Kurtosis Based Spectrum Sensing in Cognitive Radio

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Abstract

In this paper, we consider spectrum sensing in cognitive radio networks based on higher order statistics. The kurtosis, a fourth order statistic, which is a measure of deviation from Gaussianity, is used as a detection statistic. An optimum threshold is set up based on the Neyman-Pearson criterion and analytical expression for upper bound on probability of miss is derived for a single pair of primary and secondary users. Further, we also propose a collaborative spectrum sensing scheme for more than one secondary user and it is shown by simulations that the proposed kurtosis based method outperforms the energy based spectrum hole detection method significantly.

Index Terms

Cognitive radio, cooperative spectrum sensing, kurtosis, Neyman Pearson test, spectrum sensing.

I. INTRODUCTION

Evolution of the wireless systems and technologies have made a huge impact on the society. However, at the same time, it has caused an acute scarcity of frequency bands for new services. This calls for efficient management of the spectrum. In this regard, the Federal Communication Commission (FCC) published a report on the spectrum management and utilization [1], which revealed, (as is now well known) that the...
allocated spectrum is grossly under-utilized. This problem is the motivation behind the development of the cognitive radio (CR) technology. Cognitive radio is based on the concept of software defined radio and provides techniques for spectrum sensing, spectrum management, and spectrum access for unlicensed secondary users when the primary users are not utilizing the spectrum. A function of the CR is to support intelligent and efficient dynamic spectrum access. Therefore, reliable spectrum sensing is an important aspect of cognitive radio technology [2].

In the literature, a number of spectrum sensing algorithms have been suggested [3]-[6]. These can be broadly classified based on their requirements for sensing [7], as, (A) Methods requiring both noise and signal source information, like matched filter based spectrum sensing, cyclostationary based technique, (B) Methods requiring only noise power information, which is termed as semiblind detection and includes energy based detection, and, (C) Methods requiring no information about source signal or noise, which is termed as totally blind detection (covariance based approach is an example).

The matched filter based sensing is an optimum spectrum sensing technique [3][6]- provided - knowledge about the primary user signal, modulation characteristics, transmitted power, pulse shape and the packet format are known at the secondary user terminal. This implies that, the suitability of this method is limited to cases where the secondary user terminal has knowledge about the pilot signals and preambles used by the primary user transmitter. Also the matched filter techniques performance degrades with timing and carrier frequency offset between the primary transmitter and the secondary user terminal.

The cyclostationary based technique is a feature based technique [5], which exploits the fact that the man made signals exhibit second order cyclostationary property in the wide sense [8] . This has the advantage of being robust to uncertainty in the noise and also the ability to differentiate between the primary user signal and interference signal, based on their cyclic properties. However this method requires the knowledge of the signal phase and one or multiple cyclic frequencies [5] of the primary user signal. The disadvantages of the cyclostationary based approach is that it requires a high sampling rate and it is computationally complex.
Another spectrum sensing method is the energy based detector, which is the simplest of the existing spectrum sensing methods [3] [9], in terms of the computation and implementation complexity. As stated above, energy based detector is a semi blind technique, which is the optimum detector for detecting a random uncorrelated Gaussian signal [3]. The energy detector does not require any prior knowledge of the primary user signals. However this technique is susceptible to uncertainty in the noise variance [3]. For fading channels, the energy based detector is not necessarily optimal. We show in this paper that the kurtosis based method outperforms the popular energy based scheme at low SNRs.

The covariance based technique exploits the fact that the received data vector of the primary user has a non-diagonal covariance matrix [10]. Hence, use of the non-diagonal elements of the covariance matrix as a test statistic is the essence of the covariance based spectrum sensing method. Another method which uses the higher order statistics (HOS) is the entropy based spectrum sensing method [11], which is based on the principle that the entropy is greater for noise samples from additive white gaussian noise (AWGN) process than that for the primary user’s signal. This method is based on the histogram approach and likelihood ratio test is applied to make the decision. However, the authors consider the situation where the secondary user has knowledge about the primary transmitter signal, which is not feasible in practical situation.

In [12], the author proposes an algorithm for sensing the Digital Television signals in Gaussian noise using Higher Order Statistics (HOS). The algorithm performs non-Gaussianity check in the frequency domain in the vicinity of the pilot of the DTV, to make a decision about the presence or absence of the primary user signal. The scheme of [12] is backed by simulation results under different conditions. However, no theoretical analysis has been carried out to obtain an analytical expressions for probability of miss and false alarm in [12]. On similar lines, the author in [13] gives a sensing technique based on higher order statistics, in the frequency domain, using a 2048 point fast fourier transform. The bispectrum, which is a third order statistic, is used as a decision statistic in [14]. The performance of the scheme is illustrated by simulations only and no theoretical analysis has been conducted by the authors in [14].

In this paper, we have chosen kurtosis-the fourth order cumulant at zero lag as a test statistic for spectrum
sensing. Since it is difficult to obtain an accurate estimate of higher order statistics and prohibitively high
number of observation samples are required to get a reliable estimate, we utilize only the fourth order
statistic in this paper. To the best of our knowledge, no analysis has been carried out in the existing
literature to obtain expressions of probability of miss and false alarm for spectrum sensing using fourth
order statistics.

