Abstract

Necessary conditions for the existence of (3, 6) generalized Whist tournament designs on \(v\) players are that \(v \equiv 0, 1 \pmod{6}\). For \(v = 6n + 1\) it is shown that these designs exist for all \(n\). For \(v = 6n\), it is impossible to have a design for \(n = 1\), but for \(n > 1\) it is shown that designs exist, except possibly for 73 values of \(n\) the largest of which is \(n = 199\). A solution is also provided for the only unknown \((v, 6, 5)\) RBIBD, namely, \(v = 174\).

MSC: 05B05

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1. Introduction

In an earlier study [1] the concept of a generalized Whist tournament design was introduced. The definition given below is less general than that given in [1] but is sufficient for the present purposes.

**Definition 1.1.** Let \(t, k\) be integers such that \(t | k\). Let \(v \equiv 0, 1 \pmod{k}\). A \((t, k)\) generalized Whist tournament design on \(v\) players, denoted by \((t, k)\) GWhD\((v)\), is a (near) resolvable \((v, k, k - 1)\) BIBD on the \(v\)-set \(X\) whose blocks can be partitioned into \(k/t\) subblocks in such a way that (1) each subblock consists of \(t\) players, (2) players appearing in the same subblock are designated as partners (or teammates), (3) players...
appearing in the same block but not the same subblock are designated as opponents and (4) every player partners every other player exactly $t - 1$ times and every player opposes every other player exactly $k - t$ times.

The games (i.e., the blocks) of a $(t, k)$ GWhD($v$) will be displayed in the form $(a_1, \ldots, a_t; b_1, \ldots, b_t; \ldots)$ with the semicolons delineating the teams (i.e., the subblocks).

**Example 1.2.** (a) The initial round of a cyclic $(3, 6)$ GWhD$(7)$ is given by the single game $(1, 2, 4; 3, 6, 5)$ where $X = Z_7$.

(b) the initial round of a 1-rotational $(3, 6)$ GWhD$(12)$ is given by the two games $(\infty, 0, 1; 2, 4, 7)$ and $(3, 8, 10; 5, 6, 9)$ where $X = Z_{11} \cup \{\infty\}$.

GWhDs are a generalization of Whist tournament designs (which correspond to the case $(t, k) = (2, 4)$). An excellent account of these designs can be found in Chapter 11 of [5]. The case $(t, k) = (2, 6)$ was studied in [3] and the case $(t, k) = (4, 8)$ in [2,4]. Here existence questions for the case $(t, k) = (3, 6)$ are investigated. Our results are such that for $v = 6n + 1$ solutions exist for all $n$. Certainly the existence of a $(t, k)$ GWhD$(v)$ implies the existence of a $(v, t, t - 1)$ RBIBD. Therefore, no $(3, 6)$ GWhD$(6)$ can exist, since it is a well known fact that there exists no $(6, 3, 2)$ RBIBD (more generally, no $(2s, s, s - 1)$ RBIBD exists when $s$ is odd). We shall show that $(3, 6)$ GWhD$(6n)$ designs exist for all $n > 199$ and there are at most 73 possible smaller exceptions.

2. Some basics

In what follows consider the integer $e$ to be defined by $k = te$.

**Definition 2.1.** Let $(a_1, \ldots, a_k; \ldots, a_{e1}, \ldots, a_{et})$ denote any game in a $(t, k)$ GWhD$(v)$.

If every such game is rewritten in the form $(a_1, a_2, \ldots, a_{e1}, a_{12}, a_{22}, \ldots, a_{e2}, \ldots, a_{1t}, a_{2t}, \ldots, a_{et})$ and if the design consisting of these rewritten blocks is a $(v, k, 1)$ resolvable perfect Mendelsohn design [12] then the $(t, k)$ GWhD$(v)$ is said to be directed. Alternatively, the $(t, k)$ GWhD$(v)$ is said to possess the PMD Property.

Conveniently, the index $\lambda$ of a $(v, k, \lambda)$ PMD refers to the pair occurrences of points “$t$-apart” within the block; essentially a PMD is a $(v, k, \lambda(k - 1))$ BIBD with the PMD property.

Obviously then the existence of a directed $(t, k)$ GWhD$(v)$ implies the existence of a $(v, k, 1)$ RPMD. The converse of this result is stronger yet.

**Theorem 2.2.** Suppose $v \equiv 0, 1 \pmod{k}$ and there exists a $(v, k, 1)$ RPMD. Then there exists a directed $(t, k)$ GWhD$(v)$ for all $t$ such that $t | k$.

**Proof.** Let $t$ be such that $k = et$. Let $(a_1, \ldots, a_k)$ be any block of the $(v, k, 1)$ RPMD. Rearrange this block as follows:

$$(a_1, a_1 + e, a_1 + 2e, \ldots, a_1 + (t - 1)e; a_2, a_2 + e, \ldots, a_2 + (t - 1)e; \ldots; a_e, a_2e, \ldots, a_{et}).$$
Performing such rearrangements to every block of the RPMD and maintaining the (near) resolvability as in the RPMD the claim is that this collection of rearranged blocks constitutes the \((t,k)\) GWhD\((v)\). Clearly the above rearrangement indicates that any player, say \(x\), partners those players, and only those players, that are \(e\) units apart from \(x\), that are \(2e\) units apart from \(x, \ldots\), and that are \((t-1)e\) units apart from \(x\). Thus if \(y\) is any player distinct from \(x\) then it follows that \(y\) partners \(x\) exactly \(t-1\) times since in the RPMD the pair \(x,y\) occur \(e\) units apart exactly once, \(2e\) units apart exactly once, \(\ldots\), \((t-1)e\) units apart exactly once. Since the totality of blocks in which the pair \(x,y\) occur together is \(k-1\) it follows that \(y\) is an opponent of \(x\) exactly \(k-t\) times. Clearly this design is directed. \(\Box\)

An immediate corollary is the following.

**Corollary 2.3.** If \(t_i\) and \(t_2\) are such that \(t_i\mid k, i=1,2\) then there exists a directed \((t_1,k)\) GWhD\((v)\) if and only if there exists a directed \((t_2,k)\) GWhD\((v)\).

If \(v=kn+1\) is an odd prime power and if \(k=et\), it can be determined from the proof of Theorem 3.8 in [1] that the initial round of a cyclic \((t,k)\) GWhD\((v)\) is given by the games

\[
(x^a, x^{en+a}, x^{2en+a}, \ldots, x^{(t-1)en+a}, x^{n+a}, x^{(e+1)n+a}, x^{(2e+1)n+a}, \ldots, x^{(te-1)n+a})
\]

for \(a=0,1,\ldots, n-1\).

Here \(x\) denotes a primitive element for GF\((v)\). Rewriting these games in the manner prescribed in Definition 2.1 one obtains

\[
(x^a, x^{n+a}, x^{2n+a}, \ldots, x^{(te-1)n+a}), \text{ for } a=0,1,\ldots, n-1.
\]

For this rewritten design the \(j\)-apart differences, \(j=1,2,\ldots, k-1\), are given by \(x^{sm+a} (x^{jn-1})\), \(s=0,1,\ldots, te-1, a=0,1,\ldots, n-1\). These are precisely the elements of GF\((v)\)\(\backslash\{0\}\); consequently we have established the following theorem.

**Theorem 2.4.** If \(v=kn+1\) is an odd prime power then there exists a cyclic directed \((t,k)\) GWhD\((v)\) for every \(t\) such that \(t\mid k\).

In the sequel considerable use will be made of group divisible designs, transversal designs, pairwise balanced designs and frames. Definitions and pertinent information related to these structures can be found in [9]. Additional excellent references are the texts [5,8,10].

### 3. Examples and constructions

In this section we provide specific examples of \((3,6)\) GWhD\((v)\)s and \((3,6)\) GWh- Frames that are important for our subsequent existence arguments.
Example 3.1. (a) The initial round of a 1-rotational (3, 6) GWhD(24) is given by the following 4 games. We take as our point set $X = Z_{23} \cup \{\infty\}$.

$$(\infty, 0, 13; 16, 17, 8) \quad (18, 20, 21; 19, 3, 7),$$

$$(22, 4, 15; 6, 10, 12) \quad (1, 9, 14; 2, 5, 11).$$

(b) The initial round of a cyclic (3, 6) GWhD(25) is given by the following 4 games. We take as our point set $X = Z_{25}$.

$$(8, 14, 7; 22, 13, 10) \quad (17, 6, 3; 15, 23, 2),$$

$$(20, 12, 21; 1, 5, 24) \quad (19, 9, 4; 18, 16, 11).$$

(c) The initial round of a 1-rotational (3, 6) GWhD(30) is given by the following 5 games. We take as our point set $X = Z_{29} \cup \{\infty\}$.

$$(\infty, 0, 7; 18, 20, 26) \quad (11, 15, 17; 5, 16, 13) \quad (14, 4, 1; 10, 21, 25),$$

$$(28, 8, 9; 6, 22, 23) \quad (19, 24, 2; 12, 3, 27).$$

(d) The initial round of a 1-rotational (3, 6) GWhD(36) is given by the following 6 games. We take as our point set $X = Z_{33} \cup \{\infty\}$.

$$(\infty, 0, 33; 14, 23, 32) \quad (1, 25, 26; 12, 18, 31) \quad (6, 20, 21; 4, 10, 27),$$

$$(7, 11, 22; 9, 16, 19) \quad (2, 29, 34; 3, 17, 30) \quad (5, 24, 28; 8, 13, 15).$$

Definition 3.2. A $(k, k - 1)$ frame is said to be a $(t, k)$ GWhFrame (alternatively, the $(k, k - 1)$ frame is said to possess the $(t, k)$ generalized Whist property (the $(t, k)$ GWhP)), if and only if each block can be subdivided into teams of $t$ points (players) each so that every pair of points from distinct groups appear as partners (teammates) exactly $t$ times and as opponents exactly $k - t$ times. Furthermore if such a frame is directed then it will be denoted as $(k, 1)$ FPMD.

Due to the flexibility afforded by directed designs we will typically exhibit the blocks of such a design in the RPMD mode. If one wishes the associated $(t, k)$ mode then the rearrangement process described in the proof of Theorem 2.2 would be employed.

