Induced path transit function, monotone and Peano axioms

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Abstract

The induced path transit function \( J(u, v) \) in a graph consists of the set of all vertices lying on any induced path between the vertices \( u \) and \( v \). A transit function \( J \) satisfies monotone axiom if \( x, y \in J(u, v) \) implies \( J(x, y) \subseteq J(u, v) \). A transit function \( J \) is said to satisfy the Peano axiom if, for any \( u, v, w \in V \), \( x \in J(v, w) \), \( y \in J(u, x) \), there is a \( z \in J(u, v) \) such that \( y \in J(w, z) \). These two axioms are equivalent for the induced path transit function of a graph. Planar graphs for which the induced path transit function satisfies the monotone axiom are characterized by forbidden induced subgraphs.

Keywords: Transit function; Induced path; Monotone axiom; Peano axiom; JHC convexity

1. Introduction

The geodesic interval function \( I \), the induced path function \( J \) and the corresponding convexities of a connected graph have been studied extensively by various authors ([4–8,10,11,13–15,19,20,22,23]) and they have contributed significantly to the development of the area of study known as metric and related graph theory. There is also sufficient literature on all path function and all path convexity, see [2,6,21]. A number of prototype problems generalizing the notion of intervals and convexity in the case of graphs is being surveyed in [17]. In this paper we follow the terminology as it is in [17]. The notion of Interval monotone (I-monotone) graphs is introduced by Mulder in [16] and proved that if \( G \) contains no subgraph homomorphic to \( K_{2,3} \) or \( W_5 - x \), then \( G \) is I-monotone. Since \( W_5 - x \) is homeomorphic to \( K_{2,3} \), it follows that if \( G \) contains no subgraph homeomorphic to \( K_{2,3} \) [16]. Mulder conjectured that the interval regular graphs [16] are I-monotone, but Mollard and Nomura [14,20] disproved it. A characterization of I-monotone graphs using forbidden induced subgraphs still remains as an unsolved problem.

In the case of induced path transit function \( J \), a characterization using the notion of \( M \)-graphs is studied in [3], which states that

The induced path transit function \( J \) on a connected graph \( G \) is monotone if and only if it does not contain any \( M \)-graph perfectly.

This characterization also does not identify the induced subgraphs to be forbidden.

Peano’s Theorem is a well-known theorem in classical plane geometry, van de Vel [24] used this theorem as an axiom—called Peano axiom—for characterizing geometric interval operators. He has shown that Peano axiom together with Pasch axiom (also from classical plane geometry) imply geometricity. Further in [24], the convexity of Pasch–Peano spaces is characterized by Join Hull Commutative (JHC).
In this paper we attempt to characterize the $J$-monotone graphs using forbidden induced subgraphs. We determine the subgraphs to be forbidden to make $G$ a $J$-monotone graph. We have obtained a necessary condition for the $J$-monotonicity of any connected graph $G$ and a characterization is obtained for planar connected graphs. It may be noted that the induced path convexity has a very nice structure because of the JHC property and using clique separators, the induced path convex hull have a simple characterization [6,7].

In Section 2, we formally define the concept of transit function, monotone and Peano axioms. As a corollary of some results of van de Vel, cf. [24], we derive that for a transit function $R$ with JHC $R$-convexity, $R$ is monotone if and only if $R$ satisfies the Peano axiom.

In Section 3, we prove our main theorem characterizing the $J$-monotone planar graphs with forbidden induced subgraphs.

1.1. Subdivided $K_{2,3}$ with a chord

We denote the graph obtained by the subdivision of the edges of a $K_{2,3}$ by $G_1$. We call the degree three vertices of $G_1$ as $u$ and $v$. There are three $u$–$v$ paths in a $G_1$ and label them as $P_s$, $P_t$ and $P_i$ where $s$ and $t$ are the neighbours of $u$ on $P_s$ and $P_t$, respectively. We denote the neighbour of $u$ on $P$ as $f$ and the neighbour of $v$ on $P_s$ as $a$. Allow the vertex $a$ to have adjacency with at least one interior vertex of $P$, the resulting graph obtained from the subdivided $K_{2,3}$ is called a subdivided $K_{2,3}$ with a chord and is denoted by $G_2$. In this paper, we may refer the vertices $u$, $v$, $s$, $t$ and $f$ vertices that we have defined on a $G_i; i = 1, 2$, as the $u$, $v$, $s$, $t$ and $f$ vertices of the $G_i$, respectively. Once we have fixed the degree three vertices $u$ and $v$ of a $G_i$, then the $s$, $t$ and $f$ vertices follow naturally from the definition of the subdivided $K_{2,3}$ (with a chord); refer Fig. 1.