The reasons for choosing kurtosis as a test statistic for semi-blind spectrum sensing are, (a) At low
SNR, when the noise variance is high, the energy estimate tends to accumulate the noise energy, whereas,
the kurtosis based scheme will suppress the effect of noise, since the kurtosis of noise, which is assumed
to be additive Gaussian, will be zero. This can be observed from Fig. 1 in Section VII, in the Simulation
results, wherein, at low SNR range of -15dB to 11dB, the kurtosis based scheme outperforms the energy
based scheme, significantly. (b) If there are multiple secondary users terminals experiencing independent
fading, then at least one of them will have a good link to the primary terminal. Since kurtosis involves
the fourth order absolute power of the channel, the kurtosis yields a better estimate, as compared to the
energy based scheme, which uses second order absolute power of the channel. This fact is illustrated in
Fig. 3 in Section VII, where, the kurtosis based method offers significant improvement over the energy
based method with 4 secondary users. (c) The energy detector cannot distinguish among the primary user
signals, and interference. But kurtosis based method has an advantage, that, in the presence of independent
interferers, even though the individual interference signals be non Gaussian, their sum tends to Gaussian
by central limit theorem and hence the kurtosis value of the interference goes to zero and helps in
differentiating between the primary user signal and interference. The disadvantage of the kurtosis based
method is that its performance advantage diminishes when the interference is non-Gaussian. This is due
to the fact that in such a situation, the received signal will always follow a non-Gaussian distribution
irrespective of the presence or absence of the primary user signal. In a sense it becomes quite similar to
the energy based scheme - the interference noise simply adds to the metric. In the TV band where the
primary interference is the TV signal, this assumption may not be true. However, in a dynamic spectrum
sharing scenario that is commonly studied today, this assumption is quite reasonable.

In this paper we assume that the secondary receiver terminals have a perfect knowledge of the noise
variance, for the operating SNR values and also the the primary signal samples are statistically uncor-
related. In [10], the authors have shown by simulations that with perfect noise variance knowledge and
uncorrelated symbols, the energy based detector performs better than the covariance based approach.
Therefore, in our simulations, we have compared our scheme with the optimal semiblind sensing scheme,
namely the energy based detector.

The contributions of this paper are as follows. 1) A kurtosis based spectrum sensing for single pair
of primary and secondary users has been proposed. 2) We derive an upper bound of the probability of
miss, i.e, probability of missing the presence of the primary user signal. 3) The proposed kurtosis based
scheme has been extended to the collaborative spectrum sensing scenario, with a single primary user and
multiple secondary users which provides significant improvement as compared to the conventional energy
based collaborative scheme.

Rest of the paper is organized as follows. Section II describes the system model and the problem
formulation. In Section III, brief description of the energy based spectrum sensing is done. Section IV
discusses the proposed spectrum sensing scheme based on kurtosis. An upper bound of the average
probability of miss has been obtained in Section V. An extension of the proposed spectrum sensing
problem for the case of constraining the interference caused by secondary user to the primary user is
discussed in Section VI. Section VII presents the simulation results. Some conclusions are drawn in
Section VIII. This paper also contains two Appendices. The variance of the kurtosis estimate has been
given in Appendix A. In Appendix B, the unbiased property of the kurtosis estimate has been proved.

II. PROBLEM FORMULATION

We consider a model in which there is a single primary user and a single secondary user. We assume
that there is no interferer in the neighborhood of the primary user and the secondary user. Spectrum
sensing in CR is the following binary hypothesis testing problem,
Hypothesis $H_0$: Absence of a primary user;

Hypothesis $H_1$: Presence of a primary user.

The secondary user has to perform spectrum sensing and decide between the following binary hypotheses

$$ r(i) = \begin{cases} 
  n(i), & H_0 \\
  hs(i) + n(i), & H_1 
\end{cases} $$

for $i = 1, 2, ..., N_{obs}$, where, $N_{obs}$ is the total number of observation samples in the observation interval, during the spectrum sensing process, $r(i)$ is the $i^{th}$ sample of the received signal, $s(i)$ is the $i^{th}$ sample of the signal from the primary user, $h$ is the complex Gaussian channel gain, assumed to be fixed during the observation interval, $n(i) \sim N_c(0, 2\rho_n^2)$ is circular AWGN. We assume that $s(i)$ to have discrete uniform distribution with zero mean. The signal constellation can be BPSK, as in [3]. This is a valid assumption, as, the signal constellation will, in general, have a finite set of symbols, and in typical communication scenarios, it is reasonable to assume that each of the symbol has a equiprobable occurrence. Also, the fading nature of the channel ensures that the overall signal under hypothesis 1 is non-gaussian. Hence, $r(i)$ is non-Gaussian under hypothesis $H_1$ and Gaussian under $H_0$.

The secondary user has to decide the correct hypothesis during the observation interval and make a decision whether the primary user is transmitting or not. The fact that $r(i)$ is Gaussian under hypothesis $H_0$ and non-Gaussian under hypothesis $H_1$ makes kurtosis an appropriate choice as a decision statistic.

III. REVIEW OF ENERGY BASED SPECTRUM SENSING

Energy based method is one of the oldest spectrum sensing schemes used in cognitive radio [3]. In this scheme, the channel is sensed in the desired frequency band and the RF energy is measured to detect the presence of a primary user. Once the energy estimate is obtained, it is compared with a pre-determined threshold to make the binary decision.

This scheme is the best choice in the scenario, where knowledge of the primary user signal is totally absent at the CR. The drawbacks of this approach is that it is not able to differentiate between primary
users signal and interference and it also has a performance degradation in the low SNR range.

From (1), the energy estimate can be obtained as,

$$\hat{e} = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^2,$$

which is modeled in [15] [16] as,

$$\hat{e} = \begin{cases} 
\chi^2_{2N_{\text{obs}}}, & H_0 \\
\chi^2_{2N_{\text{obs}}}(2\gamma), & H_1 
\end{cases}$$

where $\chi^2_{2N_{\text{obs}}}$ denotes a central chi-square distribution with $2N_{\text{obs}}$ degrees of freedom, $\chi^2_{2N_{\text{obs}}}(2\gamma)$ denotes a non-central chi-squared distribution with $2N_{\text{obs}}$ degrees of freedom and a non-centrality parameter $2\gamma$. Here $\gamma = |h|^2 \frac{E[|s|^2]}{E[|n|^2]}$ is the instantaneous SNR. Expressions for the probability of false alarm and probability of miss have been obtained in [15].

