $\ell$ is the half-length of the...
Fig. 15. Trajectories of the error state of the distance for a delay time of 0.3 s with $m = 2$.

Fig. 16. Trajectories of the error state of the velocity for a delay time of 0.3 s with $m = 2$.

Fig. 17. Trajectory of the control input $u(t)$ for a delay time of 0.3 s with $m = 2$.

Fig. 18. Trajectories of $x_1(t)$ for our scheme and the DAFC scheme [36] with a delay time of 0.2 s and $m = 1$. 

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for $\ell$ as that used in [28], for both $\rho$ and $\omega$. Let $\tilde{x}_{ji}$ be the filter output and $x_{ji}$ the filter input, the external disturbance $d(t) = 0.05 \sin(5t)$, $f(x) = \frac{g \sin(x_1) - \frac{m \ell}{m + \rho} \cos(x_1) \sin(x_1)}{\ell (s - s + 5)}$, and $b(x) = \frac{m g \sin \theta}{\ell (s - s + 5)}$. In this case, we selected $m_c = 1$ kg, $m = 0.1$ kg, $\ell = 1$ m, the sampling time was 0.001 s, and the variable ranges were $x_1(t) \in [-\pi/6, \pi/6]$ and $x_2(t) \in [-\pi/6, \pi/6]$. We selected the same filter $L(s) = 25/(s + 5)^2$ as that used in [28], for both $\hat{e}_j$ and $\hat{e}_j$. Let the reference signal be $x_{r1} = 0$ and output delay time be $\Delta = 0.22$ s to analyze the trajectory tracking performance of the closed-loop system.

**Step 1:** The number of sub-observers was set as 2 with $m = 1$ for a system with an output delay time of $\Delta = 0.22$ s.

**Step 2:** If we require that $x_1(t) \in [-\pi/6, \pi/6]$ and $x_2(t) \in [-\pi/6, \pi/6]$, then the bound $b^{-1}_{j\min}$ can be computed as $b^{-1}_{j\min} = 1/1.12$ and we select $(\sup_{t \geq 0} |b_{ij}|) = 2$, $\delta = 0.01$, $\gamma_f = 1$, $\gamma_b = 0.01$, $\kappa_f = 0.1$, $\kappa_z = 0.05$, and the initial conditions $x_1(0) = 0.02$, $x_2(0) = 0$, $\hat{e}_{j1}(0) = \hat{e}_{j2}(0) = 0$, $\theta_f(0) = 0$, $\theta(0) = 0$, $\theta(0) = 0$, $\Delta = 1$. Let $Q_1 = 0.0001 \times I_{1 \times 2}$, $Q_2 = 0.0001 \times I_{1 \times 2}$, $j = 0$, 1, for $m = 1$ and select $\phi = 1$, $r = 0.01$, and $\rho^2 = 0.9$.

**Step 4:** From (32), we obtain $P_2$, for $m = 1$ and $m = 2$, $x_{r1} = 2 + 0.5 \sin(0.5t)$ as

$$P_2 = \begin{bmatrix} 5.7002 & 0.0444 \\ 0.0444 & 5.3076 \end{bmatrix}$$

and the observer gain vectors $K_{o\psi}$, $j = 0, 1$, $\ell_j = 1, 2$, for $m = 1$ and $m = 2$, can be obtained using

$$K_{o0} = [0.5936, 0.0038]^T, \quad K_{o1} = [0.6530, 0.0042]^T,$$

$$K_{o2} = [0.7124, 0.0046]^T, \quad K_{o12} = [0.7717, 0.0050]^T$$

Fig. 19. Trajectories of $x_2(t)$ for our scheme and the DAFC scheme [36] with a delay time of 0.2 s and $m = 1$.

Fig. 20. The inverted pendulum system.
Step 5: From (32), we obtain 
\[ P_1 = \begin{bmatrix} 0.0038 & 0.0024 \\ 0.0024 & 0.0038 \end{bmatrix} \]
and the feedback control gain vectors 
\[ K_j^0, j = 0, 1, 2, \]
for \( m = 1 \) and \( m = 2 \), can be obtained using
\[ K_0^0 = [1.9439 \quad 2.0362]^T, \quad K_1^0 = [2.1382 \quad 2.2398]^T \]

Step 6: The minimal attenuation level \( \rho^2 \) was determined as 0.65 using MATLAB’s LMI toolbox.

Step 7: By substituting \( P = \text{diag}(P_1, P_2) \) and \( \rho^2 \) back into (28), the matrix inequality holds.