The cycle formed by $P_s \cup P_t$ is called the cycle of the $G_i$. Since $K_{2,3}$ is planar and the subdivision graph of a planar graph is again planar, it follows that $G_1$ and $G_2$ are planar graphs.

All graphs in this paper are connected, simple, loopless and finite.

2. Transit function and associated convexity

A transit function on a finite set $V$ is a function $R : V \times V \to 2^V$ satisfying the three transit axioms

(t1) $u \in R(u, v)$ for any $u, v \in V$.
(t2) $R(u, v) = R(v, u)$ for all $u, v \in V$.
(t3) $R(u, u) = \{u\}$ for all $u \in V$.

If $G$ is a graph with vertex set $V$ and if $R$ is a transit function on $V$, then we say that $R$ is a transit function on $G$. In [24], van de Vel used the axioms t1 and t2 only as axioms of the interval function. But in [17] the t3 axiom is included in the definition of transit function to avoid the trivial cases.

Prime examples of transit functions on graphs are provided by the geodesic interval function,

$I(u, v) = \{w \in V | w \text{ lies on some shortest } u - v \text{ path in } G\},$

the induced path transit function,

$J(u, v) = \{w \in V | w \text{ lies on some induced } u - v \text{ path in } G\}$
and also the all paths transit function,
\[ A(u, v) = \{ w \in V | w \text{ lies on some } u - v \text{ path in } G \}. \]

A transit function \( R \) is said to satisfy the Peano axiom if, for any \( u, v, w \in V \), \( x \in R(v, w) \), \( y \in R(u, x) \) there is an \( z \in R(u, v) \) such that \( y \in R(w, z) \). A set \( W \subseteq V \) is an \( R \)-convex set if \( R(u, v) \subseteq W \), for all \( u, v \in W \). The family of \( R \)-convex sets in \( G \) is an edge. Then the monotone axiom. For, trees and complete graphs are monotone.

The induced path transit function \( A \) is any central vertex. Proposition 2. The induced path convexity \( J \)-convexity on \( V \) is called a minimal chord of the vertex \( y \) such that \( A \) is not a vertex on \( P \) and adjacent to a vertex on \( b \). We denote the transit function associated with \( P \) by \( \rightarrow \). Minimal Chord of a vertex: Let \( u, v, y \in a \rightarrow P \rightarrow b \), \( y \not= a, b \) such that \( u \in a \rightarrow P \rightarrow (y) \), \( v \in (y) \rightarrow P \rightarrow b \) and \( uv \) is an edge. Then \( uv \) is called a minimal chord of the vertex \( y \), since it forbids the path \( a \rightarrow P \rightarrow b \) being an induced path. In the same sense we say \( uv \) forbids the minimality of the path \( a \rightarrow P \rightarrow b \) and avoids \( y \) from the induced path \( a \rightarrow P \rightarrow u \rightarrow v \rightarrow P \rightarrow b \).

We can easily see that the induced path transit function \( J \) on a subdivided \( K_{2,3} \) or a subdivided \( K_{2,3} \) with a chord does not satisfy the Peano axiom, since \( v \in J(s, t) \), \( y \in J(u, v) \), \( J(u, t) = \{ u, t \} \), but there is no \( z \in J(s, u) \) with \( y \in J(z, t) \), where \( y \) is any central vertex.

Before going to our main theorems, let us examine the class of connected graphs with fewest number of vertices on which the induced path transit function \( J \) satisfies the Peano axiom or equivalently the monotone axiom.

Observation 1. The induced path transit function \( J \) on any connected graph \( G \) containing four or less vertices satisfies the monotone axiom. For, trees and complete graphs are \( J \)-monotone. Hence there remains only three graphs to be
Hence \( x \in J(a,b) \), \( y \notin J(z,b) \) for all \( z \in J(a,c) \). In particular \( y \notin J(a,b) \cup J(b,c) \cup J(c,a) \).

