Local Boundedness for Doubly Degenerate Quasi-Linear Parabolic Systems

M. SANGO
Department of Mathematics, Vista University
Private Bag X1311, Silverton 0127
Pretoria, South Africa
sango-m@marlin.vista.ac.za

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Abstract—we establish the local boundedness of weak solutions for the degenerate quasilinear parabolic systems

$$\frac{\partial u^i}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( |u|^\alpha \left| \frac{\partial u}{\partial x_j} \right|^{p-2} \frac{\partial u^i}{\partial x_j} \right) = C |u|^{\alpha-1} \left| \frac{\partial u}{\partial x} \right|^{p-1} u^i, \quad i = 1, \ldots, N; \quad N \geq 1.$$

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1. INTRODUCTION AND FORMULATION OF MAIN RESULT

In this work, we establish the local boundedness for a solution of the doubly degenerate parabolic quasilinear system

$$\frac{\partial u^i}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( |u|^\alpha \left| \frac{\partial u}{\partial x_j} \right|^{p-2} \frac{\partial u^i}{\partial x_j} \right) = C |u|^{\alpha-1} \left| \frac{\partial u}{\partial x} \right|^{p-1} u^i, \quad i = 1, \ldots, N,$$

in the cylinder $Q_T = \Omega \times (0, T)$, $0 < T < \infty$, where $\Omega$ is a bounded region in $\mathbb{R}^n$, $n \geq 2$, $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^N)$, $N \geq 1$, $|u| = [\sum_{i=1}^{N} (u^i)^2]^{1/2}$, $\frac{\partial u}{\partial x}$ denotes the gradient of $u$, and $C$ is a positive constant. For simplicity, in the sequel we shall assume that a pair of same indices will mean summation from 1 to $n$ or from 1 to $N$.

For $p \geq 2$, system (1) is degenerate when $u = 0$, or when the gradient of $u$ vanishes. We do not consider here the case when $p < 2$; this corresponds to a situation when the system can be degenerate and singular. We note that the case when $\alpha = 0$ is known to be of importance in fluid mechanics [1]. DiBenedetto and Friedman [2] have studied the case $\alpha = 0$ and established a
number of results on the regularity properties (such as the Hölder continuity) of the weak solutions and their gradients. A more recent account of these results can be found in the monograph [3], where a proof of the local boundedness of weak solutions is also provided.

The aim of this paper is to establish the local boundedness for weak solutions of (1) in the case \( \alpha > 0 \) and \( p > 2 \). Our result extends the corresponding ones obtained in [3] to doubly degenerate parabolic systems. The main ingredient for our investigation is Moser’s iteration process. We refer to the bibliography of [3] for details on the regularity theory in the scalar case \( (N = 1) \). For existence results on boundary value problems involving equation (1), we refer to the paper by Alt and Luckaus [4].

A measurable function \( u \) is a local weak solution of (1) in \( Q_T \) if

\[
 u^i \in C_{\text{loc}} \left( [0, T], L^2_{\text{loc}}(\Omega) \right) \cap W^{1,p}_{\text{loc}} (Q_T), \quad |u|^{\alpha/p - 1} u^i \in L^p_{\text{loc}} \left( 0, T, W^{1,p}_{\text{loc}}(\Omega) \right),
\]

\( i = 1, \ldots, N \), and for all numbers \( t_1, t_2 \) such that \( 0 < t_1 < t_2 < T \) and all compact subsets \( K \subset \Omega \), the integral identity

\[
 \int_K u^i \Phi^i \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \left\{ -u^i \frac{\partial \Phi^i}{\partial t} + |u|^\alpha \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u^i}{\partial x_j} \frac{\partial \Phi^i}{\partial x_k} \right\} \, dx \, dt
\]

\( i = 1, \ldots, N \). We refer to [3] for the definition of the functions spaces used above. Under the above assumptions, we see that each term in the integral identity (3) makes sense. Let \( B(x_0, \rho) \) denote the ball \( \{ x : |x - x_0| < \rho \} \).