Step 8: Obtain the observer-based VSS indirect adaptive fuzzy \( H_\infty \) tracking control law from (16). The controller rules, \( u_{h_j}, u_{s_j} \), are the same as those in Example 1, and compute the adaptive laws (27) to adjust the parameter vectors 
\[ \theta_{\hat{f}_j}, \theta_{\hat{b}_j}, j = 0, 1. \]

The responses of the tracking performance using the proposed scheme and the previously reported method [33] with a delay time of \( \Delta = 0.22 \) s for the \( H_\infty \) tracking performance with attenuation levels \( \rho^2 = 0.9 \) are shown in Figs. 21–23. Fig. 21 shows the responses of the states \( x_1 \) and \( \hat{x}_1 \) for both schemes, and Fig. 22 shows the responses of the states \( x_2 \) and \( \hat{x}_2 \). Fig. 23 shows the control input for the proposed scheme. The figures show that the system performances errors, including the tracking errors, can be decreased effectively in the presence of disturbances and uncertainties using the proposed scheme. The system required 5 s to approach the reference signal with the proposed scheme, whereas FAOB did not approach the reference signal for \( \rho^2 = 0.9 \) with FAOB [33], as shown in Figs. 21 and 22. These results clearly demonstrate the effectiveness of our proposed scheme when used to control a nonlinear system with output delays and disturbances, which was better than that using previously proposed schemes such as FAOB [33].

Fig. 21. Trajectories of \( x_1(t) \) for our proposed scheme and the FAOB scheme [33] with a delay time of 0.22 s and \( m = 1 \).

Fig. 22. Trajectories of \( x_2(t) \) for our proposed scheme and the FAOB scheme [33] with a delay time of 0.22 s and \( m = 1 \).
5. Concluding remarks

In this study, we developed a FAOB control design based on a VSS control algorithm and disturbance attenuation theory for nonlinear systems with output delays. First, the system is represented as a set of T–S fuzzy linear models with the capacity for adaptation. If the system states are not available, the states estimated from two-layer fuzzy observers are used to estimate the error state variables vector and to construct the output feedback fuzzy controller. The effects of tracking errors caused by fuzzy approximation errors are eliminated by using a VSS control algorithm. A delayed output feedback fuzzy adaptive controller was developed to overcome the nonlinearities, time delays, and external disturbances so $H_\infty$ tracking performance can be achieved. Based on a Lyapunov criterion and the LMI method, some sufficient conditions are derived so all the states of the system are UUB and the effects of external disturbance on the tracking errors can be attenuated to any prescribed level. Thus, $H_\infty$ robust tracking control is achieved. Compared with previous methods [36] based on a direct adaptive fuzzy-based controller with less restrictions on the control gain, our method can handle a broader class of nonlinear systems in the presence of plant uncertainties, output time delays, and external disturbances. This is achieved by integrating the VSS and $H_\infty$ tracking control techniques. Moreover, our proposed technique attenuates the external disturbances due to the tracking error such that the attenuation level can be made arbitrarily small. A standard VSS is also used to eliminate large fuzzy approximation errors, $\Delta F_j$ and $\Delta B_j$, in contrast to the method presented in [32,33] where the fuzzy approximation errors are attenuated. Finally, to validate the proposed control scheme, we used a magnetic levitation system and an inverted pendulum system as examples of nonlinear systems with output delays and external disturbances. Compared with the control schemes proposed in [33] and [36], our proposed control scheme was shown to be more effective and it delivered good performance results.

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Appendix A. Proof of Theorem 1

Select the Lyapunov function candidate

$$V = e_a^T P e_a + \frac{1}{\gamma_f} \bar{\theta}_f^T \dot{\bar{\theta}}_f + \frac{1}{\gamma_b} \bar{\theta}_b^T \dot{\bar{\theta}}_b + \phi^2 \int_{t-\Delta}^t e_a^T (\xi) e_a (\xi) d\xi$$

where $\phi$ is a positive constant. By taking the time derivative of $V$ along the trajectory (22), we have

$$\dot{V} = e_a^T \dot{P} e_a + \dot{e}_a^T \dot{P} e_a + \frac{2}{\gamma_f} \bar{\theta}_f^T \bar{\theta}_f + \frac{2}{\gamma_b} \bar{\theta}_b^T \bar{\theta}_b + \phi^2 \dot{e}_a^T (t) e_a (t) - \phi^2 e_a^T (t - \Delta) e_a (t - \Delta)$$

$$= \sum_{j=0}^{m} \sum_{\ell_j=1}^{N_{a_j}} w_j (|\eta|) \psi_j (\bar{e}_j) \left( (A_{a_j} \bar{e}_j + \tilde{A}_{a_j} x_{a,\Delta} + B_{a_j} (\bar{\theta}_f^T \bar{\xi}_f (\hat{x}_j) + \bar{\theta}_b^T \bar{\xi}_b (\hat{x}_j) u_j) \right)$$