Theorem 1. Let \( G \) be a connected graph with at least five vertices such that it neither contains an induced sub divided \( K_2,3 \), nor an induced sub divided \( K_{2,3} \) with a chord. Then the induced path transit function \( J \) of \( G \) satisfies the Peano axiom.

Proof. Suppose the transit function \( J \) on \( G \) does not satisfy the Peano axiom. Then there exist five vertices \( a,b,c,x,y \) and \( z \) such that \( x \in J(a,b) \), \( y \notin J(z,b) \) for all \( z \in J(a,c) \). In particular \( y \notin J(a,b) \cup J(b,c) \cup J(c,a) \).

Hence \( x \neq y \); \( x \neq a,b,c \) and \( b \neq c \). Now there exist an \( a-b \) induced path \( P_i \) containing \( x \) and a \( c-x \) induced path \( P_j \) containing \( c \). Choose \( P_s \) so that no vertex on \( y \rightarrow P_y \rightarrow (x) \) belongs to \( J(a,b) \). Let \( x_1 \) and \( x_4 \) be the neighbours of \( x \) on \( a \rightarrow P_a \rightarrow (x) \) and \( b \rightarrow P_b \rightarrow (x) \), respectively. Then by the choice of \( P_s \), the only vertex on \( a \rightarrow P_s \rightarrow (x) \) which can be adjacent to a vertex on \( y \rightarrow P_y \rightarrow x \) is \( x_1 \) and the only vertex on \( b \rightarrow P_s \rightarrow (x) \) which can be adjacent to a vertex on \( y \rightarrow P_y \rightarrow x \) is \( x_4 \). Let \( y_1x_1 \) and \( y_4x_4 \) be the chords from \( (y) \rightarrow P_y \rightarrow x \) to \( a \rightarrow P_s \rightarrow (x) \) and \( y \rightarrow P_y \rightarrow x \) to \( b \rightarrow P_s \rightarrow (x) \), respectively. By the choice of \( P_s \), it follows that if \( y_1 \neq y \) then \( y_4 = x \) and if \( y_4 \neq x \) then \( y_1 = x \). We complete the proof in two cases.

Case 1: \( x \) is the only common vertex of \( a \rightarrow P_a \rightarrow b \) and \( c \rightarrow P_y \rightarrow x \); refer Fig. 2.

Case 1.1: \( y_2 = y_3 \).

If both the cycles \( C_1 \) and \( C_2 \) are induced cycles, then \( G' \) is isomorphic to \( G_1 \) with \( s = x'_2 \), \( t = x'_1 \), \( v = x \), \( u = y_2 \) and \( f = y \). If the cycle \( C_1 \) is not an induced cycle, then \( x_1 \) is adjacent to some vertex on \( (x) \rightarrow P_s \rightarrow y \). Then \( C_2 \) is an induced cycle and \( x_1 = x'_2 \). In this case \( G' \) is isomorphic to \( G_2 \) with \( s = x'_2 \), \( t = x'_1 \), \( v = x \), \( u = y_3 \) and \( f = y \). If \( C_2 \) is not an induced cycle, then \( x_4 \) is adjacent to some vertex on \( (x) \rightarrow P_s \rightarrow y \) and \( C_1 \) is an induced cycle. Hence \( x_4 = x'_3 \). In this case \( G' \) is isomorphic to \( G_2 \) with \( s = x'_2 \), \( t = x'_1 \), \( u = y_3 \), \( v = x \) and \( f = y \).

Case 1.2: \( y_2 \in (y_3) \rightarrow P_y \rightarrow (y) \) and \( x_1 = x_2 \).

In this case \( x'_2 = x_2 \) and \( C_2 \) is an induced cycle. If \( C_1 \) is also an induced cycle and \( x_2 \) is not adjacent to any vertex on \( (y_2) \rightarrow P_y \rightarrow y_3 \), then \( G' \) is isomorphic to \( G_1 \) with \( s = x'_2 \), \( t = x'_1 \), \( u = y_2 \), \( v = x \) and \( f = y \). If \( C_1 \) is not an induced cycle or \( x_2 \) is adjacent to some vertex on \( (y_2) \rightarrow P_y \rightarrow y_3 \), then \( G' \) is isomorphic to \( G_2 \) with \( s = x'_2 \), \( v = x \), \( u \)-vertex is the last vertex on \( y_2 \rightarrow P_y \rightarrow y_3 \) which is adjacent to \( x_2 \). If \( u \neq y_3 \), then the \( t \)-vertex is the neighbour of \( u \) on \( (u) \rightarrow P_y \rightarrow y_3 \). If \( u = y_3 \), then \( t = x'_3 \).
Similarly we can prove the existence of an induced subgraph of $G$ isomorphic to a subdivided $K_{2,3}$ or a subdivided $K_{2,3}$ with a chord follows. This completes the proof.