The main result of the paper is the following.

**THEOREM 1.** Let \( u \) be a local weak solution of system (1). Then \( u(x, t) \) is locally bounded in \( Q_T \), i.e., for any ball \( B(x_0, \rho) \), \( \rho > 0 \), such that \( B(x_0, \rho) \subset \Omega \), and for every \( t_1, t_2 \) satisfying \( 0 < t_1 < t_2 < T \), the following estimate holds:

\[
 \forall r \text{ max } \{ |u(x, t)| : (x, t) \in B(x_0, \rho) \times (t_1, t_2) \} \leq M,
\]

where \( M \) is a constant depending only on the data.

**2. PROOF OF THE THEOREM**

For convenience, we shall prove the theorem with \( \rho = 1 \), \( t_1 = t_0 - 1/2 \), \( t_2 = t_0 + 1/2 \), under the assumption that the cylinder \( Q_0 = B(x_0, 2) \times (t_0 - 1, t_0 + 1) \) lies inside \( Q_T \). Since system (1) is invariant (modulo a resealing of the constant \( C \)) under the two-parameter group defined by

\[
 \tilde{u}(x, t) = Su(Lx, Mt), \quad M = S^{a+1} - 1 L^p, \quad S, L > 0,
\]

appropriate transformations show that there is no loss of generality making such a restriction.

We start by stating the following interpolation inequality in a form suitable for our purposes; we refer to [3, p. 7] for a proof. For \( \sigma, q \geq 1 \), let \( v = v(x, t) \geq 0 \) be such that \( v^\sigma(x, t) \in W^{1,p}(Q_T^2) \) with

\[
 \sup_{|t-t_0| \leq L^p} \int_{B^2_2} v^\sigma(x, t) \, dx < \infty,
\]

then

\[
 \int_{Q^2_2} v^\beta(x, t) \, dx \, dt \leq C \left( \int_{Q^2_2} \left| \frac{\partial v^\sigma}{\partial x} \right|^p \, dx \, dt \right)^{1/p} \left( \sup_{|t-t_0| \leq L^p} \int_{B^2_2} v^\sigma(x, t) \, dx \, dt \right)^{\beta/p},
\]