Example 3.3. A (3, 6) GWhD(96). We begin by constructing a directed RGDD with group type $(12)^8$. Take as point set $X = Z_{64} \cup \{\infty_1, \ldots, \infty_{12}\}$ and as groups $g_j = \{7i + j: 0 \leq i \leq 11\}$, $j = 0, \ldots, 6$ and $g_\infty = \{\infty_1, \ldots, \infty_{12}\}$. Define

$$B_1 = (33, 15, 13, 67, 17, 23), \quad B_2 = (5, 4, 29, 77, 80, 51),$$

$$B_3 = (\infty_1, 1, 14, 6, 74, 59), \quad B_4 = (\infty_4, 32, 61, 66, 63, 57),$$

$$B_5 = (\infty_7, 38, 0, 8, 68, 79), \quad B_6 = (\infty_{10}, 47, 2, 18, 20, 3).$$

The 16 blocks obtained by taking $B_i$ together with $B_i$, $25B_i + 28$ and $37B_i + 56$ for $2 \leq i \leq 6$ produces a parallel class. Here, $aB_i + x$ denotes the block obtained by multiplying all elements of $B_i$ by $a$ and then adding $x$ to them; we define $\infty_{j+1} = 25\infty_j + 28$ and $\infty_{j+2} = 37\infty_j + 56$ for $j = 1, 4, 7, 10$. Development of this parallel class modulo 84
yields the directed RGDD. Rewriting the blocks of this RGDD in the \((3,6)\) format provides 84 rounds of the \((3,6)\) GWhD(96). On each group of the RGDD construct a \((3,6)\) GWhD(12) and label the rounds as Round 1, ..., Round 11. The union of all the rounds labelled Round \(i\) produces a round of the \((3,6)\) GWhD(96). Thus we have 95 rounds in all. It is worthy of note that this \((3,6)\) GWhD(96) contains 8 mutually disjoint \((3,6)\) GWhD(12)s as sub-designs.

**Example 3.4.** A \((3,6)\) GWhD(192) and a \((3,6)\) GWhD(240). These designs are similar to the previous one (but are not directed). We construct RGDDs with group types \((24)^8\) and \((12)^{20}\) and the \((3,6)\) GWhP; then fill in each group with a \((3,6)\) GWhD(\(v\)) for \(v=24\) or 12.

For the RGDD of type \(24^8\), take as point set \(X=Z_{168} \cup \{x_1, \ldots, x_{24}\}\); groups are \(g_j = \{7i + j: 0 \leq i \leq 23\}, j=0,\ldots, 6\) and \(g_\infty = \{\infty_1, \ldots, \infty_{12}\}\). Let

\[
B_1 = (1, 81, 65; 12, 20, 52), \quad B_2 = (3, 131, 139; 58, 162, 74), \\
B_3 = (2, 62, 33; 137, 66, 85), \quad B_4 = (59, 159, 105; 113, 123, 86), \\
B_5 = (\infty_1, 15, 117; 55, 147, 130), \quad B_6 = (\infty_4, 29, 152; 63, 101, 146), \\
B_7 = (\infty_7, 82, 108; 148, 16, 77), \quad B_8 = (\infty_{10}, 84, 30; 80, 104, 89), \\
B_9 = (\infty_{13}, 13, 37; 116, 38, 99), \quad B_{10} = (\infty_{16}, 142, 144; 0, 118, 103), \\
B_{11} = (\infty_{19}, 17, 127; 42, 124, 67), \quad B_{12} = (\infty_{22}, 126, 83; 92, 88, 135).
\]

The required base blocks to be cycled mod 168 for this RGDD are then \(B_1, B_2\) plus \(B_i, 25B_i + 56\) and \(121B_i + 112\) for \(3 \leq i \leq 12\); these blocks form a parallel class. For this purpose, we define \(\infty_{j+1} = 25\infty_j + 56\) and \(\infty_{j+2} = 121\infty_j + 112\) for \(j=1, 4, 7, \ldots, 22\).

For the RGDD of type \(12^{20}\), take as point set \(X=Z_{228} \cup \{x_1, \ldots, x_{12}\}\); with \(g_j = \{19i + j: 0 \leq i \leq 11\}, j=0,\ldots, 18\) and \(g_\infty = \{\infty_1, \ldots, \infty_{12}\}\) as the groups. Let

\[
B_1 = (1, 125, 45; 4, 44, 180), \quad B_2 = (3, 123, 46; 156, 71, 127), \\
B_3 = (77, 189, 111; 161, 141, 86), \quad B_4 = (218, 64, 97; 182, 153, 81), \\
B_5 = (165, 209, 75; 212, 68, 31), \quad B_6 = (148, 66, 7; 22, 124, 23), \\
B_7 = (26, 122, 109; 92, 214, 89), \quad B_8 = (91, 157, 159; 2, 206, 95), \\
B_9 = (164, 146, 178; 188, 130, 195), \quad B_{10} = (187, 29, 152; 198, 98, 56), \\
B_{11} = (\infty_1, 110, 47; 208, 176, 137), \quad B_{12} = (\infty_4, 211, 39; 65, 5, 80), \\
B_{13} = (\infty_7, 136, 114; 6, 52, 67), \quad B_{14} = (\infty_{10}, 151, 72; 21, 61, 140).
\]

The required base blocks to be cycled mod 228 for this RGDD are then \(B_1\) plus \(B_i, 49B_i + 76\) and \(121B_i + 152\) for \(2 \leq i \leq 14\); these blocks form a parallel class. For this purpose, we define \(\infty_{j+1} = 49\infty_j + 76\) and \(\infty_{j+2} = 121\infty_j + 152\) for \(j=1, 4, 7, 10\).

We define a \((k, \lambda)\) GDD of type \(g^u u^1\) to be partially resolvable if its blocks can be partitioned into (1) a number of parallel classes plus (2) some partial parallel classes which miss only the group of size \(u\). Counting the number of replications of each point reveals that the number of parallel classes must be \(\lambda(v-u)/(k-1)\), and the number of partial parallel classes must be \(\lambda(u-g)/(k-1)\); since this last value must be \(\geq 0\), we require \(u \geq g\) for such a design. If further, \((u, k, \lambda)\) and \((g, k, \lambda)\) RBIBDs
exist, then such a GDD can be used to obtain a \((gm + u, k, \lambda)\) RBIBD in which the parallel classes are of 3 types. Firstly, \(\lambda(v - u)/(k - 1)\) parallel classes come from the GDD. Secondly, \(\lambda(g - u)/(k - 1)\) classes are obtained by combining a partial parallel class from the GDD with a parallel class from a \((u, k, \lambda)\) BIBD on the size \(u\) group. There then remain \(\lambda(g - 1)/(k - 1)\) unused parallel classes in this BIBD; each of these unused parallel classes is then combined with 1 parallel class from a \((g, k, \lambda)\) BIBD on each of the other groups. It is also not hard to see that if the GDD plus the filling \((u, k, \lambda)\) and \((g, k, \lambda)\) RBIBDs all possess the \((t, k)\) GWhP, then so does the resultant \((gm + u, k, \lambda)\) RBIBD.

The next lemma gives some GDDs of this type.

**Lemma 3.5.** \((6, 5)\) GDDs with the \((3, 6)\) GWhP exist of types \(g^m u^1\) for the following values of \(g, m, u\): (1) \(g = 12, m = 13, u = 18, 24, 30\); (2) \(g = 24, m = 7, u = 30\); (3) \(g = 12, m = 19, u = 30, 36\).

**Proof.** These designs are constructed like those in Example 3.4. In each case, we take the point set as \(X = Z_{gm} \cup \{\infty_1, \infty_2, \ldots, \infty_k\}\); for \(gm = 156, 168\) and 228, respectively, we take \(w = 61, 25\) and 49 as a cube of unity in \(Z_{gm}\). In each case we give some initial base blocks \(B_i\); base blocks to be cycled mod \(gm\) are then \(B_i \cdot w \cdot B_i + gm/3\), and \(w^2 \cdot B_i + 2gm/3\). For this purpose, for \(i = 1, 4, 7, \ldots, u - 2\) we define \(\infty_{i+1} = w \cdot \infty_i + gm/3\) and \(\infty_{i+2} = w^2 \cdot \infty_i + 2gm/3\). For some values of \(i\), \(B_i = w \cdot B_i + gm/3\) in which case these 3 blocks are identical and should be included once only. Also, in each case we indicate which base blocks generate the partial parallel classes; in each of these blocks all points are distinct mod 6 and for \(0 \leq x \leq 5\), adding \(x, x + 6, x + 12, \ldots, x + gm - 6\) to such a block produces a partial parallel classes missing the infinite points. The remaining base blocks form a parallel class.