IV. PROPOSED SPECTRUM SENSING SCHEME

The kurtosis, $k$, for a circular zero mean random variable $r$ is defined as

$$k = E[|r|^4] - 2(E[|r|^2])^2.$$ 

(4)

It can be established that the kurtosis of a Gaussian random variable is zero while it is non-zero for a non-Gaussian random variable [17]. Therefore, the kurtosis is zero under hypothesis $H_0$, and non-zero under hypothesis $H_1$. The magnitude of the kurtosis, therefore, serves as a good candidate for the decision statistic of the binary hypothesis test.

A. Estimation of Kurtosis using Finite Observation Samples

The kurtosis can be estimated as [18]

$$\hat{k} = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^4 - 2\left(\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^2\right)^2.$$

(5)

It has been shown in [20], that asymptotically, $\hat{k}$ tends to a Gaussian distribution. For medium SNR range of around 15dB, the kurtosis estimate calculated using finite number of observation samples ($N_{\text{obs}} = 100$)
exhibit this Gaussian property. Hence, we use the finite observation kurtosis estimate, $\hat{k}$, as the decision statistic in the spectrum sensing problem.

**B. Decision Statistic and Cost Function**

Since (5) is an unbiased estimate \(^1\), under hypothesis $H_0$ it has a zero mean and under hypothesis $H_1$ it has a non-zero mean. The separation of the mean values can be used as a suitable decision statistic, as follows,

$$|\hat{k}| \begin{cases} \geq_{H_1} & \lambda, \\ \leq_{H_0} & \lambda \end{cases}$$

(6)

where, $\lambda$ is a decision threshold which has to be chosen by the minimization of a cost function. The kurtosis estimates have the following statistics: $|\hat{k}|_{H_0} \sim N(\mu_0, \sigma_0^2)$ and $|\hat{k}|_{H_1} \sim N(\mu_1, \sigma_1^2)$.\(^2\)

The probability of false alarm, $P_f$, indicates the under utilization of the frequency spectrum by the secondary user terminal. On the other hand, the probability of miss, $P_m$, implies that the secondary user has missed sensing the presence of primary user signal and starts transmitting its signal in the same frequency band as that of the primary user signal, causing interference to the licensed primary user. The main virtue of the secondary user is not to interfere at all with the primary transmission. This is critical from the viewpoint of the primary user. The sum of $P_m$ and $P_f$ could have been chosen as a cost function. However, this does not guarantee a desired $P_f$. Therefore, we choose we choose $P_m$ as the cost function and minimize it, by fixing the value of $P_f$. Formally,

$$\min_{\lambda} P_m(\lambda),$$

such that $P_f \leq \alpha'_0$

(7)

where $P_f$ is the probability of False alarm, the probability of deciding in favour of a primary user’s signal when it is absent.

\(^1\)Unbiased property of $\hat{k}$ has been proved in Appendix- B

\(^2\)The expressions for the mean and variance are given in the Appendix- A
We can use the Neyman-Pearson criterion to find the optimized threshold, $\lambda^*$ [21]. From (7), setting $P_f = \alpha_0$, we have,

$$\int_{-\infty}^{\lambda^*} f_{\hat{k}|H_0}(z)dz + \int_{\lambda^*}^{\infty} f_{\hat{k}|H_0}(z)dz = \alpha_0,$$

which reduces to,

$$2Q\left(\frac{\lambda^*}{\sigma_0}\right) = \alpha_0,$$

where we have used the fact that $\mu_0 = 0$, since $\hat{k}|H_0$ is an unbiased estimate of noise kurtosis. Hence $\lambda^* = \sigma_0 Q^{-1}(\frac{\alpha_0}{2})$ is the optimized threshold. It is important to note that for deriving the optimum threshold, no knowledge of the channel SNR is required.

C. Collaborative Spectrum Sensing

If the secondary users are far away from the primary users then there might be deep fading and/or shadowing which might prevent the secondary users from detecting the primary user signal. In such a case, the decision of a single secondary user will not be reliable. In such scenarios, collaborative spectrum sensing techniques can be used [22]. The proposed kurtosis based scheme can be extended to the case of a collaborative CR model. We shall consider the case where the participating secondary users make a hard decision i.e., the remaining collaborating secondary users send their binary decisions about the presence or absence of a primary user to a centralized fusion center and this fusion center makes a final decision based on some voting rule [16]. In particular, one of the rule is the OR fusion rule, where, the fusion center considers the binary decisions of the participating secondary users and decides $H_1$, if, at least one of the binary decisions is 1. The probability of detection and false alarm under such a scenario is given as [22],

$$Q_{m,avg} = (P_{m,avg})^N,$$

$$Q_{f,avg} = 1 - (1 - P_{f,avg})^N,$$

where $P_{m,avg}$ and $P_{f,avg}$ are the average probability of miss and the probability of false at the secondary user, given in the Section V. Since $P_{m,avg} < 1$ and $P_{f,avg} < 1$, for $N > 1$, $Q_{d,avg} < P_{m,avg}$ and
using binomial approximation [23, Eq. (1.110)] $Q_{m,\text{avg}} \approx NP_{f,\text{avg}}$. Hence, for collaborative scheme, the probability of miss decreases, but at the cost of increase in the value of probability of false alarm. We have chosen the OR-rule because the OR-rule gives more emphasis to minimizing the probability of miss. This is attributed to the fact that the OR-rule is a minority rule for detecting the presence of a primary user, in the sense, even if at least one of the participating secondary users detects the presence of the primary user signal, the fusion center gives the decision in favor of hypothesis $H_1$.

