Differential subordination and argumental property

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ABSTRACT

For analytic functions \( f(z) \) in the open unit disk \( \mathbb{E} \) and convex functions \( g(z) \) in \( \mathbb{E} \), Ch. Pommerenke [Ch. Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc. 114 (1) (1965) 176–186] has proved one theorem which is a generalization of the result by K. Sakaguchi [K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11 (1959) 72–75]. The object of the present paper is to generalize the theorem due to Pommerenke.

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1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{E} = \{ z \in \mathbb{C} \mid |z| < 1 \} \). A function \( f(z) \in \mathcal{A} \) is said to be convex in \( \mathbb{E} \) if and only if it satisfies the condition

\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{E}).
\]

We denote by \( \mathcal{C} \) the subclass of \( \mathcal{A} \) consisting of all such functions. A function \( f(z) \in \mathcal{A} \) is said to be starlike of order \( \alpha \) in \( \mathbb{E} \) if and only if it satisfies the condition

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{E})
\]

for some \( \alpha (0 \leq \alpha < 1) \). We denote by \( \mathcal{S}^*(\alpha) \) the subclass of \( \mathcal{A} \) consisting of all such functions. It is well known that if \( f(z) \in \mathcal{C} \), then

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \frac{1}{2} \quad (z \in \mathbb{E}),
\]

so that \( f(z) \in \mathcal{S}^*(\frac{1}{2}) \).
This result was obtained by Marx [1] and Strohhäcker [2]. If $f(z) \in \Delta^*$ satisfies the condition

$$\text{Re} \left( \frac{f(z)}{zf'(z)} \right) > \beta \quad (z \in \mathbb{E})$$

where $0 \leq \beta < 1$, then $f(z)$ is said to be starlike of reciprocal order $\beta$.

**Example 1.** Let us consider a function $f(z)$ given by

$$f(z) = \frac{z}{(1 - z)^{2(1-\alpha)}} \quad (0 < \alpha < 1).$$

Then we see that $f(z) \in \Delta^*(\alpha) \subset \Delta^*$ and

$$f(z) = \frac{1 - z}{1 + (1 - 2\alpha)z^2}.$$ 

This implies that

$$\left| \frac{f(z)}{zf'(z)} - \frac{1 - \alpha}{\alpha} \right| < \frac{1 - \alpha}{\alpha} \quad (z \in \mathbb{E}),$$

that is,

$$\text{Re} \left( \frac{f(z)}{zf'(z)} \right) > 0 \quad (z \in \mathbb{E}).$$

Thus $f(z)$ is starlike of reciprocal order $0$ in $\mathbb{E}$.

**Example 2.** Let us define the function $f(z)$ by

$$f(z) = ze^{(1-\alpha)z} \quad (z \in \mathbb{E})$$

with $0 < \alpha < 1$. This gives us that

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) = \text{Re} \left( 1 + (1 - \alpha)z \right) > \alpha \quad (z \in \mathbb{E}).$$

Therefore, we see that $f(z) \in \Delta^*(\alpha)$.

Furthermore, we have that

$$\frac{f(z)}{zf'(z)} = \frac{1}{1 + (1 - \alpha)z^2}.$$ 

It follows that

$$\frac{f(z)}{zf'(z)} = 1 \quad (z = 0)$$

and

$$\text{Re} \left( \frac{f(z)}{zf'(z)} \right) = \text{Re} \left( \frac{1}{1 + (1 - \alpha)e^{i\theta}} \right) > \frac{1}{2 - \alpha} \quad (z = e^{i\theta}).$$

Therefore, we conclude that $f(z) \in \Delta^*(\alpha)$ and starlike of reciprocal order $\frac{1}{2 - \alpha}$ in $\mathbb{E}$.

Pommerenke [3] proved the following theorem. If $f(z)$ is analytic in $\mathbb{E} = \{z : |z| < 1\}$ and $g(z)$ is convex in $\mathbb{E}$ and

$$\left| \arg \frac{f'(z)}{g'(z)} \right| \leq \frac{\alpha \pi}{2} \quad (0 \leq \alpha \leq 1)$$

then

$$\left| \arg \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} \right| \leq \frac{\alpha \pi}{2} \quad (|z_1| < 1 \text{ and } |z_2| < 1).$$

This theorem is a generalization of Sakaguchi’s result [4]. We will generalize the above theorem in the next section.
2. Main result

**Theorem 1.** Let \( f(z) \in \mathcal{A} \), \( g(z) \in \mathcal{C} \) and \( g(z) \) is starlike of reciprocal order \( \beta \) and suppose that
\[
\left| \frac{\arg f'(z)}{g'(z)} \right| < \alpha \frac{\pi}{2} + \tan^{-1} \frac{\alpha \beta}{1 + \alpha} \quad (z \in \mathbb{E}),
\]
where \( 0 < \alpha \leq 1 \) and \( 0 \leq \beta < 1 \).