Type 1213181: Here \(B_i = 61B_i + 52\) for \(i = 1, 2, 3\), while \(B_1\) generates the partial parallel classes.

\[
\begin{align*}
B_1 &= (1, 113, 81; 2, 18, 58), & B_2 &= (7, 11, 99; 10, 38, 30), \\
B_3 &= (49, 77, 69; 83, 123, 67), & B_4 &= (68, 152, 73; 5, 106, 127), \\
B_5 &= (72, 22, 34; 0, 142, 81), & B_6 &= (15, 107, 78; 101, 47, 70), \\
B_7 &= (\infty_1, 79, 33; 13, 103, 96), & B_8 &= (\infty_4, 46, 94; 131, 89, 18), \\
B_9 &= (\infty_7, 86, 23; 90, 128, 59), & B_{10} &= (\infty_{10}, 41, 143; 148, 16, 25), \\
B_{11} &= (\infty_{13}, 42, 112; 56, 12, 9), & B_{12} &= (\infty_{16}, 105, 116; 119, 109, 84).
\end{align*}
\]

Type 1213241: Here \(B_i = 61B_i + 52\) for \(i = 1, 2\), while \(B_1, B_2\) generate the partial parallel classes.

\[
\begin{align*}
B_1 &= (1, 113, 81; 2, 18, 58), & B_2 &= (7, 11, 99; 10, 38, 30), \\
B_3 &= (2, 44, 105; 35, 102, 85), & B_4 &= (37, 83, 155; 121, 107, 46), \\
B_5 &= (\infty_1, 47, 113; 13, 77, 106), & B_6 &= (\infty_4, 38, 0; 92, 8, 19), \\
B_7 &= (\infty_7, 7, 109; 134, 14, 87), & B_8 &= (\infty_{10}, 110, 60; 31, 63, 66), \\
B_9 &= (\infty_{13}, 95, 86; 141, 41, 62), & B_{10} &= (\infty_{16}, 26, 80; 36, 144, 51), \\
B_{11} &= (\infty_{19}, 120, 118; 4, 132, 45), & B_{12} &= (\infty_{22}, 133, 108; 153, 143, 112).
\end{align*}
\]
Lemma 3.6. If \( v \in \{ 180, 186, 198, 258, 264 \} \), then a (3,6) GWhD(v) exists. Also a (174,6,5) RBIBD exists.
Proof. Each of these designs are obtainable from one of the partially resolvable GDDs in the previous lemma; for \( v \neq 174 \), we can use \((3, 6)\) GWhD\((t)\)s for \( t = 12, 24, 30, 36 \) to fill in the groups. For \( v = 174 \), we use the partially resolvable GDD of type 1213181 and fill in the groups with \((12, 6, 5)\) and \((18, 6, 5)\) RBIBDs; \((18, 6, 5)\) RBIBD solutions can be found in [6, 11]. □

In Theorem 7.30 of [3], it was noted that a \((v, 6, 5)\) RBIBD exists for all positive integers \( v \equiv 0 \mod 6 \) except possibly \( v = 174 \). Updating this result, we now have the following theorem:

**Theorem 3.7.** For every positive integer \( v \equiv 0 \mod 6 \), a \((v, 6, 5)\) RBIBD exists.

**Example 3.8.** A \((3, 6)\) GWhFrame with group type \( 6^7 \). The point set is \( X = \{Z_5 \cup \{\infty\}\} \times Z_7 \) and the groups are \( g_i = \{Z_5 \cup \{\infty\}\} \times \{i\} \), \( 0 \leq i \leq 6 \). Let \( B_1 = \{(\infty, 4), (1, 1), (4, 3), (0, 5), (2, 2), (3, 6)\} \) and \( B_2 = \{(\infty, 3), (2, 5), (3, 1), (0, 2), (1, 6), (4, 4)\} \). Multiply \( B_1 \) and \( B_2 \) by \((1, y)\) for \( y = 1, 2, 4 \) to obtain a partial parallel class that misses the group \( g_0 \). Development of this partial parallel class modulo \((5, 7)\) produces 35 partial parallel classes for which each group is missed by exactly 5 of these partial parallel classes. The set of 6 blocks \( B_x \) where \( B_x = \{(x, 1), (x, 2), (x, 4), (x, 5), (x, 3)\} \) for \( x \in \{Z_5 \cup \{\infty\}\} \) constitutes a partial parallel class that misses \( g_0 \). Development of this partial parallel class modulo \((-, 7)\) yields a 6th partial parallel class for each group. It is easy to show that this frame possesses the \((3, 6)\) GWhP.

**Example 3.9.** A \((3, 6)\) GWhFrame with group type \( 6^9 \). The point set is \( X = \{Z_4 \times Z_{12}\} \cup \{\infty_1, \ldots, \infty_6\} \), and the groups are \( g_\infty = \{\infty_1, \ldots, \infty_6\} \) plus \( g_{ij} = \{(0 + i, 2k + j) : k = 0, 1, 2, 3, 4, 5\} \) for \( 0 \leq i \leq 3 \) and \( j = 0, 1 \). A partial parallel class that misses the group \( g_\infty \) is obtained by adding \((0, 3x)\) and \((1, 3x)\) for \( x = 0, 1, 2, 3 \) to the block \( ((0, 6), (2, 1), (3, 11); (2, 6), (0, 1), (1, 11)) \). A second partial parallel class that misses the group \( g_\infty \) is obtained by adding the same 8 ordered pairs to the block \( ((0, 0), (2, 11), (3, 7); (2, 6), (0, 5), (1, 1)) \). 6 partial parallel classes that miss the group \( g_\infty \) are obtained by adding \((0, y), y = 0, 1, 2\) to each of these partial parallel classes. A partial parallel class that misses the group \( g_{00} \) is given by the following collection of 8 blocks.

\[
\begin{align*}
((2, 0), (2, 7), (3, 1); (1, 0), (3, 4), (0, 3)), \\
((2, 10), (2, 9), (1, 6); (0, 1), (3, 8), (3, 5)), \\
(\infty_1, (3, 2), (3, 9); (2, 1), (0, 11), (1, 1)), \\
(\infty_2, (0, 7), (2, 11); (1, 2), (1, 11), (2, 2)), \\
(\infty_3, (2, 4), (3, 10); (3, 3), (1, 3), (2, 5)), \\
(\infty_4, (1, 7), (2, 6); (3, 0), (1, 10), (0, 9)), \\
(\infty_5, (1, 9), (3, 6); (3, 7), (1, 8), (2, 3)), \\
(\infty_6, (0, 5), (2, 8); (3, 11), (1, 5), (1, 4)).
\end{align*}
\]

Developing this latter partial parallel class modulo \((4, 12)\) completes the frame. It is not difficult to verify that this frame possesses the \((3, 6)\) GWhP.
Example 3.10. A (3, 6) GWhFrame with group type $6_{10}$. The point set is $X = \{Z_2 \times Z_3 \times Z_3 \times Z_3 \} \cup \{\infty_1, \ldots, \infty_6\}$. For ease of notation an element of the form $((i, j, k, l))$ will be abbreviated to $ijkl$. The groups are $g_{xy}$, $x, y \in Z_3$ with $g_{xy} = \{0xy0, 1xy0, 0xy1, 1xy1, 0xy2, 1xy2\}$ and $g_\infty = \{\infty_1, \ldots, \infty_6\}$. Basic blocks are

- $B_1 = (0101, 0222, 0011; 1111, 1022, 1202)$,
- $B_2 = (0101, 1200, 1011; 0121, 0022, 1110)$,
- $B_3 = (0200, 0210, 1012; 0110, 0102, \infty_1)$,
- $B_4 = (0010, 1210, 1111; 1120, 1100, \infty_4)$.

Define an automorphism $T$ of order 3 on the points by $T(z, a, b, c) = (z, 2a + b, 2a + c)$ and $T(\infty_i) = \infty_{i+1}$, $i = 1, 2, 4, 5$; $T(\infty_3) = \infty_4$, $T(\infty_6) = \infty_4$. Apply the group of order 3 generated by $T$ to $B_2, B_3, B_4$ to obtain 9 blocks which form a partial parallel class that misses the group $g_{00}$. Development of this partial parallel class via the elements of $Z_2 \times Z_3 \times Z_3 \times Z_3$ produces 54 partial parallel classes precisely 6 of which miss each group $g_{xy}$. Indeed, noting that the groups $g_{xy}$ occur in additive inverse pairs with $g_{00}$ a self inverse, it is clear that if $w$ denotes the additive inverse of an element in $g_{xy}$ then adding $w$ to each element of $g_{00}$ produces the group that is the additive inverse pair to $g_{xy}$. Adding the 9 values $00xy, x, y \in Z_3$ to the block $B_1$ yields a partial parallel class that misses $g_\infty$. Developing this latter partial parallel class modulo $(2, 3, -, -)$ gives 6 partial parallel classes that miss $g_\infty$. It is a straightforward exercise to verify that this frame possesses the $(3, 6)$ GWhP.

Example 3.11. There exist $(3, 6)$ GWhFrames with group type $1^{6s+1}$ for every $s$ for which there exists a $(3, 6)$ GWhD$(6s+1)$. Indeed, if $X$ denotes a set such that $|X| = 6s + 1$ then the groups of the frame are $\{x\}, x \in X$ and the partial parallel class that misses the group $\{x\}$ is the round of the $(3, 6)$ GWhD$(6s + 1)$, constructed on $X$, that omits $x$. Note, in particular, that there exists a $(3, 6)$ GWhFrame with group type $1^{37}$.

Example 3.12. A directed $(3, 6)$ GWhFrame with group type $1^{3671}$ can be obtained as follows. Take as point set $X = Z_{36} \cup \{\infty_1, \ldots, \infty_7\}$ and for groups take $g_x = \{x\}, x \in Z_{36}$ and $g_\infty = \{\infty_1, \ldots, \infty_7\}$. One partial parallel class that misses the group $g_\infty$ is given by the six blocks obtained by adding $0, 1, 2, 3, 4, 5$ to the block $(0, 6, 12, 18, 24, 30)$. A second partial parallel class that misses the group $g_\infty$ is given by the 6 blocks obtained by adding $0, 6, 12, 18, 24, 30$ to the block $(0, 22, 21, 32, 7, 23)$. If we label this latter partial parallel class as $A$ then 5 additional partial parallel classes that miss the group $g_\infty$ are obtained by adding $1, 2, 3, 4, 5$ to $A$. A partial parallel class that misses the group $g_0$ is given by the following 7 blocks.

- $(4, 6, 18, 27, 9, \infty_1), (10, 12, 3, 24, 5, \infty_2), (26, 21, 13, 25, 29, \infty_3), (7, 11, 1, 34, 17, \infty_4), (30, 2, 28, 35, 22, \infty_5), (8, 23, 16, 19, 33, \infty_6), (14, 20, 15, 31, 32, \infty_7)$.

If we denote this latter partial parallel class by $B$ then adding $x$ to $B, x \in Z_{36}$, produces a partial parallel class that misses the group $g_x$. It is a routine matter to show that this frame is directed.
Example 3.13. A directed (3, 6) GWhFrame with group type $1^{35}7^2$ can be obtained by first forming a 7-GDD($1^{35}7^2$) which is obtained by forming a resolvable (49, 7, 1) BIBD, and taking 2 blocks in one of its parallel classes as groups. On each block in this GDD, we form a directed (3, 6) GWhD($1^7$) (or directed (3, 6) GWhD($7$)). For any point $x$, take the partial parallel classes in the directed (3, 6) GWhD($7$)s on all blocks from 7-GDD($1^{35}7^2$) that contain a point in the same group as $x$; these constitute a partial parallel class missing the group containing $x$. There are 1 or 7 such partial parallel classes, depending on whether $x$ is in a group of size 1 or 7. It is clear that the frame constructed is a directed (3, 6) GWhFrame. A more general form of this construction is given later in Theorem 4.10.