V. DERIVATION OF UPPER BOUND OF AVERAGE PROBABILITY OF MISS

A. Bound on Probability of Miss

The probability of miss, conditioned on the channel, is expressed as,

$$P_{m|\gamma,\theta} = 1 - \int_{\lambda^*}^{\infty} f_{k|H_1}(z)dz - \int_{-\infty}^{\lambda^*} f_{k|H_1}(z)dz,$$

$$= 1 - P_{m1|\gamma,\theta} - P_{m2|\gamma,\theta}, \quad (11)$$

where $P_{m1|\gamma,\theta} = \int_{\lambda^*}^{\infty} f_{k|H_1}(z)dz$, and $P_{m2|\gamma,\theta} = \int_{-\infty}^{\lambda^*} f_{k|H_1}(z)dz$. Here $\gamma = |h|^2 \frac{E[|s|^2]}{E[|n|^2]}$ is the instantaneous SNR, and $\theta = tan^{-1}\left(\frac{\text{Im}[h]}{\text{Re}[h]}\right)$, where $\text{Im}[X]$ and $\text{Re}[X]$ denote the imaginary and real parts of $X$ respectively, is the instantaneous phase angle of the channel. Since the kurtosis estimate follows a Gaussian distribution under hypothesis $H_1$, i.e., $\hat{k} \sim \mathcal{N}(\mu_1, \sigma_1^2(\gamma, \theta))$, (11) can be rewritten as,

$$P_{m|\gamma,\theta} = 1 - P_{m1|\gamma,\theta} - P_{m2|\gamma,\theta}, \quad (12)$$

$$P_{m|\gamma,\theta} = 1 - \frac{1}{2}\text{erfc}\left(\frac{\lambda^* - \mu_1}{\sqrt{2}\sigma_1(\gamma, \theta)}\right) - \frac{1}{2}\text{erfc}\left(\frac{\lambda^* + \mu_1}{\sqrt{2}\sigma_1(\gamma, \theta)}\right),$$

where $P_{m1|\gamma,\theta} = \frac{1}{2}\text{erfc}\left(\frac{\lambda^* - \mu_1}{\sqrt{2}\sigma_1(\gamma, \theta)}\right)$, $P_{m2|\gamma,\theta} = \frac{1}{2}\text{erfc}\left(\frac{\lambda^* + \mu_1}{\sqrt{2}\sigma_1(\gamma, \theta)}\right)$, and $\text{erfc}(.)$ is the complementary error function [Eq. (8.253)][23]. The exact expression for $\sigma_1^2(\gamma, \theta)$ in terms of $\gamma$ and $\theta$ is given in the Appendix A.

The conditional probability, $P_{m|\gamma,\theta}$, can be averaged with respect to the random variables $\gamma$ and $\theta$. For a circular Gaussian channel $h$, $\gamma$ follows an exponential distribution, i.e., $\gamma \sim \xi(\frac{1}{\bar{\gamma}})$, where $\bar{\gamma}$ is the average
SNR, and θ is uniformly distributed between [0, 2π]. Here, ξ(·) denotes the exponential distribution. The average value, \( P_{m,\text{avg}} \), after averaging \( P_{m|\gamma,\theta} \) with respect to \( \gamma \) and \( \theta \), can be written as,

\[
P_{m,\text{avg}} = 1 - P_{m_1,\text{avg}} - P_{m_2,\text{avg}},
\]

where,

\[
P_{m_1,\text{avg}} = \int_{\gamma=0}^{\infty} \int_{\theta=0}^{2\pi} P_{m|\gamma,\theta} f_{\gamma,\theta}(\gamma,\theta) d\theta d\gamma,
\]

\[
= \int_{\gamma=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{K}{\sigma_1(\gamma,\theta)} \left( \frac{1}{2\pi^2} e^{-\frac{\gamma^2}{2\sigma_1^2}} \right) d\theta d\gamma,
\]

where, \( K = \frac{\xi(\cdot) - \mu_1}{\sqrt{2}} \). In (14), \( f_{\gamma,\theta}(\gamma,\theta) = f_{\gamma}(\gamma) f_{\theta}(\theta) = \frac{1}{2\pi\gamma} e^{-\frac{\gamma^2}{2\sigma_1^2}} \) is the joint pdf of \( \gamma \) and \( \theta \), which are statistically independent. We can simplify integral given in (14) as follows,

\[
P_{m_1,\text{avg}} = c \int_{\gamma=0}^{x_0} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma + \]
\[
c \int_{\gamma=x_0}^{1} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma + \]
\[
c \int_{\gamma=1}^{\infty} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma, \tag{15}
\]

where \( x_0 < 1 \), \( I_1 = c \int_{\gamma=0}^{x_0} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma, I_2 = c \int_{\gamma=x_0}^{1} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma, I_3 = \]

\[
c \int_{\gamma=1}^{\infty} \int_{\theta=0}^{2\pi} \text{erfc} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right) e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma, \text{ and } c \text{ is a constant that is independent of the variables of integration.}
\]