Then we have
\[
\left| \frac{\arg f(z)}{g(z)} \right| < \alpha \frac{\pi}{2} \quad (z \in \mathbb{E}).
\]

**Proof.** Let us put
\[
p(z) = \frac{f(z)}{g(z)}, \quad p(0) = 1.
\]
Then it follows that
\[
\frac{f'(z)}{g'(z)} = p(z) + p'(z) \frac{g(z)}{g'(z)} = p(z) \left( 1 + \left( \frac{zp'(z)}{p(z)} \right) \left( \frac{g(z)}{zg'(z)} \right) \right).
\]
(1)

If there exists a point \( z_0 \in \mathbb{E} \) such that
\[
\left| \arg p(z_0) \right| < \alpha \frac{\pi}{2} \quad \text{for } |z| < |z_0|
\]
and
\[
\left| \arg p(z_0) \right| = \alpha \frac{\pi}{2},
\]
then from Nunokawa’s result [5], we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \alpha K
\]
(2)
where
\[
K \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when } \arg p(z_0) = \alpha \frac{\pi}{2}
\]
and
\[
K \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when } \arg p(z_0) = -\alpha \frac{\pi}{2}
\]
where
\[
p(z_0)^\frac{1}{a} = \pm ia \quad \text{and } 0 < a.
\]
From Marx–Strohhäcker’s theorem, we have
\[
\Re \left( \frac{zg'(z)}{g(z)} \right) > \frac{1}{2} \quad (z \in \mathbb{E})
\]
and therefore, we have
\[
\left| \frac{g(z)}{zg'(z)} - 1 \right| < 1 \quad (z \in \mathbb{E}),
\]
(3)
which implies that
\[
\left| \Im \left( \frac{g(z)}{zg'(z)} \right) \right| < 1 \quad (z \in \mathbb{E}),
\]
and from the assumption, we have
\[
\Re \left( \frac{g(z)}{zg'(z)} \right) > \beta \quad (z \in \mathbb{E}).
\]
(4)
For the case \( \arg p(z_0) = \frac{\alpha \pi}{2} \), from (1) to (4), it follows that

\[
\arg \left( \frac{f'(z_0)}{g'(z_0)} \right) = \arg p(z_0) + \arg \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)} \left( \frac{g(z_0)}{z_0 g'(z_0)} \right) \right)
\]

\[
= \frac{\alpha \pi}{2} + \arg \left( 1 + i\alpha K \left( \frac{\text{Re} \frac{g(z_0)}{z_0 g'(z_0)}}{\text{Im} \frac{g(z_0)}{z_0 g'(z_0)}} + i \left( \frac{\text{Re} \frac{g(z_0)}{z_0 g'(z_0)}}{\text{Im} \frac{g(z_0)}{z_0 g'(z_0)}} \right) \right) \right)
\]

\[
= \frac{\alpha \pi}{2} + \arg \left( 1 - \alpha K \left( \text{Im} \frac{g(z_0)}{z_0 g'(z_0)} - i \alpha K \left( \frac{\text{Re} \frac{g(z_0)}{z_0 g'(z_0)}}{\text{Im} \frac{g(z_0)}{z_0 g'(z_0)}} \right) \right) \right)
\]

\[
\geq \frac{\alpha \pi}{2} + \tan^{-1} \left( \frac{\alpha K \text{Re} \frac{g(z_0)}{z_0 g'(z_0)}}{1 + \alpha K} \right)
\]

\[
\geq \frac{\alpha \pi}{2} + \tan^{-1} \left( \frac{\alpha \beta K}{1 + \alpha K} \right)
\]

\[
\geq \frac{\alpha \pi}{2} + \tan^{-1} \left( \frac{\alpha \beta}{1 + \alpha} \right)
\]

This contradicts the assumption of Theorem 1 and for the case \( \arg p(z_0) = -\frac{\alpha \pi}{2} \), applying the same method as the above, we have a contradiction. This completes the proof of Theorem 1.

\[\square\]

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