Using a method somewhat analogous to that employed in Example 3.13 one can obtain a (3, 6) GWhFrame with group type $1^{36}7^1$ by adding a new point, say $\infty$, to each group of the (3, 6) GWhFrame of type 67 given in Example 3.8. The frame given in Example 3.12 is preferred since it is directed. A directed $(t, k)$ frame of any type implies existence of $k-1$ HMOLS of that type; thus this frame gives 5 HMOLS of type $1^{36}7^1$ or 5 IMOLS(43, 7).

The three (3, 6) GWhD frames of types $1^{37}$, $1^{36}7^1$ and $1^{35}7^2$ obtained in Examples 3.11–3.13 are crucial ingredients for our asymptotic existence results in the case $v=6n$ (see Theorems 5.27 and 5.29 below). They also provide an essential tool for the following “Oval Point Construction”. Note that, in the proof of Theorem 3.14, the production of the TD($q + 1, q$) can be brought about in a variety of ways. The approach employed here is in keeping with some of the specific (3, 6) GWhD($v$) constructed later (see Example 5.8).

Theorem 3.14. Suppose that $q$ is an odd prime power such that $q \equiv 5 \pmod{6}$ and $q > 73$. Then

(a) if (3, 6) GWhD($v$) exist for each of $v=q+1, q+7, q+13$ then (3, 6) GWhD($z$) exist for all $37q + 1 \leq z \leq 37q + 445$, $z \equiv 0 \pmod{6}$,

(b) if (3, 6) GWhD($v$) exist for each of $v=q+1, q+7$ then (3, 6) GWhD($z$) exist for all $37q + 1 \leq z \leq 37q + 223$, $z \equiv 0 \pmod{6}$,

(c) if (3, 6) GWhD($v$) exist for each of $v=q+7, q+13$ then (3, 6) GWhD($z$) exist for all $37q + 223 \leq z \leq 37q + 445$, $z \equiv 0 \pmod{6}$,

(d) if (3, 6) GWhD($v$) exist for each of $v=q+1, q+13$ then (3, 6) GWhD($z$) exist for all $z=37q + 1, 37q + 13, 37q + 25, \ldots, 37q + 445$.

Proof. Begin with the projective plane PG(2, $q$) and let $\mathcal{O}$ denote an oval in this projective plane. Of course $\mathcal{O}$ contains $q + 1$ points. Select a non-oval tangent point, say $x$, and eliminate this point from the design using its lines to define the groups of a TD($q + 1, q$). Note that, in the original PG(2, $q$), $x$ belonged to 2 tangent lines and $s=(q-1)/2$ secant lines, thus there are $s$ groups in the TD($q + 1, q$) that contain 2 oval points and 2 groups that contain 1 oval point. Since $s>37$ one obtains a TD(37, $q$) with 37 groups each having 2 oval points simply by deleting groups until there are 37 left, being careful to ensure that the remaining groups are those that have 2 oval points.
Next one gives \( r \) of the oval points a weight of 7, \( 0 \leq r \leq 74 \), and proceeds precisely as in the proof of Theorem 5.29 using the construction contained in Theorem 5.27. Since the details are elaborated below we will not repeat them here. \( \square \)

**Remark 3.15.** If \( q < 73 \) then one can still utilize the approach employed in Theorem 3.14 with shortened ranges for the \( z \). A special case to be employed later is that of \( q = 71 \). Here the \( TD(q + 1, q) \) will contain 2 groups with 1 oval point and \((q - 1)/2 = 35\) groups with 2 oval points. In addition here, we have \((3, 6) GWhD(v)\) for \( v = q + 7, q + 13 \), but not \( q + 1 \). Thus one constructs the \( TD(37, q) \) by retaining these 37 groups and deleting all others.

**Remark 3.16.** In the proof of Theorem 3.14, we pivoted on a non-oval tangent point. If we had chosen an oval point to pivot on, then we would have constructed a \( TD(q + 1, q) \) with \( q \) groups containing a single oval point, and one group containing no oval points when \( q \) is odd (and \( q + 1 \) groups containing a hyperoval point when \( q \) is even).

**Example 3.17.** A \((3, 6) GWhFrame\) with group type \( 6^3 18^1 \). The point set is \( X = Z_{186} \cup \{ \infty, \ldots, \infty_{18} \} \) and the groups are \( g_i = \{ 0 + i, 31 + i, 62 + i, \ldots, 155 + i \}, \ i = 0, 1, \ldots, 30 \) and \( g_\infty = \{ \infty, \ldots, \infty_{18} \} \). Consider the following set of blocks.

\[
\begin{align*}
B_1 &= (0, 34, 33; 122, 5, 1), & B_2 &= (33, 97, 181; 117, 69, 152), \\
B_3 &= (100, 132, 105; 130, 166, 45), & B_4 &= (184, 42, 142; 46, 74, 71), \\
B_5 &= (99, 125, 170; 53, 143, 30), & B_6 &= (96, 110, 157; 18, 10, 1), \\
B_7 &= (79, 139, 98; 38, 118, \infty_1), & B_8 &= (172, 50, 113; 3, 43, \infty_2), \\
B_9 &= (147, 29, 59; 26, 14, \infty_3), & B_{10} &= (183, 63, 60; 49, 11, \infty_4), \\
B_{11} &= (126, 36, 76; 73, 84, \infty_5), & B_{12} &= (119, 169, 162; 77, 182, \infty_6).
\end{align*}
\]

Adding \( 6i, \ i = 0, 1, \ldots, 30 \) to \( B_1 \) produces 31 blocks that constitute a partial parallel class that misses \( g_\infty \). Denoting this partial parallel class by \( A_1 \) and defining \( A_2, A_3 \) by \( A_2 = 25A_1 \) and \( A_3 = 67A_1 \) with all arithmetic modulo 186, we find that both \( A_2 \) and \( A_3 \) form partial parallel classes that miss \( g_\infty \). Adding \( 0, 1, 2, 3, 4, 5 \) to each of \( A_1, A_2, A_3 \) produces 18 parallel partial classes that miss \( g_\infty \). If, for \( 1 \leq j \leq 6 \), we define \( 25\infty_j = \infty_{j+6}, \ 67\infty_j = \infty_{j+12} \) and denote by \( B \) the collection of 33 blocks \( B_1, 25B_1, 67B_1, i = 2, 3, \ldots, 12 \), then \( B \) constitutes a partial parallel class that misses \( g_0 \). Development of \( B \) via the elements of \( Z_{186} \) produces 186 partial parallel classes exactly 6 of which miss \( g_i, \ i = 0, \ldots, 30 \). Once again it is a straightforward task to show that this frame possesses the \((3, 6) GWhP\).

**Example 3.18.** A \((3, 6) GWhFrame\) with group type \((12)^3 (36)^1\). The point set is \( X = Z_{228} \cup \{ \infty, \ldots, \infty_{36} \} \) and the groups are \( g_i = \{ 0 + i, 19 + i, 38 + i, \ldots, 209 + i \}, \ i = 0, \ 1, \ldots, 18 \) and \( g_\infty = \{ \infty, \ldots, \infty_{36} \} \). Consider the following set of blocks.

\[
\begin{align*}
B_1 &= (0, 166, 9; 158, 41, 73), & B_2 &= (0, 70, 125; 105, 7, 188), \\
B_3 &= (24, 192, 118; 120, 84, 197), & B_4 &= (1, 135, 79; 103, 137, 146), \\
B_5 &= (4, 34, 153; 154, 66, \infty_1), & B_6 &= (113, 11, 174; 29, 177, \infty_2),
\end{align*}
\]
Adding 6i, \( i = 0, 1, \ldots, 37 \) to \( B_1 \) produces 38 blocks that constitute a partial parallel class that misses \( g_\infty \). Denote this partial parallel class by \( A_1 \). Performing the exact same addition process to the block \( B_2 \) produces another partial parallel class that misses \( g_\infty \). Denote this latter parallel class by \( A_4 \). Define \( A_2 = 49A_1 \), \( A_3 = 121A_1 \), \( A_5 = 49A_4 \) and \( A_6 = 121A_4 \) with the arithmetic performed modulo 228, we find that each of \( A_1 \) through \( A_6 \) form partial parallel classes that miss \( g_\infty \). Adding 0, 1, 2, 3, 4, 5 to each of these partial parallel classes produces 36 partial parallel classes that miss \( g_\infty \). If we define \( 49 \infty_j = \infty_{j+12}, 121 \infty_j = \infty_{j+24} \) for \( 1 \leq j \leq 12 \), and denote by \( B \) the collection of 42 blocks \( B_1, 49B_1, 121B_1, i = 3, 4, \ldots, 16 \), then \( B \) constitutes a partial parallel class that misses \( g_0 \). Development of \( B \) via the elements of \( Z_{228} \) produces 228 partial parallel classes exactly 12 of which miss \( g_i \), \( i = 0, 1, 2, \ldots, 18 \). Once again it is a straightforward task to show that this frame possesses the \((3,6)\) GWhP.

4. Existence for the case \( v = 6n + 1 \)

The following two results already appear in the literature and considerably facilitate obtaining new existence results for \((3,6)\) GWhD(6n + 1).

Theorem 4.1 (Abel et al. [1, Theorem 4.6]). Suppose there exists a PBD(K, v) such that for each \( u \in K \), \( u = 6n + 1 \) and there exists a \((t, k)\) GWhD(\( u \)). Then there exists a \((t, k)\) GWhD(\( v \)). Further, if the \((t, k)\) GWhD(\( u \))s are directed for all \( u \in K \), then so is the \((t, k)\) GWhD(\( v \)).