1) Evaluation of \( I_1 \): The range of \( \gamma \) is \( 0 \leq \gamma \leq x_0 \). Using the series expansion of \( \text{erfc}(\cdot) \) [Eq. (8.253)][23],

\[
\text{erfc}(z) = 1 - \left( \frac{2}{\sqrt{\pi}} \right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{2k-1}}{(2k-1)(k-1)!} \]

\[
\tag{16}
\]

using (16) in \( I_1 \),

\[
I_1 = \frac{c}{4\pi^2} \int_{\gamma=0}^{x_0} \int_{\theta=0}^{2\pi} \left[ 1 - \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left( \frac{K}{\sigma_1(\gamma,\theta)} \right)^{2k-1}}{(2k-1)(k-1)!} e^{-\frac{\gamma^2}{2\sigma_1^2}} d\theta d\gamma, \tag{17}
\]

For \( 0 \leq \gamma \leq x_0 \), \( \sigma_1(\gamma,\theta) \) in (39) may be approximated by

\[
\sigma_1(\gamma,\theta) \approx \sqrt{a_{00} + a_{10}\gamma} = \sqrt{a_{00} (1 + b_0\gamma)} \]

\[
\tag{18}
\]
where \( a_{00} \) and \( a_{10} \) are constants which depend on the modulation scheme being employed under hypothesis \( H_1 \), and whose values are given in the Appendix A, for 16-QAM constellation, and \( b_0 = \frac{a_{10}}{a_{00}} \). The \( k \)-th term of the integrand, \( t_k \), of \( I_1 \) in (17) is given by,

\[
t_k = \left( -\frac{2}{\sqrt{\pi}} \right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left( \frac{K}{\pi a_{00} \sqrt{1+b_0 \gamma}} \right)^{2k-1}}{(2k-1)(k-1)!} e^{-\frac{\gamma}{2}},
\]

(19)

for \( k \geq 1 \). Using (18), (19) can be rewritten as

\[
t_k = \left( -\frac{2}{\sqrt{\pi}} \right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left( \frac{K}{\sqrt{a_{00} \gamma}} \right)^{2k-1}}{(2k-1)(k-1)!} e^{-\frac{\gamma}{2}}
\]

(20)

Since \( |b_0 \gamma| < 1 \) for the range of \( \gamma \) considered above, using the binomial expansion [Eq. (1.110)][23] for \( (1 + b_0 \gamma)^{-k+\frac{1}{2}} \), (20) can be further simplified as,

\[
t_k = \left( -\frac{2}{\sqrt{\pi}} \right) \left( \frac{K}{\sqrt{a_{00}}} \right)^{2k-1} \frac{(-1)^{k+1}}{(2k-1)(k-1)!} \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p (2k-1)(2k-3)...(2k-(2p-3)) b_0^p \gamma^p}{2^p p!} \right\} e^{-\frac{\gamma}{2}}
\]

(21)

Therefore, \( I_1 \) in (17) is a double integration with a double summation. The \((k,p)\)-th term of the double summation, where \( k \) and \( p \) are the indices of the first and second summation, respectively, after integration yields,

\[
I_{1,k,p} = \int_{\gamma=0}^{x_0} \int_{\theta=0}^{2\pi} (\gamma)^p e^x p (-\frac{\gamma}{2\gamma}) d\theta d\gamma = 2\pi \gamma L(p+1, \frac{x_0}{\gamma}),
\]

(22)

where, \( \gamma L(\cdot, \cdot) \) is the lower incomplete Gamma function [Eq. (8.350.1)][23]. Therefore, \( I_1 \) in (17) can be written as,

\[
I_1 = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} I_{1,k,p}
\]

\[
= \sum_{k=1}^{\infty} \frac{2c}{\sqrt{\pi}} \left( \frac{K}{\pi a_{00}} \right)^{2k-1} \frac{(-1)^{k+1}}{(2k-1)(k-1)!} \left\{ \frac{1}{2^{\gamma p}} \gamma L(p+1, \frac{x_0}{\gamma}) \delta(p) + \sum_{p=1}^{\infty} \frac{(-1)^p (2k-1)(2k-3)...(2k-(2p-3)) \gamma^p b_0^p}{2^p p!} \gamma L(p+1, \frac{x_0}{\gamma}) \right\},
\]

where \( \delta(.) \) is the Kronecker delta function.
2) Evaluation of $I_2$: The range of $\gamma$ is $x_0 \leq \gamma \leq 1$. In this range of $\gamma$, since $a_{10}\gamma$ is the dominant term in the expression for $\sigma(\gamma, \theta)$ in (39), it can be approximated as,

$$\sigma_1(\gamma, \theta) = \sqrt{a_{10} \gamma};$$

(23)

where $a_{10}$ is a constant which depends on the modulation format and is given in the Appendix A. Using the series expansion of $\text{erfc}(\cdot)$ in $I_2$ in (17) and proceeding in similar lines as for $I_1$, the $k$-th term integrand for $I_2$ in (17) can be written as,

$$t_k = \left( -\frac{2}{\sqrt{\pi}} \right) (-1)^{k+1} \left( \frac{K}{\sqrt{a_{10}}} \right)^{2k-1} \gamma^{-k+\left(\frac{1}{2}\right)} e^{x_0 (-\frac{\gamma}{\gamma})}$$

(24)

Using the series expansion for $\exp(-\frac{2}{\gamma})$ [Eq. (1.211.1)][23], (24) reduces to,

$$t_k = \left( -\frac{2}{\sqrt{\pi}} \right) (-1)^{k+1} \left( \frac{K}{\sqrt{a_{10}}} \right)^{2k-1} \gamma^{-k+\left(\frac{1}{2}\right)} \sum_{p=0}^{\infty} \frac{(-\gamma)^p}{p!}$$

(25)

Therefore the $(k, p)$-th term in the integrand in $I_2$ will be

$$t_{k,p} = \left( \frac{2}{\sqrt{\pi}} \right) (-1)^{k+p+1} \left( \frac{K}{\sqrt{a_{10}}} \right)^{2k-1} \gamma^{p-k+\frac{1}{2}}$$