**Remark 1.** Let $G$ be a connected graph with at least five vertices. Suppose the induced path transit function $J$ of $G$ does not satisfy the Peano axiom. Then by Theorem 1, we have proved the existence of an induced subgraph of $G$. 

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**Fig. 3.**

**Fig. 4.**

**Fig. 5.**
isomorphic to a subdivided $K_{2,3}$ or a subdivided $K_{2,3}$ with a chord. In the above proof of Theorem 1, we can see that either $y_3 \in c \rightarrow P_y \rightarrow y_2$ or $y_2 \in c \rightarrow P_y \rightarrow y_3$ and one of the cycles $C_1: x'_2 \rightarrow P_s \rightarrow x \rightarrow P_y \rightarrow y_2 \rightarrow x'_2$ or $C_2: x'_2 \rightarrow P_s \rightarrow x \rightarrow P_y \rightarrow y_1 \rightarrow x'_2$ is an induced cycle. If we assume that $C_2$ is an induced cycle and $y_3 \in c \rightarrow P_y \rightarrow y_2$, then we can observe the following things. $y_3 \neq y_2$ implies $x'_2 = x_1$. The subgraph induced by $x'_2 \rightarrow P_s \rightarrow x'_1 \rightarrow x \rightarrow P_y \rightarrow y_3$ is isomorphic to a subdivided $K_{2,3}$ or a subdivided $K_{2,3}$ with a chord. Their $s,t,u,v$ and $f$ vertices are as follows: $s = x'_2$, $v = x$ and $f = y$. When $y_2 \neq y_3$, $u$-vertex is the last vertex on $y_2 \rightarrow P_y \rightarrow y_3$ which is adjacent to $x_2$. If $u \neq y_3$, then the $t$-vertex is the neighbour of $u$ on $(u) \rightarrow P_y \rightarrow y_3$. If $u = y_3$, then $t = x_1$. When $y_2 = y_3$, the cycle $C$ of $G_i$ is $x'_2 \rightarrow P_s \rightarrow x'_1 \rightarrow y_2 \rightarrow x'_2$. If $y_2 \neq y_3$, the cycle $C$ of $G_i$ is $x'_2 \rightarrow P_s \rightarrow x'_1 \rightarrow y_3 \rightarrow P_y \rightarrow y_2 \rightarrow x'_2$. In all cases the central axis is $(u) \rightarrow P_y \rightarrow (v)$. An important observation is that no vertex on $f \rightarrow P_y \rightarrow (x)$ belongs to $J(a,b)$.

We can observe that if $G$ is not $J$-monotone, then there exists five vertices $a, b, u, v, z$ and $a \rightarrow b$ induced paths $P_1$ and $P_2$ containing $a$ and $v$, respectively, and a $a \rightarrow b$ induced path $P$ containing $z$ so that there is no $a \rightarrow b$ induced path containing $z$. Let $a'$ and $b'$ be the vertices common to $P_1$ and $P_2$ so that $a' \rightarrow P_1 \rightarrow a \rightarrow b' \rightarrow b \rightarrow b' \rightarrow P_2 \rightarrow v \rightarrow P_2 \rightarrow a'$ is a cycle, say $C$. If $G$ is planar, then every $a \rightarrow b$ induced path containing $z$ contains a subpath connecting two vertices on $C$. Hence for a planar graph, the $J$-monotonicity can be characterized by the induced subgraph formed by $V(C) \cup V(P)$.

When $G$ is planar, the cycle $C$ is not sufficient to prove the $J$-monotonicity of $G$ which implies the non-existence of induced $G_1$ and $G_2$.

In the rest of the discussion, we shall choose $f$ as the neighbour of $x$ on $P_y$. Let us denote the vertices $x'_2$ and $x'_1$ by $x_2$ and $x_3$, respectively, so that $y_2 x_2$ is the chord from $y \rightarrow P_y \rightarrow c$ to $x \rightarrow P_x \rightarrow a$ and $y_3 x_3$ is the chord from $y \rightarrow P_y \rightarrow c$ to $x \rightarrow P_x \rightarrow b$.