where \( \beta = (p/n)q + \sigma \), \( C \) is a constant depending only on the data.
Let
\[ f_h(x,t) = \frac{1}{h} \int_t^{t+h} f(x,\tau) d\tau, \quad f_{h_k}(x,t) = \frac{1}{h} \int_{t-h}^{t} f(x,\tau) d\tau \]
be some Steklov averaging functions. Let \( K \) be a compact subset of \( \Omega \), \( 0 < \tau_1 < \tau_2 < T \) and denote the cylinder \( K \times (\tau_1, \tau_2) \) by \( Q_T \). We consider the functions
\[ \Phi^i(x,t) = \left( u_{h,k}^i(x,t) \psi \left( |u_h|^2 \right) \right)^{\frac{1}{r}} |\xi^i|^2(x,t), \quad i = 1, \ldots, N, \]
where \( \psi = \psi(s) \) is a continuously differentiable nonnegative function in \( R \), and \( \xi = \xi(x,t) \in C^1(Q_T) \) such that \( 0 \leq \xi \leq 1 \), and \( \xi(x,t) = 0 \) outside \( K \times (\tau_1, \tau_2) \); an appropriate choice of \( \xi \) will be done later on. The vector-function \( \Phi(x,t) \) defined by (6) is an admissible test function for the integral identity (4). Substituting it in (4) and performing some well-known calculations involving Young’s inequality along the same lines as in [5], we get
\[ \frac{1}{2} \int_\Omega \psi (|u|^2) \xi^i(x,t)|\xi^i|^2 \, dx + \nu \int_{Q_T} |u|^q |\frac{\partial u}{\partial x}|^p \xi^p \left[ \psi \left( |u|^2 \right) + |u|^2 \psi' \left( |u|^2 \right) \right] \, dx \, dt \leq \frac{p}{2} \int_{Q_T} \psi \left( |u|^2 \right) \xi^{p-1} \frac{\partial \xi}{\partial t} \, dx \, dt + C \int_{Q_T} |u|^{\alpha + p} \psi' \left( |u|^2 \right) \left[ \frac{\partial |u|}{\partial x} \right]^p + \xi^p \right) \, dx \, dt. \]
Let us choose the function \( \xi(x,t) \) as \( \xi(x,t) = \zeta(x)\eta(t) \), where \( \zeta \) is a continuously differentiable function in \( \mathbb{R}^n \), such that \( 0 \leq \zeta(x) \leq 1 \), \( \zeta(x) = 1 \) in the ball \( B_1^{(h)} = B(x_0, 1 + h) \) and \( \zeta(x) = 0 \) outside the ball \( B_2^{(h)} = B(x_0, 2 + h) \), \( \eta(t) \) is a continuously differentiable function in \( \mathbb{R} \), such that \( 0 \leq \eta(t) \leq 1 \), \( \eta(t) = 1 \) for \( |t - t_0| \leq h^p/2^p \), \( \eta(t) = 0 \) for \( |t - t_0| > h^p \), \( h \) is a positive number chosen in such a way that \( B(x_0, 1 + 2h) \subset \Omega \) and \( [t_0 - h^p, t_0 + h^p] \subset (0, T) \). Furthermore, we assume that \( \frac{\partial \zeta}{\partial x} \leq C/h \) and \( \frac{\partial \eta}{\partial t} \leq C/h^p \), \( C \) is a constant. Therefore, \( \xi(x,t) = 1 \) in the cylinder \( Q_1^{(h)} = B_1 \times [t_0 - h^p/2^p, t_0 + h^p/2^p] \) and \( \xi(x,t) = 0 \) outside \( Q_2^{(h)} = B_2 \times [t_0 - h^p, t_0 + h^p] \).

We choose the function \( \psi \) as
\[ \psi(s) = \begin{cases} \frac{s^{r/2}}{r}, & \text{if } |s| \leq k^2, \\ \left( \frac{r}{2} + 1 \right) s^{r/2} - k^{r/2} \frac{s^{r/2}}{r}, & \text{if } |s| > k^2, \end{cases} \]
with \( r \geq p \), where \( k \) is an arbitrary real number. We have
\[ \frac{\partial |u|}{\partial x_j} = u^i \frac{\partial u^i}{\partial x_j}, \]
thus,
\[ \left| \frac{\partial |u|}{\partial x} \right|^2 = \sum_{j=1}^n \left( \frac{\partial |u|}{\partial x_j} \right)^2 = \sum_{j=1}^n \frac{u^i}{|u|} \frac{\partial u^i}{\partial x_j} \frac{u^l}{|u|} \frac{\partial u^l}{\partial x_j}. \]
By Cauchy-Schwarz inequality, we have \( |\frac{\partial |u|}{\partial x_j}| < |\frac{\partial u}{\partial x_j}| \). Using this inequality together with relation (7) and the interpolation inequality (5) with \( q = r + 2 \), \( \sigma = (r + \alpha)(1/p) + 1 \), \( \beta = (r + 2)(1 + p/n) + \alpha + p - 2 \) and passing to the limit as \( k \to \infty \), we obtain
\[ \int_{Q_1^{(h)}} w^{\gamma q + \theta}(x,t) \, dx \, dt \leq C \left( 1 + h^{-p} \right)^\gamma \left( \int_{Q_2^{(h)}} w^{\gamma q + \theta}(x,t) + \text{meas} Q_2^{(h)} \right)^\gamma, \]
where \( \gamma = p/n + 1 \), \( \theta = \alpha + p - 2 \). We rewrite this inequality as
\[ \left( \int_{Q_1^{(h)}} w^{\gamma q + \theta}(x,t) \, dx \, dt \right)^{1/(\gamma q)} \leq C \left( 1 + h^{-p} \right)^{1/q} \left( \int_{Q_2^{(h)}} w^{\gamma q + \theta}(x,t) + h^p(1 + 2h)^n \right)^{1/q}. \]
Inequality (8) constitutes the basis for Moser’s iteration process. For \( l = 0, 1, 2, \ldots \), we introduce the sequences of numbers