Theorem 4.2 (Baker [7, Table 3.18]). If \( v = 6n + 1 \) and \( v \not\in S = \{55, 115, 145, 205, 235, 265, 319, 355, 391, 415, 445, 451, 493, 649, 667, 685, 697, 745, 781, 799, 805, 1315\} \), then there exists a PBD(K, v) such that for all \( u \in K \), \( u \) is a prime power of the form \( 6m + 1 \).

Combining Theorems 2.4, 4.1 and 4.2 leads to the following.

Theorem 4.3. Let \( S \) be the set defined in Theorem 4.2. If \( v = 6n + 1 \) and \( v \not\in S \) then there exists a directed \((3,6)\) GWhD(\( v \)).

The following theorem is proven in [3, Theorem 6.8].

Theorem 4.4. If \( v \in T = \{55, 115, 145, 445, 685, 745, 799, 805\} \), then there exists a directed \((2,6)\) GWhD(\( v \)).
It follows from Corollary 2.3 that there exist directed \((3,6)\) GWhD\((v)\) for all \(v \in T\) where \(T\) is the set given in Theorem 4.4. Several of the remaining cases can be settled utilizing the following theorem.

**Theorem 4.5** (Abel et al. [1, Theorem 4.15]). Suppose we have an RTD\((km+1, kn+1)\) which is given by a km + 1 by kn + 1 difference matrix over an Abelian group, \(G\), of order \(kn + 1\) and a \((kn + 1, k, k - 1)\) NRBIBD which is generated by a difference family over \(G\). Suppose also, \(0 \leq w \leq n\), and a \((km, k, k - 1)\) RBIBD, a \((km + 1, k, k - 1)\) NRBIBD plus a \((kw + 1, k, k - 1)\) NRBIBD all exist. Then \(a (km(kn + 1) + kw + 1, k, k - 1)\) NRBIBD exists. Furthermore if the RBIBD and all the input NRBIBDs have the \((t,k)\) generalized Whist property then so does the resultant NRBIBD.

**Theorem 4.6.** If \(v \in \{235, 319, 391, 451, 781, 1315\}\) then there exists a \((3,6)\) GWhD\((v)\).

**Proof.** All indicated cases can be handled via an application of Theorem 4.5. In each case \(k = 6\) and the Abelian group is \(GF(6n + 1)\). In all cases it is a fact that \(6n + 1\) is a prime or a prime power greater than or equal to 13 (and, of course, greater than or equal to \(6m + 1\)) thus the existence of the required \(km + 1\) by \(kn + 1\) difference matrix is guaranteed by Corollary 2.5.4 in [10]. We indicate the appropriate values as 4-tuples in the form \((v; km, kn + 1, kw + 1)\).

\[
(235; 12, 19, 7), \quad (319; 12, 25, 19), \quad (391; 12, 31, 19), \\
(451; 12, 37, 7), \quad (781; 12, 61, 49), \quad (1315; 12, 103, 79). \quad \square
\]

**Theorem 4.7.** If a \((t,k)\) GWhFrame of type \((s_1, s_2, \ldots, s_n)\) and a resolvable TD\((k,g)\) both exist, then a \((t,k)\) GWhFrame of type \((g s_1, g s_2, \ldots, g s_n)\) exists.

**Proof.** Let \(S_i, (i = 1, \ldots, n)\) be the groups of sizes \(s_i\) in the original frame, for TD\((k,g)\) we take the groups as \(x \times G\) where \(x \in I_k\) and \(G\) is a set of size \(g\). The required frame of type \((g s_1, g s_2, \ldots, g s_n)\) will have groups of the form \(S_i \times G\); its blocks are then \(((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k))\) for each block \((b_1, b_2, \ldots, b_k)\) in the original frame and each block \((1, c_1), (2, c_2), \ldots, (k, c_k)\) in TD\((k,g)\). It is not hard to see that if the original frame possesses the \((t,k)\) GWhP then so does the required frame. Further, if FP is a partial parallel class missing \(S_i\) in the original frame, and TP is a parallel class in RTD\((k,g)\) then the blocks of the form \(((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k))\) for \((b_1, b_2, \ldots, b_k) \in FP\) and \((1, c_1), (2, c_2), \ldots, (k, c_k) \in TP\) form a partial parallel class missing \(S_i \times G\) in the required frame. \(\square\)

**Theorem 4.8.** Suppose there exist a \((t,k)\) GWhD\((ks_i + 1)\) for \(1 \leq i \leq n\), and a \((t,k)\) GWhFrame of type \((ks_1, ks_2, \ldots, ks_n)\). Then there exists a \((t,k)\) GWhD\((v)\) for \(v = (\sum_{i=1}^{n} ks_i) + 1\).

**Proof.** Add a new point, say \(\infty\), to the frame, and on each group of size \(ks_i\) \((1 \leq i \leq n)\) form a \((t,k)\) GWhD\((ks_i + 1)\). The partial parallel class missing \(\infty\) is obtained by combining the partial parallel classes missing \(\infty\) from each \((t,k)\) GWhD\((ks_i + 1)\). The partial parallel class missing any other point \(x\), in a group of size \(ks_i\), is obtained by
combining a partial parallel class from the frame missing this group and the partial parallel class missing \(x\) in the \((t,k)\) GWhD\((ks_i + 1)\) on this group plus \(\infty\).

**Theorem 4.9.** If \(v \in \{205, 265\}\), then there exists a \((3,6)\) GWhD\((v)\).

**Proof.** These designs are obtained using Theorem 4.8. For \(v = 205, 265\), take the \((3,6)\) GWhFrames of types 631181 and 1219361 in Examples 3.17 and 3.18; add an infinite point and form a \((3,6)\) GWhD\((w)\) for \(w = 7, 13, 19, 37\) on each group of the frame plus an infinite point. 

Our main tool for resolving the final \(v = 6n + 1\) cases is a variation of Wilson’s fundamental construction.

**Theorem 4.10** (Abel et al. [1, Theorem 4.6]). Suppose we have a “master” \((K,1, GDD\) with group type vector \((|G_j| : j = 1, \ldots, g)\) and a weighting that assigns a positive weight of \(w(x)\) to each point \(x\). Let \(W(B_i)\) be the weight vector of the \(i\)th block. If for every block \(B_i\), we have an ingredient \((k, \lambda)\) frame with a group type vector \(W(B_i)\), then there exists a \((k, \lambda)\) frame with a group size vector \((\sum_{x \in G_j} w(x) : j = 1, \ldots, g)\). Furthermore, if all the ingredient frames possess the \((t,k)\) generalized Whist property, then, so does the resultant frame.

**Theorem 4.11.** If \(v \in \{355,415,493,649,667,697\}\), then a \((3,6)\) GWhD\((v)\) exists.

**Proof.** Take a TD\((k,m)\) for \(k = 8\) or \(9\), and truncate one group to size \(s\). Then give all points a weight of \(6\) in Theorem 4.10 to obtain \((3,6)\) GWhFrames of type \((6m)^{k-1}(6s)^1\) using \((3,6)\) GWhFrames of type \(6^p\), \((n=7,8,9)\) as ingredients; finally apply Theorem 4.8. These ingredients are given in Examples 3.8 and 3.9 for \(n = 7, 9\); for \(n = 8\), delete 1 point from AG\((2,7)\) and using a \((3,6)\) GWhD\((7)\) (treated as a \((3,6)\) GWhFrame of type \(1^7\)) as the ingredient in Theorem 4.10 to give a \((3,6)\) GWhFrame of type \(6^8\). The values of \((k,m,s)\) needed are \((8,8,3), (8,9,6), (8,11,5), (9,13,4), (9,13,7), (8,16,4)\).

Combining all of the materials of this section enables us to state the following existence result for the case \(v = 6n + 1\).

**Theorem 4.12.** A \((3,6)\) GWhD\((v)\) exists for all \(v \equiv 1 \mod 6\).

### 5. Existence for the case \(v = 6n\)

Towards the end of this section we prove the asymptotic result that \((3,6)\) GWhD\((6n)\) exist for all \(6n \geq 6624\). First, we focus on those cases with \(6n < 6624\) using several constructions that already appear in literature along with examples and constructions introduced in this study. We point out that for economy of space, we have employed a presentation that is not strictly sequential; some examples require information that is found...
in examples occurring later in the exposition. Recall that a \((3, 6)\) GWhD(12) appears in Example 1.2 and that \((3, 6)\) GWhD(6n) for \(n=4, 5, 6, 16, 30, 31, 32, 33, 40, 43, 44\) appear in Section 3. Note that the designs for \(n=16, 30, 31, 40, 43\) and 44 contain several disjoint \((3, 6)\) GWhD(12) subdesigns.

**Theorem 5.1** (Abel et al. [1, Lemma 4.12]). If a \((km, k, k−1)\) RBIBD exists and an RTD\((k, kn + 1)\) exists and a \((kn + 1, k, k−1)\) NRBIBD exists, then there exists a \((km(kn + 1), k, k−1)\) RBIBD containing \(kn + 1\) disjoint \((km, k, k−1)\) RBIBDs as subdesigns. Furthermore if the input RBIBD and NRBIBD possess the \((t, k)\) generalized Whist property then so does the resultant RBIBD.

**Example 5.2.** The data listed below reflects those \(v\) for which one can obtain a \((3, 6)\) GWhD\((v)\) using Theorem 5.1. The data are in the form \((v; 6m, 6n + 1)\).

<table>
<thead>
<tr>
<th>(v)</th>
<th>(6m)</th>
<th>(6n + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>84; 12, 7</td>
<td>(156; 12, 13),</td>
<td>(168; 24, 7),</td>
</tr>
<tr>
<td>228; 12, 19,</td>
<td>(252; 36, 7),</td>
<td>(300; 12, 25),</td>
</tr>
<tr>
<td>372; 12, 31,</td>
<td>(390; 30, 13),</td>
<td>(444; 12, 37),</td>
</tr>
<tr>
<td>468; 36, 13,</td>
<td>(516; 12, 43),</td>
<td>(546; 78, 7),</td>
</tr>
<tr>
<td>588; 12, 49,</td>
<td>(600; 24, 25),</td>
<td>(744; 24, 31),</td>
</tr>
<tr>
<td>876; 12, 73,</td>
<td>(888; 24, 37),</td>
<td>(930; 30, 31),</td>
</tr>
<tr>
<td>1008; 144, 7,</td>
<td>(1014; 78, 13),</td>
<td>(1020; 12, 85),</td>
</tr>
<tr>
<td>1092; 12, 91,</td>
<td>(1116; 36, 31),</td>
<td></td>
</tr>
</tbody>
</table>

**Theorem 5.3** (Abel et al. [1, Lemma 4.13]). If a \((km, k, k−1)\) RBIBD exists and a RTD\((k, kn)\) exists and a \((kn, k, k−1)\) RBIBD exists then there exists a \((kmkn, k, k−1)\) RBIBD containing \(kn\) disjoint \((km, k, k−1)\) RBIBDs as subdesigns and containing \(km\) disjoint \((kn, k, k−1)\) RBIBDs as subdesigns. Furthermore if the input RBIBDs possess the \((t, k)\) generalized Whist property then so does the resultant RBIBD.