**Theorem 2.** The induced path transit function $J$ on a connected planar graph $G$ satisfies the Peano axiom if and only if $G$ has no induced sub graph isomorphic to neither a subdivided $K_{2,3}$ nor an induced subgraph isomorphic to a subdivided $K_{2,3}$ with a chord such that there is no induced path in $G$ connecting their $s, t$ vertices and containing the $f$ vertex.

**Proof.** If $G$ has an induced sub graph isomorphic to a subdivided $K_{2,3}$ or a subdivided $K_{2,3}$ with a chord such that there is no induced path in $G$ connecting their $s, t$ vertices and containing the $f$ vertex. Then $f \notin J(s,t)$ but $f \in J(u,v)$ and $u,v \in J(s,t)$.

Therefore $G$ is not $J$-monotone, equivalently $J$ does not satisfy the Peano axiom on $G$. So to complete the proof, we have to prove the sufficiency part alone. For that, let us assume that, $J$ does not satisfy the Peano axiom.

Hence by Theorem 1, we have the following:

(i) there exist vertices $a, b, c, x$ and $y$ of $G$ with $a \neq b; x \neq y; x, y \neq a, b, c$; two induced paths $P$ and $Q$ connecting $a$ to $b$ and $c$ to $x$, respectively, so that $x$ is on $P$ and $y$ is on $Q$.

(ii) there exists another set of vertices $x_1, x_2, x_3$ and $y_4$ on $P$ and $y_1, y_2, y_3$ and $y_4$ on $Q$, so that $x_1$ and $x_4$ form the neighbours of $x$ on $a \rightarrow P_s \rightarrow x \rightarrow b$ and $b \rightarrow P_s \rightarrow x \rightarrow a$, respectively. $y_2 x_2$ forms the chord from $(y) \rightarrow P_y \rightarrow c$ to $(x) \rightarrow P_y \rightarrow a$ and $y_3 x_3$ forms the chord from $(y) \rightarrow P_y \rightarrow c$ to $(x) \rightarrow P_y \rightarrow b$. In this case either $y_2 \in (y) \rightarrow P_y \rightarrow y_1$ or $y_2 \in (y) \rightarrow P_y \rightarrow y_2$. Without loss of generality, we let assume that $y_2 \in (y) \rightarrow P_y \rightarrow y_1$. Also assume that $C_2: x \rightarrow P_v \rightarrow x_3 \rightarrow y_3 \rightarrow P_y \rightarrow x$ is an induced cycle. By the Remark 1, it follows that $y_2 \neq y_3$ implies $x_1 = x_2$. Also, the subgraph induced by the vertices of $P_x$ or $Q_x$ has an induced subgraph isomorphic to $G_1$ or $G_2$. Their $s, t, u, v$ and $f$ vertices are as follows: $s = x_2$, $v = x$ and $f = y$. When $y_2 \neq y_3$, $u$-vertex is the last vertex on $y_2 \rightarrow P_y \rightarrow y_3$ which is adjacent to $x_2$. If $u \neq y_3$, then the $t$-vertex is the neighbour of $u$ on $(u) \rightarrow P_y \rightarrow y_3$. If $u = y_3$, then $t = x_1$. When $y_2 = y_3$, the cycle $C$ of $G_i$ is $x_2 \rightarrow P_x \rightarrow x_3 \rightarrow y_2 \rightarrow x_2$. When $y_2 \neq y_3$, the cycle $C$ of $G_i$ is $x_2 \rightarrow P_x \rightarrow x_3 \rightarrow y_1 \rightarrow P_y \rightarrow y_2 \rightarrow x_2$. In all cases the central axis is $(u) \rightarrow P_y \rightarrow (v)$. By the Remark 1 no vertex on $f \rightarrow P_y \rightarrow (x)$ belongs to $J(a,b)$.
Also, without loss of generality, we can assume that $f$ is adjacent to $x$. Consider any planar embedding of $G$. We can prove that $f \notin J(s,t)$. Suppose not, then there is an $s-t$ induced path $\mu$ in $G$ containing $f$. We now define four vertices $p_1, p_2, q_1$, and $q_2$ as follows. Suppose that we are traversing from $s$ to $t$ along $\mu$, let us assume that $p_1$ is the last vertex of $s \rightarrow \mu \rightarrow f$ and $p_2$ is the first vertex of $f \rightarrow \mu \rightarrow t$ lying on $C$. Similarly, let $q_1$ be the first vertex of $s \rightarrow \mu \rightarrow f$, $q_2$ be the last vertex of $f \rightarrow \mu \rightarrow t$ lying on $(u) \rightarrow P_x \rightarrow (v)$ (Fig. 7). By the definition of the chords $y_2x_2$ and $y_3x_3$ at least one of the subpaths $p_1 \rightarrow q_1$ or $q_2 \rightarrow p_2$ is of length greater than one. Also $p_1 \neq p_2$, $p_1, p_2 \neq q_1, q_2$, $q_1, q_2 \neq s, t$, $p_2 \neq s$, and $p_1 \neq t$.