(26)

Therefore, using (26), the expression for $I_2$ becomes,

$$I_2 = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{\frac{2}{\sqrt{\pi}} (-1)^{k+p+1} \left( \frac{K}{\sqrt{a_{10}}} \right)^{2k-1} (1 - x_0^{p-k+\frac{1}{2}})}{(2k-1)(k-1)! p! (p-k+\frac{1}{2})^{p+1}}$$

(27)

3) Evaluation of $I_3$: $\gamma > 1$ In this range of $\gamma$ we can neglect the lower order terms retaining the highest order term of $\gamma$ in the expression for $\sigma_1(\gamma, \theta)$ in (39). Hence

$$\sigma_1(\gamma, \theta) \approx \sqrt{[a_{40} + a_{41} \sin^2 2\theta - a_{41} \sin^4 4\theta] \gamma^4}$$

$$\approx \sqrt{[a_{40} + a_{41} \sin^2 4\theta (1 - \sin^2 2\theta)] \gamma^4}$$

$$\approx \sqrt{a_{40} [1 + b_1 \sin^2 4\theta] \gamma^4}$$

(28)

where $b_1 = \frac{a_{41}}{4a_{40}}$, and $a_{40}, a_{41}$ are constants, given in the Appendix A. Therefore, the $k$-th term in the integrand of $I_3$ can be written as

$$t_k = \left( \frac{2}{\sqrt{\pi}} \right) (-1)^{k+1} \left( \frac{K}{\sqrt{a_{10}}} \right)^{2k-1} \left[ 1 + b_1 \sin^2 4\theta \right]^{-k+\frac{1}{2}} \gamma^{-4k+2} e^{-\frac{\gamma}{\gamma}}$$

(29)
Since $|b_1| < 1$ the term $[1 + b_1 \sin^2 4\theta]^{-k + \frac{1}{2}}$ can be expanded as a binomial series,

$$[1 + b_1 \sin^2 4\theta]^{-k + \frac{1}{2}} = 1 + \sum_{p=1}^{\infty} \frac{(-1)^p(2k - 1)(2k - 3)\ldots(2k - (2p - 3))}{2^p p!} b_1^p \sin^{2p} 4\theta$$  \hspace{1cm} (30)

The $(k, p)$-th term integral of $I_3$ will be

$$I_{3,k,p} = \int_{\gamma=1}^{\infty} \int_{\theta=0}^{2\pi} \sin^{2p} 4\theta \gamma^{-4k+2} e^{-\frac{\gamma}{4}} d\theta d\gamma = \frac{2\sqrt{\pi} \Gamma(p + \frac{1}{2})}{\Gamma(p + 1)} E_{4k-2}(\frac{1}{\gamma})$$ \hspace{1cm} (31)

where $\Gamma(\cdot)$ is the Gamma function [Eq. (8.310.1)]\cite{23} and $E_{(\cdot)}(\cdot)$ is the Exponential integral [Eq. (12), pp xxxv]\cite{23}. Therefore,

$$I_3 = \sum_{k=1}^{\infty} \frac{1}{2\gamma\sqrt{\pi}} \sum_{p=1}^{\infty} \left( \frac{c}{\gamma^2} \right) \left( \frac{2k - 1}{(2k - 1)(k - 1)!\Gamma(p + 1)} \left( E_{4k-2}(\frac{1}{\gamma}) \right) \delta(p) \right)$$

$$+ \frac{1}{2\gamma\sqrt{\pi}} \sum_{p=1}^{\infty} \left( \frac{(-1)^p(2k - 1)(2k - 3)\ldots(2k - (2p - 3))}{2^p p!} \right) \times$$

$$b_1^p \frac{\Gamma(p + \frac{1}{2})}{\Gamma(p + 1)} E_{4k-2}(\frac{1}{\gamma})$$ \hspace{1cm} (32)

An expression for $P_{m2,avg}$ can be derived by following the same analysis with $K = \frac{\lambda^* + \mu_1}{2}$. Therefore, the expression for $P_{m,avg}$ can be written as,

$$P_{m,avg}(\lambda^*) = 1 - \sum_{k=1}^{\infty} A_k \left[ \left( \frac{\lambda^* - \mu_1}{2} \right)^{2k-1} + \left( \frac{\lambda^* + \mu_1}{2} \right)^{2k-1} \right].$$ \hspace{1cm} (33)

where $A_k$ is the term independent of $\lambda^*$.

**B. Probability of False Alarm**

For the decision statistic $\hat{|k|}$, the probability of false alarm, $P_f$, is the probability of deciding in the favor of presence of the primary user signal when it is actually not present. Since this probability does not depend on the channel statistics, the conditional (on the channel) probability of false alarm, $P_{f|\gamma,\theta}$, and the average probability of false alarm, $P_{f,avg}$, are identical. Hence,

$$P_{f,avg} = \int_{\lambda^*}^{\infty} f_{k|H_0}(z)dz + \int_{-\infty}^{-\lambda^*} f_{k|H_0}(z)dz,$$ \hspace{1cm} (34)
Since for hypothesis $H_0$, $\hat{k} \sim N(\mu_0, \sigma_0^2)$, (34) reduces to,

$$P_{f,avg} = \text{erfc}\left(\frac{\lambda^* - \mu_0}{\sqrt{2\sigma_0}}\right) + \text{erfc}\left(\frac{\lambda^* + \mu_0}{\sqrt{2\sigma_0}}\right)$$  \hspace{1cm} (35)

In (35), the quantity $\sigma_0$ depends on the noise variance $\rho_n^2$ only.