\[
q = 2^l, \quad h_l = \frac{1}{2}, \quad \rho_l = 1 + h_l, \quad \tau^+_l = t_0 \pm s_l, \quad s_l = \frac{1}{2} + \frac{1}{2^l},
\]

and the cylinders \( Q_l = B(x_0, \rho_l) \times (\tau^-_l, \tau^+_l), Q_\infty = B(x_0,1) \times (t_0 - 1/2, t_0 + 1/2), Q_0 = B(x_0,2) \times (t_0 - 1, t_0 + 1). \) We have \( \rho_{l-1} = 1 + h_{l-1} = 1 + 2h_l \geq \rho_l, \quad \tau^+_{l-1} = t_0 \pm (1/2 + 1/2^l(l-1)) = t_0 \pm (1/2 + 2^l/2^{l-1}); \) thus, \( \tau^+_{l-1} \geq \tau^+_l \) and \( \tau^-_{l-1} \leq \tau^-_l. \) Hence, the sequence of cylinders \( Q_l \) is decreasing, i.e., \( Q_{l+1} \subset Q_l. \) We set

\[
\Theta_l = \left( \int_{Q_l} w^{2^l + \theta}(x,t) \, dx \, dt \right)^{1/(2^l)}. \tag{10}
\]

Recalling the definition of the cylinders \( Q_1^{(h)} \) and \( Q_2^{(h)} \) and the definition of \( \Theta_l, \) we derive from the estimate (8) the recurrent inequalities

\[
\Theta_{l+1} \leq [2C]^{p_l/(2^l)} \left[ \Theta_l + \left( \frac{1}{2^l} \right)^{1/(2^l)} \right], \quad l = 0, 1, 2, \ldots. \tag{11}
\]

Iterating these inequalities, we get

\[
\Theta_{l+1} \leq (2C)^{\sum_{j=1}^{\infty} p_j/(2^j)} \left[ \Theta_0 + \sum_{l=0}^{\infty} \left( \frac{1}{2^l} \right)^{1/(2^l)} \right].
\]

Since \([1/2^l]^{1/(2^l)} \leq [1/2^l]^{1}, \) by some simple calculations, we get

\[
\Theta_{l+1} \leq (2C)^{-(p_l/2)(1-n/\gamma)-1} \left[ \Theta_0 + \frac{1}{1 - 2^{-p}} \right].
\]

Passing to the limit in the left-hand side of this inequality, we get by Fatou’s lemma,

\[
vrai \max_{(x,t) \in Q_\infty} w(x,t) \leq C \left[ \int_{Q_0} w^{2^l + \theta}(x,t) \, dx \, dt + 1 \right].
\]

Recalling the definition of \( \theta, \) and that \( w = |u|, \) we get

\[
vrai \max_{(x,t) \in Q_\infty} |u(x,t)| \leq C \left[ \int_{Q_0} |u(x,t)|^{a+p} \, dx \, dt + 1 \right]
\leq C \left[ \int_{Q_0} \left| \frac{\partial u^{a/p+1}}{\partial x} \right|^p \, dx \, dt + 1 \right]
\leq C \left[ \int_{Q_0} u^a \left| \frac{\partial u}{\partial x} \right|^p \, dx \, dt + 1 \right].
\]

The last integral is bounded by the relations (2). Hence, the theorem is proved.

REFERENCES