Let us observe some other properties of $p_1, p_2, q_1$ and $q_2$.

(a1): If $q_1 \neq q_2$, then $p_1 \in x \rightarrow P_x \rightarrow p_2 \Rightarrow q_1 \in q_2 \rightarrow P_y \rightarrow y$.

Suppose $q_1 \neq q_2$ and $p_1 \in x \rightarrow P_x \rightarrow p_2$. If $p_1 = x$, then $x$ and $y$ are vertices of $\mu$ and $xy$ is an edge. Therefore $xy$ is an edge of $\mu$. Therefore $f = q_1$. Now suppose $p_1 \neq x$. We can prove that $q_1 \in (q_2) \rightarrow P_y \rightarrow (x)$. Suppose not, then $q_1 \in (q_2) \rightarrow P_y \rightarrow (y)$. Since $p_1 \neq x, x_1 \neq x_2$. Therefore $y_2 = y_3$. Let $C'$ be the cycle $x \rightarrow P_x \rightarrow x_3 \rightarrow y_3 \rightarrow x_2 \rightarrow P_x \rightarrow p_2 \rightarrow q_2 \rightarrow P_y \rightarrow x$.

Evidently $p_1$ is an exterior vertex and $q_1$ is an interior vertex of $C'$. Let $S: p_1 \rightarrow \mu \rightarrow q_1$. Since $q_1 \neq q_2$ and both $S$ and $p_2 \rightarrow \mu \rightarrow q_2$ are subpaths of $\mu$, they cannot have common vertices. By the definition and choice of $p_1$ and $q_1$, $S$ cannot have vertices in common with $(q_2) \rightarrow P_y \rightarrow x \rightarrow P_x \rightarrow x_3$. Hence we get the contradiction that some edge of $S$ crosses an edge of $p_2 \rightarrow \mu \rightarrow q_2$. This proves (a1).

(a2): $p_1 \in y_2 \rightarrow P_y \rightarrow p_2 \Rightarrow q_1 \in (y_2) \rightarrow P_y \rightarrow q_2$.

The proof of (a2) is similar to that of (a1).

Using (a1) and (a2), let us complete the proof of the theorem in two cases.

Case 1: $t = x_3$.

Since $t = x_3, y_3$ is adjacent to $x_2$ and $s = x_2$. If $\mu$ contains $y_3$, then either $f \in x_2 \rightarrow \mu \rightarrow y_3$ or $f \in x_3 \rightarrow \mu \rightarrow y_3$ and which gives the contradiction that either $f = x_2, x_3$ or $y_3$. Hence, $\mu$ cannot contain the vertex $y_3$. Therefore $p_1, p_2 \in x_3 \rightarrow P_x \rightarrow x_2$.

Case 1.1: Both $p_1, p_2 \in x \rightarrow P_x \rightarrow x_2$ (Fig. 8).

Therefore, either $p_1 \in x \rightarrow P_x \rightarrow p_2$ or $p_2 \in x \rightarrow P_x \rightarrow p_1$.

First, let us consider the case when $p_1 \in x \rightarrow P_x \rightarrow p_2$. Therefore, by (a1), $q_1 \in q_2 \rightarrow P_y \rightarrow y$.