VI. SPECTRUM SENSING - THE DUAL PROBLEM

In section IV, the spectrum sensing problem was formulated by constraining the probability of false alarm, $P_{f,avg}$ to $\alpha_0$, and minimizing the probability of miss, $P_{m,avg}$. This implies that the secondary user can use only a fraction($\alpha_0$) of the times the primary user channel is free. However, in many scenarios, it is required that the interference caused by the secondary user to the primary user should not exceed a predetermined level. Motivated by this fact, we formulate the dual spectrum sensing problem namely minimizing $P_{f,avg}$, putting a constraint on $P_{m,avg}$, i.e.,

$$\min_{\lambda} P_{f,avg}(\lambda),$$

such that, $P_{m,avg} \leq \beta_0$  \hspace{1cm} (36)

It is important to note that we are putting a constraint on the $P_{m,avg}$ (averaged over the primary user channel) and not the instantaneous $P_m$. With the limited processing capabilities of the secondary users, it is not very practical to assume that they will have the instantaneous channel SNR. Hence, the optimal thresholds, $\lambda^*_{dual}$ will only depend on the average channel SNR and it is reasonable to assume the average channel SNR knowledge at the secondary user terminal.

By NP criterion,

$$P_{m,avg}(\lambda^*_{dual}) = \beta_0$$  \hspace{1cm} (37)

Using the expression for $P_{m,avg}$ derived in 33, we have,

$$P_{m,avg}(\lambda^*_{dual}) = 1 - \sum_{k=1}^{\infty} A_k \left[ \left(\frac{\lambda^*_{dual} - \mu_1}{2}\right)^{2k-1} + \left(\frac{\lambda^*_{dual} + \mu_1}{2}\right)^{2k-1} \right] = \beta_0,$$  \hspace{1cm} (38)

where, $\mu_1$ is the kurtosis mean value. $\lambda^*_{dual}$ can be obtained by solving (3) numerically. Comparing the two thresholds, $\lambda^*$ and $\lambda^*_{dual}$, it can be seen that, $\lambda^*$, is totally independent of the instantaneous channel
SNR(\(\gamma\)) and average channel SNR, (\(\gamma_{\text{avg}}\)), whereas, \(\lambda_{\text{dual}}^*\), is dependent \(\gamma_{\text{avg}}\). Hence, \(\lambda_{\text{dual}}^*\) will be more sensitive to the channel SNR (in an average sense), and we can choose \(\beta_0\) in a sensible manner depending on \(\gamma_{\text{avg}}\).

VII. Simulation Results

In our simulations, we have compared the kurtosis based semi-blind scheme with the known semiblind detector, namely, the energy based detector [7]. The energy detector cannot distinguish among the primary user signals, secondary user signals, and interference. However, kurtosis based method has an advantage that in the presence of independent interferers, the sum of the interference signals tends to become Gaussian distributed, by the central limit theorem. Hence, the kurtosis goes to zero and helps in differentiating between the primary user signal and interference. The simulations have been done for the slow fading Rayleigh channel, 16-QAM constellation and zero mean Gaussian noise with unit variance. The SNR is given by the signal power. Fig. 1 compares the performance of the proposed kurtosis and the conventional energy based scheme. It can be seen that, for the single secondary user case, under low SNR range of -15dB to 11dB, kurtosis based scheme outperforms Energy based scheme. This is intuitively appealing as in the low SNR region, noise dominates and increases the energy in the absence of a primary user signal whereas the cumulant based kurtosis scheme tends to suppress the Gaussian noise and hence shows an improved performance. For \(P_{m,\text{avg}} = 10^{-2}\), the kurtosis based scheme gives an improvement of 1.5dB over the energy based scheme.

The complementary Receiver Operating Characteristics (ROC) curve is depicted in Fig. 2 with the average SNR as a parameter. For a fixed probability of false alarm, lower value of probability of miss and therefore better performance is observed at a higher average SNR, for the proposed scheme.

The proposed spectrum sensing scheme is compared with the energy based scheme in a collaborative system with 1, 2, and 4 secondary users and a single primary user. It can be seen from Fig. 3 that increasing the number of secondary users users from two to four leads to a considerable improvement in the kurtosis based scheme compared to the Energy based scheme. This is based on the fact that kurtosis has absolute
fourth power of random variables as compared to absolute second power for the energy method. If there is at least one good channel path among the independent fading paths for the different secondary users, kurtosis will be more reliable than the energy estimate and hence provides better performance. Fig. 4 is a plot of the probability of miss curve obtained by simulation and also the analytical upper bound, derived in Section V.

The fundamental problem in the proposed kurtosis method is that the kurtosis estimate requires more number of observation samples to have a better estimate than that required for the energy based. Fig. 5 is an illustration of this fact. To achieve a probability of false alarm of 0.1, 250 samples are required for the energy based scheme as compared to 350 by the kurtosis based scheme. But the huge advantage that is obtained by the proposed kurtosis method, in the collaborative scenario outweighs the fact that it requires more samples for obtaining a good estimate.

The ROC curves for the dual spectrum sensing problem is plotted in Fig. 6. Here $\lambda_{\text{dual}}$ is obtained numerically by fixing the value of $P_{m,\text{avg}}$ to $\beta_0$. The ROC curves have been plotted for $\gamma_{\text{avg}} = -5dB$ and $\gamma_{\text{avg}} = 0dB$. If it is required that the interference caused by the secondary user to the primary user be as low as $\beta_0 = 0.005$, it can be seen from the curves that, at -5dB, the secondary user will utilize the free channel for 60% of the times and for the 0dB case it will use the free channel with 99.5%. Hence depending on the average SNR, the utilization of the primary channel and hence the secondary user traffic can be determined in the dual spectrum sensing case.