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From (25) and the adaptive law (27), we obtain

\[
\dot{\mathbf{x}}_{a\Delta} = \begin{bmatrix} \mathbf{x}_{F}(t - \Delta) - \mathbf{x}_{F} \end{bmatrix} - \begin{bmatrix} \mathbf{e}_{F}(t - \Delta) \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{rF}(t - \Delta) \end{bmatrix} \]

For convenience, we define

\[
\begin{align*}
\mathbf{e}_{F\Delta} &= \begin{bmatrix} \mathbf{e}_{F}(t - \Delta) \\
\mathbf{e}_{F} \end{bmatrix}, \\
\mathbf{x}_{rF\Delta} &= \begin{bmatrix} \mathbf{x}_{rF}(t - \Delta) \\
\mathbf{x}_{rF} \end{bmatrix}
\end{align*}
\]

Thus, we have

\[
\dot{\mathbf{V}} = \sum_{j=0}^{m} \sum_{\ell_{j}=1}^{N_{o_{j}}} \left| w_{j}(\eta) \right| v^{f}\left(\mathbf{\hat{e}}_{j}\right) \left( \mathbf{e}_{a_{j}}^{T}\left(\mathbf{A}_{a_{j}}^{T} \mathbf{P} + \mathbf{P}\mathbf{A}_{a_{j}} + \phi^{2}\mathbf{I}\right)\mathbf{e}_{a_{j}} + 2\mathbf{B}_{a}^{T}\mathbf{B}_{a}\mathbf{e}_{a_{j}} + 2\mathbf{B}_{a}^{T}\mathbf{B}_{a}(\Delta B_{j} + 1)\mathbf{u}_{h_{j}} \right)
\]

From (25) and the adaptive law (27), we obtain

\[
\dot{\mathbf{V}} \leq \sum_{j=0}^{m} \sum_{\ell_{j}=1}^{N_{o_{j}}} \left| w_{j}(\eta) \right| v^{f}\left(\mathbf{x}_{j}\right) \left( \mathbf{e}_{a_{j}}^{T}\left(\mathbf{A}_{a_{j}}^{T} \mathbf{P} + \mathbf{P}\mathbf{A}_{a_{j}} + \phi^{2}\mathbf{I}\right)\mathbf{e}_{a_{j}} + \left(1 + \frac{2}{\phi^{2}} + \frac{1}{\rho^{2}}\right)\mathbf{P}\mathbf{A}_{a_{j}} \mathbf{A}_{a_{j}}^{T} \mathbf{P} + \left(\frac{1}{\rho^{2}} - \frac{2}{r}\right)\mathbf{B}_{a}^{T}\mathbf{B}_{a} + \phi^{2}\mathbf{I} \right)\mathbf{e}_{a_{j}} + \rho^{2}d^{2} - \left(\sqrt{0.5\dot{\phi}\mathbf{e}_{F\Delta} - \frac{1}{\sqrt{0.5\dot{\phi}}} \mathbf{A}_{a_{j}}^{T} \mathbf{P}\mathbf{e}_{a_{j}}\right)\]

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Therefore, whenever
\[
V(t) \leq \sum_{j=0}^{m} \sum_{\ell_j=1}^{N_{\ell_j}} w_j (|\eta|) v^{f_j}(\hat{e}_j) (e_{aj}^T A_{aj}^T P + P A_{aj} + \left(1 + \frac{2}{\phi^2} + \frac{1}{\rho^2} \right) P A_{aj} A_{aj}^T P + 2\phi^2 I)
\]
\[+ \left(\frac{1}{\rho^2} - \frac{2}{\rho} \right) B_a^T B_a P e_{aj} + \rho^2 (d^2 + x_{rf,d}^T x_{rf,d}) + 2e_{aj}^T P B_a (\Delta B_j + 1) u_{s_j}
\]
\[+ 2e_{aj}^T P B_a (\dot{A} F_j + z_j + \dot{A} B_j (-\dot{f}_j - K^T_c \hat{e}_j + x_{r}^{(n)}))}\]
(42)

Since |\Delta B_j + 1| \leq \kappa_{bj} + 1 and from (26), we have
\[
2(e_{aj}^T P B_a (\Delta B_j + 1) u_{s_j} + e_{aj}^T P B_a (\dot{A} F_j + z_j + \dot{A} B_j (-\dot{f}_j - K^T_c \hat{e}_j + x_{r}^{(n)})))
\]
\[\leq 2 \left( -\frac{2\alpha_j}{1 + \kappa_{bj}} (\kappa_{bj} + 1) |B_a^T P e_{aj}| + \alpha_j |B_a^T P e_{aj}| \right) \leq 0 \]
(43)