Then $a \rightarrow P_x \rightarrow p_2 \rightarrow \mu \rightarrow q_2 \rightarrow P_y \rightarrow x \rightarrow P_x \rightarrow b$ is an $a-b$ path containing $f$. To forbid the minimality of the path, there must exist a chord $t_1t_2$ (say) from $(p_2) \rightarrow \mu \rightarrow (q_2)$ to $(y) \rightarrow P_y \rightarrow x \rightarrow P_x \rightarrow b$. In that case, $t_1$ is an interior vertex of the cycle $C'$: $x_2 \rightarrow P_x \rightarrow p_1 \rightarrow \mu \rightarrow f \rightarrow P_y \rightarrow x \rightarrow P_x \rightarrow x_3 \rightarrow y_2 \rightarrow x_2$ and $t_2$ lies exterior to $C'$.

Hence $t_1t_2$ must cross at least one edge of $C_1$ which affects the planarity of $G$. Similar contradictions can be derived when $p_2 \in x \rightarrow P_x \rightarrow p_1$ and both $p_1$ and $p_2$ are vertices of $x \rightarrow P_x \rightarrow x_3$. 
Case 1.2: \( p_1 \in x_2 \rightarrow P_x \rightarrow x \) and \( p_2 \in x \rightarrow P_x \rightarrow x_3 \).

In this case, \( a \rightarrow P_x \rightarrow p_1 \rightarrow \mu \rightarrow P_x \rightarrow P_x \rightarrow b \) is an \( a \rightarrow b \) path containing \( f \). To forbid the minimality of the path, there must exist a chord \( t_3t_4 \) from \( a \rightarrow P_x \rightarrow (p_1) \rightarrow (f) \rightarrow \mu \rightarrow (p_2) \) or a chord \( t_5t_6 \) from \( (p_1) \rightarrow \mu \rightarrow (f) \rightarrow (p_2) \rightarrow \mu \rightarrow b \). If \( t_3t_4 \) exists, then \( t_3 \) is an interior vertex of the cycle \( C_2^1 : x_2 \rightarrow P_x \rightarrow x \rightarrow P_x \rightarrow y_2 \rightarrow x_2 \), whereas \( t_4 \) is an exterior vertex of \( C_2^1 \), hence \( t_3t_4 \) must cross at least one edge of \( C_2^1 \). Similarly, if \( t_5t_6 \) exists it must cross at least one edge of the cycle \( x \rightarrow P_x \rightarrow x \rightarrow y_3 \rightarrow P_y \rightarrow x_2 \rightarrow x \), again a contradiction. Similar contradictions can be derived when \( p_2 \in x_2 \rightarrow P_x \rightarrow x \) and \( p_1 \in x \rightarrow P_x \rightarrow x_3 \). Hence \( f \not\in J(s,t) \).

Case 1.2: \( t \neq x_3 \).

In this case, \( y_2 \neq y_3 \), \( x_2 = x_1 \). Also the \( u \)-vertex is the last vertex on \( y_2 \rightarrow P_y \rightarrow y_3 \) which is adjacent to \( x_2 \), and the \( t \)-vertex is the neighbour of \( u \) on \( (u) \rightarrow P_y \rightarrow y_3 \), and \( x_2 \) is the only vertex on \( x_2 \rightarrow P_x \rightarrow a \) which can be adjacent to a vertex on \( y_2 \rightarrow P_y \rightarrow (y_3) \). Here we have the following cases:

Case 2.1: Both \( p_1, p_2 \in x_2 \rightarrow P_x \rightarrow x_2 \).

In Case 2.1, we can derive the contradiction \( f \in J(a,b) \). The proof is similar to the proof of Cases 1.1 and 1.2. So let us prove the remaining cases.

Case 2.2: Refer Fig. 9.

Since \( p_1, p_2 \in y_3 \rightarrow P_y \rightarrow y_2 \), by \( (x_2) \), we get \( q_1 \in (y_2) \rightarrow P_y \rightarrow q_2 \).

Let \( x'y' \) be the chord from \( a \rightarrow P_x \rightarrow x_2 \) to \( p_1 \rightarrow P_y \rightarrow y_2 \). Then \( a \rightarrow P_x \rightarrow x' \rightarrow y' \rightarrow P_y \rightarrow p_1 \rightarrow \mu \rightarrow q_1 \rightarrow \mu \rightarrow f \rightarrow x \rightarrow P_x \rightarrow b \) is an \( a \rightarrow b \) path containing \( f \). Since \( (p_1) \rightarrow \mu \rightarrow (q_1) \) lies interior to the cycle.
In all cases we have obtained contradictions and which shows the non-existence of an $a - b$ induced path containing $f$, a contradiction.