VIII. CONCLUSION

A spectrum sensing scheme based on the higher order statistics has been proposed which exploits the fact that kurtosis of Gaussian noise is zero. An analytical expression is derived for an upper bound of the probability of miss, assuming that the probability of false alarm is fixed to some acceptable level. The dual problem is also addressed. It has been shown by simulations that the proposed algorithm outperforms the energy based scheme in the low SNR region in the case of a single secondary user as well as in the case of multiple secondary users.
APPENDIX A

KURTOSIS ESTIMATE STATISTICS

The expressions for the mean and variance of the cumulants of a complex random variable, \( r \), has been derived in [19]. Using those expressions, we have the following:

\[
\begin{align*}
\mu_1 &= E(|r|^4) - 2\{E(|r|^2)\}^2 \\
\sigma^2_1(\gamma, \theta) &= a_{00} + a_{10}\gamma + [a_{20} + a_{21}\sin(2\theta)]\gamma^2 + [a_{30} + \\
&+ a_{31}\sin(2\theta)]\gamma^3 + [a_{40} + a_{41}\sin(2\theta) + a_{42}\sin^4(2\theta)]\gamma^4, \\
\sigma^2_0(\gamma, \theta) &= c_{00}
\end{align*}
\]

(39)

where \( a_{i,j} \)s are constants which depend on the type of modulation scheme being employed and number of observation samples used to estimate the kurtosis. \( \gamma = \frac{\sigma^2_s}{\sigma^2_n} |h|^2 \) is the instantaneous SNR and \( \theta = \tan^{-1}\left(\frac{\text{Im}[h]}{\text{Re}[h]}\right) \) is the instantaneous phase angle of the channel. The values of \( a_{i,j} \)s for 16-QAM constellation are evaluated and given in the Table 1. It is important to note that unlike second order statistics which requires only the instantaneous SNR, the higher order statistics depend on the phase information of the channel also.

| TABLE I |

<table>
<thead>
<tr>
<th>Modulation Scheme</th>
<th>Modulation Constants</th>
</tr>
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<tbody>
<tr>
<td>16-QAM</td>
<td></td>
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<tr>
<td>( a_{00} = \frac{24\rho^8}{N} ), ( a_{10} = \frac{96\rho^6}{N} ), ( a_{20} = \frac{46.08\rho^8}{N} ), ( a_{21} = -\frac{48.96\rho^8}{N} ), ( a_{30} = \frac{33.28\rho^8}{N} ), ( a_{31} = \frac{128.64\rho^6}{N} ), ( a_{40} = \frac{10.33\rho^8}{N} ), ( a_{41} = -\frac{1.93\rho^8}{N} ), ( a_{42} = \frac{1.74\rho^8}{N} ), ( c_{00} = \frac{24\rho^8}{N} )</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX B

UNBIASED PROPERTY OF THE KURTOSIS ESTIMATE

The kurtosis estimate in (5) is an unbiased estimate. This can be proved by taking the expectation on both sides of (5),

\[
E[\hat{k}] = E \left[ \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^4 - 2 \left( \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^2 \right)^2 \right]
\]

\[
= E \left[ \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^4 - 2 \left( \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^2 \right) \left( \frac{1}{N_{\text{obs}}} \sum_{j=1}^{N_{\text{obs}}} |r(j)|^2 \right) \right],
\]

\[
= E \left[ \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |r(i)|^4 - 2 \left( \frac{1}{N_{\text{obs}}^2} \sum_{i=1}^{N_{\text{obs}}} \sum_{j=1}^{N_{\text{obs}}} |r(i)|^2 |r(j)|^2 \right) \right],
\]

(40)

where, \( E[\cdot] \) is the expectation operator. Taking the expectation operator inside in summation, in (40), and combining the like terms and unlike terms, we obtain,

\[
E[\hat{k}] = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} E[|r(i)|^4] - \frac{2}{N_{\text{obs}}^2} \left[ N_{\text{obs}} E[|r(i)|^4] + (N_{\text{obs}} - 1) \{ E[|r(i)|^2] \}^2 \right],
\]

(41)

Since the expectation operator is independent of the index \( i \), we can rewrite (41) as,

\[
E[\hat{k}] = \left( 1 - \frac{2}{N_{\text{obs}}} \right) E[|r|^4] - 2 \left( 1 - \frac{1}{N_{\text{obs}}} \right) \{ E[|r|^2] \}^2.
\]

(42)

For large \( N_{\text{obs}} \), (42) can be approximated as \( E[\hat{k}] \approx k \), and hence, \( \hat{k} \) is an asymptotically unbiased estimate.

It has been verified by simulations that for low to medium SNR values, choosing around \( N_{\text{obs}} = 50 \) samples are sufficient to produce an almost unbiased estimate, \( E[\hat{k}] = k \).

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Fig. 1. Comparison of energy and proposed kurtosis based spectrum sensing method in terms of probability of miss, for $\alpha_0=0.01$ and $N_{obs}=350$.

Average SNR = 8.82dB  
Average SNR = 7.32dB

Fig. 2. Complementary ROC curves for two different average SNR values using $N_{obs} = 500$ samples for kurtosis based scheme.
Fig. 3. Probability of miss vs average SNR for collaborative scheme using OR-fusion rule, for $N_{obs} = 350$, which depicts the considerable improvement offered by kurtosis based scheme over energy based scheme for 4 collaborative CRs.

Fig. 4. Curves of analytical upper bound on probability of miss, derived in Section V and that obtained by simulation for $N_{obs} = 50$. 