Hence, from (42), we have
\[
V(t) \leq \sum_{j=0}^{m} w_j (|\eta|) (e_{aj}^T (-Q) e_{aj} + \rho^2 (d^2 + x_{rf,d}^T x_{rf,d}))
\]
(44)

Therefore, whenever
\[
\|e_{aj}\| \geq \frac{\rho \sqrt{d^2 + x_{rf,d}^T x_{rf,d}}}{\sqrt{\lambda_{\min}(Q)}}\]
(45)

for all j, where \(\lambda_{\min}(Q)\) denotes the minimal eigenvalue of \(Q\), we have \(V(t) \leq 0\). From (24), we have
bounded and we can obtain
\[
\lim_{t \to \infty} \| e_{aj} \| = 0
\]
and consequently, \[e_{aj} \in L_\infty\]. From Assumption 2, we have \(d \in L_\infty\) and \(x_r \in U_r\) is bounded. Hence, the right-hand side of (46) is bounded and we can obtain \(e_{aj} \in L_\infty\). Since \(e_{aj} \in L_\infty\), then \(\hat{e}_F + \hat{e}_r\), \(\hat{e}_F \in L_\infty\). From \(e_{aj} = \hat{e}_F + \hat{e}_r\), we obtain \(e_{aj} \in L_\infty\). Since \(x_r \in U_r\) is bounded and \(x_F = e_{aj} + x_r\), then \(x_F \in L_\infty\) and similar to \(x_F\), it can be shown that \(x_F(t - \Delta) \in L_\infty\). From Assumption 1, we know that \(\| \theta_j \| \leq M_{\theta_j}\) and \(\| \theta_{b_j} \| \leq M_{\theta_{b_j}}\). Using triangle inequality, we have \[\| \theta_j \| \leq 2M_{\theta_j}\] and \[\| \theta_{b_j} \| \leq 2M_{\theta_{b_j}}\], since \(\| \theta_j \| \leq M_{\theta_j}\) and \(\| \theta_{b_j} \| \leq M_{\theta_{b_j}}\). It can be concluded that all the variables on the right-hand side of (22) are bounded, therefore \(e_{aj} \in L_\infty\). Using Barbalat’s lemma [40], we have \[\lim_{t \to \infty} \| e_{aj} \| = 0\] and \(\lim_{t \to \infty} e_{aj_i} = 0\). From (21), we have
\[
\omega^2 e_{aj} = \omega^2 e_{aj_i} + \omega^2 e_{aj_{i+1}} + \cdots + \omega^2 e_{aj_{2n}}
\]
then,
\[
\omega^2 e_{aj} = \omega^2 e_{aj_{2n-1}} + \omega^2 e_{aj_{2n-2}} + \cdots + \omega^2 e_{aj_0}
\]
Since \(e_{aj} \in L_\infty\), \(e_{aj_{i+1}} \in L_\infty\) and \(e_{aj_{i}} \in L_\infty\), it is implied that \(e_{aj_{i+1}} \in L_\infty\). Moreover, due to \(\int_0^\infty e_{aj_{i+1}} dt = \lim_{t \to \infty} e_{aj_{i+1}} = 0\), it is implied that \(\lim_{t \to \infty} e_{aj_{i+1}} = 0\). In addition, because
\[
\omega^2 e_{aj} = \omega^2 e_{aj_i} + \omega^2 e_{aj_{i+1}} + \cdots + \omega^2 e_{aj_0}
\]
can be rewritten as
\[
\omega^2 e_{aj} = \omega^2 e_{aj_i} + \omega^2 e_{aj_{i+1}} + \cdots + \omega^2 e_{aj_0}
\]
and consequently,
\[
\lim_{t \to \infty} \omega^2 e_{aj} = \lim_{t \to \infty} \left( \omega^2 e_{aj_i} + \omega^2 e_{aj_{i+1}} + \cdots + \omega^2 e_{aj_0} \right)
\]
Hence, because \(\lim_{t \to \infty} \| e_{aj} \| = 0\) and \(\lim_{t \to \infty} \| e_{aj_i} \| = 0\), \(i = 0, 1, 2, \ldots, 2n\), we obtain \(\lim_{t \to \infty} \| e_{aj} \| = 0\). Therefore, the \(H_\infty\) tracking control performance is obtained with a prescribed attenuation level \(\rho\) [39,40]. This completes the proof. □

References