**Example 5.4.** This data reflects those \(v\) for which \((3, 6)\) GWhD\((v)\) can be constructed via an application of Theorem 5.1. The data are in the form \((v; 6m, 6n)\).

<table>
<thead>
<tr>
<th>(v)</th>
<th>(6m)</th>
<th>(6n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>144; 12, 12,</td>
<td>(288; 12, 24),</td>
<td>(360; 12, 30),</td>
</tr>
<tr>
<td>432; 12, 36,</td>
<td>(864; 24, 36).</td>
<td></td>
</tr>
</tbody>
</table>

**Theorem 5.5** (Abel et al. [1, Lemma 4.20]). Suppose a \((km + 1, k, k−1)\) NRBIBD, a RTD\((k, kn − 1)\), and a \((kn, k, k−1)\) RBIBD all exist. Then there exists a \((km(kn + 1) + 1, k, k−1)\) RBIBD containing a \((kn, k, k−1)\) RBIBD subdesign. Furthermore, if the NRBIBD and the input RBIBD have the GWhP, then so does the resultant design.

The following theorem is a special case of Theorem 5.5 that guarantees, under the conditions specified, that the resultant design is cyclic. An analog of this theorem is established in [3]. The terminology “Z-cyclic” means that the underlying point sets are, respectively, \(Z_Q \cup \{\infty\}\), \(Z_P\) and \(Z_{QP} \cup \{\infty\}\).

**Theorem 5.6.** Let \(Q>5\), \(Q \equiv 5 \pmod{6}\), \(P \equiv 1 \pmod{6}\), where Z-cyclic \((3, 6)\) GWhD\((Q + 1)\) and \((3, 6)\) GWhD\((P)\) and a \((Q, 7; 1)\)-DM exist. Then a Z-cyclic \((3, 6)\) GWhD\((QP\)
The latter design contains a \((Q+1, 6, 5)\) RBIBD subdesign that possesses the \((3, 6)\) GWhP.

**Example 5.7.** The following data reflect those values of \(v\) for which a \((3, 6)\) GWhD\((v)\) can be constructed via Theorem 5.5. The data are in the form \((v; Q, P, a)\), with \(a=Q\) if Theorem 5.6 applies, and in the form \((v; 6n - 1, 6m + 1, a)\), with \(a=n\) otherwise. Note that Theorem 5.6 covers the same parameter sets as Theorem 5.5.

\[
\begin{align*}
(78; 11, 7, Q), & \quad (162; 23, 7, Q), & \quad (204; 29, 7, Q), & \quad (210; 11, 19, Q), \\
(246; 35, 7, n), & \quad (276; 11, 25, Q), & \quad (300; 23, 13, Q), & \quad (342; 11, 31, Q), \\
(378; 29, 13, Q), & \quad (408; 11, 37, Q), & \quad (438; 23, 19, Q), & \quad (474; 11, 43, Q), \\
(540; 11, 49, Q), & \quad (552; 29, 19, Q), & \quad (576; 23, 25, Q), & \quad (582; 83, 7, n), \\
(666; 95, 7, n), & \quad (738; 11, 67, Q), & \quad (804; 11, 73, Q), & \quad (852; 23, 37, Q), \\
(870; 11, 79, Q), & \quad (900; 29, 31, Q), & \quad (936; 11, 85, n), & \quad (990; 23, 43, Q), \\
(1002; 11, 91, Q), & \quad (1068; 11, 97, Q), & \quad (1074; 29, 37, Q), & \quad (1080; 83, 13, n), \\
(1086; 35, 31, n), & \quad (1128; 23, 49, Q), & \quad (1134; 11, 103, Q).
\end{align*}
\]

**Example 5.8.** By applying Theorem 3.14(c) with \(q=71\) together with the adjustments mentioned in Remark 3.15, one obtains \((3, 6)\) GWhD\((v)\) for all \(v \equiv 0 \pmod{6}\) in the range \([37.71 + 37.6 + 1, 37.71 + 35.12 + 2.6 + 1]\), that is \([2850, 3060]\).

**Definition 5.9.** An incomplete RBIBD is a triple \((X, W, B)\) that satisfies the following: (1) \(X\) is a finite set of points and \(W \subset X\); (2) \(B\) is a collection of subsets of \(X\), called blocks, such that every pair of points containing at least one member in \(X \setminus W\) occurs in exactly \(\lambda\) blocks; (3) no block contains two members of \(W\); and (4) the blocks can be resolved into parallel classes or partial parallel classes that miss \(W\). As notation, suppose \(|X| = v, |W| = w\) and that each block is of size \(k\), then the incomplete RBIBD is denoted by \((v, k, \lambda)\) IRBIBD\(_w\).

A straightforward mechanism for constructing a \((v, k, \lambda)\) IRBIBD\(_m\) is to begin with a \((v, k, \lambda)\) RBIBD that contains a \((w, k, \lambda)\) RBIBD as a subdesign and remove this subdesign. For constructions of this type, Theorems 5.1, 5.3, 5.5 and Theorem 5.6 are useful. Indeed the data associated with the application of these latter theorems are the basis for the subsequent results obtained through the use of IRBIBDs. For convenience we adopt the notation that the set \(P\) denotes those integers \(n\) for which there exists a \((3, 6)\) GWhD\((6n)\) and IP\(_{6n}\) denotes those integers \(m\) for which there exists a \((3, 6)\) GWhD\((6m)\) missing a \((3, 6)\) GWhD\((6s)\) subdesign. That is to say there exists an \((6m, 6, 5)\) IRBIBD\(_{6s}\) that has the \((3, 6)\) GWhP.

**Theorem 5.10** (Abel et al. [1, Theorem 4.9]). Suppose \(S = \sum_{j=1}^{n} s_j\) and there exists a \((k, \lambda)\) frame on \(S\) points with group sizes \(s_i, 1 \leq i \leq n\). Suppose also, \(w \geq 1\), and for each \(i \geq 2\), a \((s_i + w, k, \lambda)\) IRBIBD\(_w\) exists. Then a \((S + w, k, \lambda)\) IRBIBD\(_{s_i+w}\) exists.

If, in addition, a \((s_1 + w, k, \lambda)\) RBIBD exists then there exists a \((S + w, k, \lambda)\) RBIBD, which contains the design on \(s_1 + w\) points as a subdesign. Furthermore if the first frame plus the filling IRBIBDs on \(s_1 + w\) points (\(i \geq 2\)) and the filling design on \(s_1 + w\)
Apply Theorem 5.11 with \( v \) for \( n \) to obtain the resulting RBIBD.

**Theorem 5.11.** Suppose there exist \((3,6)\) GWhFrames with group types \( 6^t \) for \( r \leq t \leq r + n \) and a TD\((r+n,m)\) exists. Let \( w > 0 \) and let \( s_1, \ldots, s_n \) be such that \( 0 \leq s_i \leq m \). Set \( S = s_1 + \cdots + s_n \). If \( m + w \in \text{IP}_{6w} \) and all the \( s_i \) except possibly \( s_n \), satisfy \( s_i + w \in \text{IP}_{6w} \), then \( (rm + S + w) \in \text{IP}_{6w + 6w} \). Furthermore, if \( s_n + w \in P \), then \( (rm + S + w) \in P \).

**Proof.** Begin with the TD\((r+n,m)\) and truncate \( n \) of the groups to sizes \( s_1, s_2, \ldots, s_n \). Give all points in this design a weight of \( 6 \). For an application of Wilson’s Fundamental Construction (i.e., Theorem 4.10) ingredient frames of types \( 6^t \), \( r \leq t \leq r + n \) are needed and exist by hypothesis. Hence we obtain a \((3,6)\) GWhFrame of type \( (6m)' \times (6s_1)^1 \times \cdots \times (6s_n)^1 \). Fill these groups with \( 6w \) infinite points and apply Theorem 5.10 to obtain the \((3,6)\) GWhD\((6(rm + S + w))\) missing a \((3,6)\) GWhD\((6(s_n + w))\) subdesign. If \( s_n + w \in P \), then we can fill in this missing design to obtain \((rm + S + w) \in P \).

Note that the construction embodied in the proof of Theorem 5.11 requires \( n + 1 \) consecutive integers \( t \) for which \((3,6)\) GWhFrames of type \( 6^t \) exist. The next few examples make extensive use of the frames with group types \( 6^7, 6^9, 6^{10} \), (given in Section 3) and type \( 6^8 \) which was given in the proof of Theorem 4.11.

**Corollary 5.12.** Suppose \( m \geq 41 \) and there exist a TD\((10,m)\) plus a \((6m + 12,6,5)\) IRBIBD\(_{12}\) with the \((3,6)\) GWhP. Then for \( 22 \leq s \leq 123 \), there exists a \((3,6)\) GWhD\((v)\) for \( v = 6(7m + s) + 12 \).

**Proof.** Apply Theorem 5.11 with \( r = 7 \), \( n = 3 \), \( w = 2 \). In the table below, for each \( s = s_1 + s_2 + s_3 \), \( 22 \leq s \leq 121 \), suitable values of \( s_1, s_2, s_3 \) are indicated. In each case, \( \{s_1 + 2, s_2 + 2\} \subset \text{IP}_{12} \), and \( s_3 + 2 \in P \). The data are displayed in the format \((s; s_1, s_2, s_3)\).