Case 2.3: $p_1 = x_2$ and $p_2 \in y_3 \rightarrow P_3 \rightarrow (y_2)$. Refer Fig. 10.

Now $a \rightarrow P_x \rightarrow x_1 \rightarrow \mu \rightarrow p_2 \rightarrow P_3 \rightarrow y_3 \rightarrow x_3 \rightarrow P_3 \rightarrow b$ is an $a - b$ path containing $f$. Here each vertex of $(p_1) \rightarrow \mu \rightarrow (q_1)$ lies interior to the cycle $x_2 \rightarrow P_x \rightarrow x \rightarrow P_3 \rightarrow y_2 \rightarrow x_2$ and each vertex of $(p_2) \rightarrow \mu \rightarrow (q_2)$ lies interior to the cycle $x \rightarrow P_3 \rightarrow x_3 \rightarrow y_3 \rightarrow P_y \rightarrow x$. Hence, to forbid the minimality of the path, the only possibility is the existence of a chord $y_3x_3$ from $y_3 \rightarrow P_y \rightarrow p_2$ to $a \rightarrow P_3 \rightarrow x_2$. If $y_3 \neq y_3$, then $a \rightarrow P_x \rightarrow x_3 \rightarrow y_3 \rightarrow P_y \rightarrow p_2 \rightarrow \mu \rightarrow f \rightarrow x \rightarrow P_x \rightarrow b$ is an $a - b$ path containing $f$.

To forbid the minimality of the path there must exist a chord $t_7t_6$ from $(p_2) \rightarrow \mu \rightarrow (y)$ to $x \rightarrow P_x \rightarrow x_3$ and which gives the $a - b$ induced path $a \rightarrow P_x \rightarrow x_1 \rightarrow \mu \rightarrow t_7 \rightarrow t_8 \rightarrow P_3 \rightarrow b$ containing $f$, a contradiction. If $y_5 = y_3$, then by Case 1, the existence of an induced subdivided $K_{2,3}$ or an induced subdivided $K_{2,3}$ with a chord follows.

Case 2.4: $p_1 \in x_3 \rightarrow P_3 \rightarrow x$ and $p_2 \in y_3 \rightarrow P_y \rightarrow y_2$. Refer Fig. 11.

In this case it follows that $p_1 \neq x$, since $x$ and $s$ are adjacent.

Evidently $a \rightarrow P_x \rightarrow x_2 \rightarrow y_2 \rightarrow P_y \rightarrow p_2 \rightarrow \mu \rightarrow p_1 \rightarrow P_x \rightarrow b$ is an $a - b$ path containing $y$. To forbid the minimality of the path, the only possibility is the existence of a chord $t_7t_10$ from $(p_2) \rightarrow \mu \rightarrow (f)$ to $(p_1) \rightarrow P_x \rightarrow x_3$ and which in turn will give the $a - b$ induced path $a \rightarrow P_x \rightarrow x \rightarrow P_y \rightarrow q_2 \rightarrow \mu \rightarrow t_9 \rightarrow t_10 \rightarrow P_x \rightarrow b$ containing $f$, again a contradiction. Similar contradictions can be derived when $p_2 \in x \rightarrow P_x \rightarrow y_4$ and $p_1 \in y_4 \rightarrow P_y \rightarrow y_3$. Thus in all cases we have obtained contradictions and which shows the non-existence of an $s - t$ induced path containing $f$. The argument of the existence of the $f$-vertex is possible due to the planarity assumption of $G$. This proves the existence of an induced subdivided $K_{2,3}$ or an induced subdivided $K_{2,3}$ with a chord; which completes the sufficiency part of the theorem. \[\square\]
We pose the following problem as an unsolved problem.

**Problem 1.** The induced path transit function $J$ on a connected graph $G$ satisfies the Peano axiom if and only if $G$ has no subdivided $K_{2,3}$ nor a subdivided $K_{2,3}$ with a chord as an induced subgraph.

**Remark 2.** Using the approach of the Peano axiom, we were able to give a forbidden induced subgraph characterization for the planar $J$-monotone graphs. Using the direct approach, we feel that the forbidden structure may not be so clear as we have noted in the case of the $M$-graph in [3]. However, one may obtain a forbidden induced subgraph characterization using the monotone axiom directly.

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**References**