In all cases, \( 0 \leq s_i \leq 41 \) for all \( i \).

\[
\begin{align*}
(22; 0,11,11), & \quad (23; 0,11,12), & \quad (24; 0,12,12), & \quad (25; 0,11,14), \\
(26; 0,12,14), & \quad (27; 0,24,3), & \quad (28; 0,24,4), & \quad (29; 12,14,3), \\
(30; 14,14,2), & \quad (31; 14,14,3), & \quad (32; 14,14,4), & \quad (33; 0,22,11), \\
(34; 0,22,12), & \quad (35; 0,24,11), & \quad (36; 0,24,12), & \quad (37; 0,12,25), \\
(38; 0,12,26), & \quad (39; 0,14,25), & \quad (40; 0,14,26), & \quad (41; 14,24,3), \\
(42; 0,28,14), & \quad (43; 0,29,14), & \quad (44; 28,14,2), & \quad (45; 29,14,2), \\
(46; 0,22,24), & \quad (47; 22,22,3), & \quad (48; 0,24,24), & \quad (49; 22,24,3), \\
(50; 22,24,4), & \quad (51; 24,24,3), & \quad (52; 24,24,4), & \quad (53; 29,12,12), \\
(54; 29,11,14), & \quad (55; 11,22,22), & \quad (56; 12,22,22), & \quad (57; 11,22,24), \\
(58; 12,22,24), & \quad (59; 11,24,24), & \quad (60; 12,24,24), & \quad (61; 12,24,25), \\
(62; 29,22,11), & \quad (63; 29,22,12), & \quad (64; 28,22,14), & \quad (65; 29,22,14), \\
(66; 22,22,22), & \quad (67; 28,28,11), & \quad (68; 29,28,11), & \quad (69; 29,29,11), \\
(70; 24,22,24), & \quad (71; 24,22,25), & \quad (72; 28,22,22), & \quad (73; 29,22,22), \\
(74; 28,24,22), & \quad (75; 29,24,22), & \quad (76; 28,24,24), & \quad (77; 28,24,25), \\
(78; 28,28,22), & \quad (79; 29,28,22), & \quad (80; 28,28,24), & \quad (81; 28,28,25), \\
(82; 28,28,26), & \quad (83; 29,28,26), & \quad (84; 28,28,28), & \quad (85; 29,28,28),
\end{align*}
\]
Example 5.13. Applying Theorem 5.11 with \( m = 89, 99, 109, 121, 128, 137, \) and 143, one can obtain \((3, 6)\) GWhD(v) for \( v \) in the range [3882, 6756]. Below we indicate the range for \( v \) covered by each \( m \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>Range for ( v )</th>
<th>( m )</th>
<th>Range for ( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>[3882, 4488]</td>
<td>99</td>
<td>[4302, 4908]</td>
</tr>
<tr>
<td>109</td>
<td>[4722, 5328]</td>
<td>121</td>
<td>[5226, 5832]</td>
</tr>
<tr>
<td>128</td>
<td>[5520, 6126]</td>
<td>137</td>
<td>[5898, 6504]</td>
</tr>
<tr>
<td>143</td>
<td>[6150, 6756]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Corollary 5.14. Suppose a TD(10, \( m \)) plus a \((6m + 12, 6, 5)\) IRBIBD_{12} with the \((3, 6)\) GWhP both exist. Then there exists a \((3, 6)\) GWhD(v) for \( v = 6(7m + s) + 12 \) if either (1) \( 22 \leq s \leq 87 \) and \( m \geq 29 \) (2) \( 22 \leq s \leq 102 \) and \( m \geq 36 \) or (3) \( 22 \leq s \leq 123 \) and \( m \geq 41 \).

Proof. As in Example 5.13, we may apply Theorem 5.11 with \( r = 7, n = 3, w = 2 \). For each \( s = s_1 + s_2 + s_3 \), \( 22 \leq s \leq 123 \), suitable values of \( s_1, s_2, s_3 \) were indicated in the table used in Corollary 5.12. In all cases with \( 22 \leq s \leq 87 \), we have \( 0 \leq s_i \leq 29 \) for all \( i \); similarly, for \( 22 \leq s \leq 99 \), we have \( 0 \leq s_i \leq 36 \) for all \( i \).

Example 5.15. Applying Corollary 5.14 with \( m = 29, 36, 41 \), noting that Wojtas has constructed a TD(10, 36) [13], one can obtain \((3, 6)\) GWhD(v) for \( v \) in the ranges [1362, 1752], [1656, 2136] and [1866, 2472].

Example 5.16. Applying Theorem 5.11 with \( n = 1, r = 7 \) one can obtain \((3, 6)\) GWhD(v) in the cases listed below. For \( m = 24 \), the required TD(8, 24) is given in [14]. The data are displayed in the format \((v; w, m, s_1)\).

\[
\begin{align*}
(486; 2, 11, 2), & \quad (492; 2, 11, 3), \quad (498; 2, 11, 4), \quad (996; 4, 23, 1), \\
(1038; 2, 24, 3), & \quad (1044; 4, 23, 9), \quad (1050; 4, 23, 10), \quad (1062; 4, 23, 12), \\
(1104; 2, 24, 14), & \quad (1110; 4, 23, 20), \quad (1122; 4, 23, 22).
\end{align*}
\]

Example 5.17. Applying Theorem 5.11 with \( n = 1, r = 8 \) one can obtain \((3, 6)\) GWhD(v)'s for the following \( v \). The data are in the form \((v; w, m, s_1)\).

\[
\begin{align*}
(558; 2, 11, 3), & \quad (564; 2, 11, 4), \quad (1182; 4, 23, 9), \quad (1188; 4, 23, 10).
\end{align*}
\]
Theorem 5.18. Let $q$ be a prime power and let \( \{ q - 2 + w, q - 1 + w \} \subset \text{IP}_{6w} \cap P \) with $w > 0$. Then:

(a) $t \in P$ when $9(q - 2) + w \leq t \leq 10(q - 1) + w$ and $q \geq 19$.
(b) $t \in \text{IP}_{6w} \cap P$ when $10(q - 2) + w \leq t \leq 10(q - 1) + w$ and $q \geq 11$.

Proof. The result (a) relies on successive members of $P$ being within 9 of one another. We start with a TD(10, $q$), truncate one group and remove oval points from the other 9 groups, ensuring that each of these 9 groups loses at least one point. The construction employed in Theorem 3.14 ensures that these 9 groups each contain 2 oval points.

In Theorem 3.4 we pivoted on a non-oval tangent point to produce our TD($q + 1, q$), although any non-oval point would work for $q \geq 19$. If we pivot instead on an oval point, our TD($q + 1, q$) will contain $q$ groups with a single oval point, so in part (b) we remove a block from a TD(10, $q$), then remove 0 or 1 oval points from each group.

Finally, we give all points of these designs a weight of 6 in Theorem 4.10, then fill each group of the resulting frame with 6$w$ infinite points and then apply Theorem 5.10 to obtain the desired result.

Example 5.19. Applying Theorem 5.18(b) for $q = 13$ and Theorem 5.18(a) for the other $q$'s in the table below covers the following ranges.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$w$</th>
<th>Range for $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>2</td>
<td>[672, 732]</td>
</tr>
<tr>
<td>43</td>
<td>2</td>
<td>[2226, 2532]</td>
</tr>
</tbody>
</table>

Theorem 5.20. Let $q \geq 23$ be a prime power and let \( \{ q - 3 + w, q - 1 + w \} \subset \text{IP}_{6w} \cap P \) with $w > 0$. Then $t \in \text{IP}_{6w} \cap P$ when $10(q - 3) + w \leq t \leq 10(q - 1) + w$ and $t \equiv w \pmod{2}$.

Proof. We start by removing a block of TD(10, $q$) then remove a pair of oval points from as many groups as required. Finally, we give all points of these designs a weight of 6 in Theorem 4.10, then fill each group of the resulting frame with 6$w$ infinite points and then apply Theorem 5.10 to obtain the desired result.

Example 5.21. Applying Theorem 5.20 covers the following (partial) ranges.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$w$</th>
<th>Range for $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>2</td>
<td>[2652, 2772] with $v \equiv 0 \pmod{12}$,</td>
</tr>
<tr>
<td>49</td>
<td>2</td>
<td>[2772, 2892] with $v \equiv 0 \pmod{12}$,</td>
</tr>
</tbody>
</table>

Theorem 5.22. If $606 \leq v \leq 660$, then there exists a $(3, 6)$ GWhD($v$) containing a 12 point subdesign.

Proof. For $612 \leq v \leq 648$, start with a TD(10, 13) and delete 17, 18 or 19 points from two concurrent lines; we delete the point of intersection and either 8 or 9 other points of each line, making sure that every group loses at least one point, and the last group loses two points. Then, for $7 \leq x \leq 9$ or $x = 11$, we delete $x$ further points from the last
group giving it a size of $s$ with $s \in \{0, 2, 3, 4\}$. For $v=654,660$, start with a TD$(9,13)$ and delete all points in 1 block (plus 1 extra point for $v=654$), and for $v=660$, we delete two parallel blocks from TD$(9,13)$.

Finally, we give all points of these designs a weight of 6 in Theorem 4.10, then fill each group of the resulting frame with 12 infinite points and then apply Theorem 5.10 to obtain the desired result. For $v \neq 606,654,660$, the intersection group only lost one point, so one of our filling designs is a GWhD(84); therefore although the filling process can destroy one of its 12 point subdesigns, there are at least 6 other disjoint ones remaining. For $v=606,654,660$, every filling design contains 84 or 78 points, so again a $v=12$ subdesign exists. □

**Theorem 5.23.** Suppose there exist $(3,6)$ GWhFrames with group types $6^l$ for $r^* - b \leq t \leq r^* + n$ and a TD$(r^* + n, m^* + b)$ containing $b$ mutually disjoint blocks exists. Let $w \geq 0$ and let $s_1, \ldots, s_n$ be such that $0 \leq s_i \leq m$. Set $S = s_1 + \cdots + s_n$. If $m^* + w \in \Pi_{6w}$ and all the $s_i$, except possibly $s_n$, satisfy $s_i + w \in \Pi_{6w}$, then $(r^* m^* + S + w) \in \Pi_{6s + 6w}$. Furthermore if $s_n + w \in P$ then $(r^* m^* + S + w) \in P$.

**Proof.** Begin with the TD$(r^* + n, m^* + b)$ and remove $b$ complete blocks; thereafter the proof follows that of Theorem 5.11. □

**Remark 5.24.** For a TD$(k,m)$ to contain disjoint blocks, it is necessary that $k \leq m$. This condition is sufficient for the TD to contain $b$ mutually disjoint blocks when $b \leq 3 \leq m$.

**Example 5.25.** The following table gives some $(3,6)$ GWhD$(v)$’s obtained by applying Theorem 5.23 with $b=1$, $n=2$ and $r^*=8$. The data are displayed in the form $(v;w, m^*, s_1, s_2)$.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$(v;w, m^*, s_1, s_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1140</td>
<td>(1140;2,22,12,0)</td>
</tr>
<tr>
<td>1158</td>
<td>(1158;2,22,11,4)</td>
</tr>
<tr>
<td>1176</td>
<td>(1176;2,22,14,4)</td>
</tr>
<tr>
<td>1212</td>
<td>(1212;2,22,22,2)</td>
</tr>
<tr>
<td>1230</td>
<td>(1230;2,24,0,11)</td>
</tr>
<tr>
<td>1248</td>
<td>(1248;2,24,12,2)</td>
</tr>
<tr>
<td>2538</td>
<td>(2538;2,48,33,4)</td>
</tr>
<tr>
<td>2556</td>
<td>(2556;2,48,36,4)</td>
</tr>
<tr>
<td>2574</td>
<td>(2574;2,46,33,26)</td>
</tr>
<tr>
<td>2592</td>
<td>(2592;2,48,22,24)</td>
</tr>
<tr>
<td>2610</td>
<td>(2610;2,48,24,25)</td>
</tr>
<tr>
<td>2628</td>
<td>(2628;2,48,48,4)</td>
</tr>
<tr>
<td>2646</td>
<td>(2646;2,46,46,25)</td>
</tr>
<tr>
<td>2682</td>
<td>(2682;2,46,44,33)</td>
</tr>
<tr>
<td>2718</td>
<td>(2718;2,46,44,39)</td>
</tr>
<tr>
<td>2754</td>
<td>(2754;2,48,33,40)</td>
</tr>
<tr>
<td>2790</td>
<td>(2790;2,48,46,33)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v$</th>
<th>$(v;w, m^*, s_1, s_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1146</td>
<td>(1146;2,22,11,2)</td>
</tr>
<tr>
<td>1164</td>
<td>(1164;2,22,12,4)</td>
</tr>
<tr>
<td>1200</td>
<td>(1200;2,22,11,11)</td>
</tr>
<tr>
<td>1218</td>
<td>(1218;2,22,22,3)</td>
</tr>
<tr>
<td>1236</td>
<td>(1236;2,24,0,12)</td>
</tr>
<tr>
<td>1254</td>
<td>(1254;2,24,12,3)</td>
</tr>
<tr>
<td>2544</td>
<td>(2544;2,48,36,2)</td>
</tr>
<tr>
<td>2562</td>
<td>(2562;2,46,33,24)</td>
</tr>
<tr>
<td>2580</td>
<td>(2580;2,46,36,24)</td>
</tr>
<tr>
<td>2598</td>
<td>(2598;2,48,22,25)</td>
</tr>
<tr>
<td>2616</td>
<td>(2616;2,48,24,26)</td>
</tr>
<tr>
<td>2634</td>
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<td>2658</td>
<td>(2658;2,46,33,40)</td>
</tr>
<tr>
<td>2694</td>
<td>(2694;2,46,46,33)</td>
</tr>
<tr>
<td>2730</td>
<td>(2730;2,48,44,25)</td>
</tr>
<tr>
<td>2766</td>
<td>(2766;2,48,36,39)</td>
</tr>
<tr>
<td>2802</td>
<td>(2802;2,48,48,33)</td>
</tr>
<tr>
<td>2814</td>
<td>(2814;2,48,44,39)</td>
</tr>
</tbody>
</table>
produces a design that possesses the GWhP.  

Example 5.26. The following table gives some (3, 6) GWhD(v)’s obtained by applying Theorem 5.23 with $b=2$, $n=1$ and $r^*=9$. Note that all these designs inherit multiple disjoint subdesigns on 12 points from the (3, 6) GWhD(96) used to fill most groups. The data are displayed in the form $(v;w,m^*,s_1)$.

$$(2826;2,48,46,39), \quad (2838;2,48,48,39), \quad (3066;2,58,33,12),$$
$$\quad (3078;2,58,33,14), \quad (3090;2,58,24,25), \quad (3102;2,58,12,39),$$
$$\quad (3114;2,58,14,39), \quad (3126;2,58,11,44), \quad (3138;2,58,11,46).$$

Theorem 5.27. Suppose there exists a GDD($\mathcal{X}, \mathcal{G}, \mathcal{B}$) such that

(a) $|\mathcal{X}|=w \equiv k-1 \pmod{k}$,
(b) for each $G_i \in \mathcal{G}$, we have $|G_i| \equiv k-1 \pmod{k}$ and there exists a $(t,k)$ GWhD($|G_i|+1$) and
(c) for each $B_j \in \mathcal{B}$, $|B_j| \equiv 1 \pmod{k}$, the following designs exist:

(i) a $(t,k)$ GWhFrame of type $1^{[|B_j|]}$;
(ii) a $(t,k)$ GWhFrame of type $1^{[|B_j|]-1}$
(iii) a $(t,k)$ GWhFrame of type $1^{[|B_j|]-2}$

Furthermore assume there are two groups, say $G_\ell, G_m$ such that
(d) for all integers $j$, $0 \leq j \leq |G_\ell|$, there exists a $(t,k)$ GWhD($|G_\ell|+ks+1$) and
(e) for all integers $u$, $0 \leq u \leq |G_m|$, there exists a $(t,k)$ GWhD($|G_m|+ku+1$).

Then there exists a $(t,k)$ GWhD(v), $v=w+ks+ku+1$, for all $0 \leq s \leq |G_\ell|, \ 0 \leq u \leq |G_m|$. 

Proof. Construct a new design as follows. In the two groups $G_\ell, G_m$ highlight $s$ points in $G_\ell$ and $u$ points in $G_m$ as “selected points” subject to the conditions $0 \leq s \leq |G_\ell|, \ 0 \leq u \leq |G_m|$. Replace each selected point by $k+1$ new points both in the group containing the selected point and in every block that contains the selected point. In the resulting design one designates the groups by $G_\ell^*$ and the “blocks” by $B_j^*$ where $|G_\ell^*| = |G_\ell| + ks$, $|G_m^*| = |G_m| + ku$, $|G_i^*| = |G_i|$, $i \neq \ell, m$. Also $|B_j^*| = |B_j|$, $|B_j| + k$, or $|B_j| + 2k$ depending on whether $B_j$ contains zero, one or two selected points. In each case replace the “block” $B_j^*$ by the blocks of the appropriate $(t,k)$ GWhFrame aligning the groups of order $k+1$ with the new points. Each partial parallel class of this latter frame is to be assigned a label. Observe first that the frame index is $k-1$ thus the number of partial parallel classes coincides with the order of the group that these partial parallel classes miss. Consequently we can arbitrarily label the partial parallel classes, one each, by the points in the missed group. This done, we next adjoin a single infinite point to each $G_i^*$ and construct a $(t,k)$ GWhD($|G_i^*|+1$) on the set $G_i^* \cup \{\infty\}$ labeling the rounds by the elements of $G_i^*$. To see that we now have constructed the desired $(t,k)$ GWhD(v) let $x \in \mathcal{X}^*$. The union of all the rounds and partial parallel classes labeled $x$ yields a round of the $(t,k)$ GWhD(v). It is clear that the construction produces a design that possesses the GWhP. \[\square\]
Of course, one can employ the construction described in the proof of Theorem 5.27 even if the hypotheses regarding \( s \) and \( u \) are not completely satisfied. That is to say, suppose it is known that \((t,k)\) GWhD\((|G_i| + ks + ku + 1)\) exist only for \( s \in S \) and that \((t,k)\) GWhD\((|G_m| + ku + 1)\) exist only for \( u \in U \) (with \( s,u \) subject to the same inequalities as stated in the theorem) then it follows that \((t,k)\) GWhD\((v)\) exist for those \( v \) of the form \( v = w + ks + ku + 1, s \in S, u \in U \). This latter observation is illustrated in the next example.

**Example 5.28.** Set \( k = 6 \) and consider a TD\((37,83)\) (so that \(|G_i| = 83\) for all \( i \)). One can discern from the data of this section that \((3,6)\) GWhD\((83 + 6s + 1)\) exist for each \( s \in \{0,2,10,12,13,14,16,17,18,19,20,21,24,26,27,28,29,30,32,34,36,38,43,46,48,49,51,54,58,59,60,62,64,65,67,68,69,72,76,77,78,79,80,81,82,83\} \). Certainly this set of values applies to \( u \) also. Thus with \(|B_j| = 37\) for all \( j \) and using the frames of Examples 3.11, 3.12 and 3.13 an application of the construction described in the proof of Theorem 5.27 yields (3,6) GWhD\((v)\), for all \( v \) such that \( v \equiv 0 \pmod{6} \) and \( 3072 \leq v \leq 4068 \) except for 8 values of \( v \) in this range, namely, \( v = 3078, 3090, 3102, 3108, 3114, 3120, 3126, 3138 \). The reason for these missing cases is, of course, due to the fact that of the known solutions of the form \( 83 + 6s + 1, 83 + 6u + 1 \) there is no appropriate combination of \( s,u \) for which \( 37.83 + 6s + 6u + 1 \) is equal to any of the missing cases. Note that the exceptions \( v = 3108 \) and 3120 are covered by an application of Theorem 3.14.d with \( q = 83 \); for the others, see Example 5.25.

In the following theorem we apply Theorem 5.27 in the special case for which \( k = 6 \), each group \( G_i \) is of the same size and \(|B_j| = 37\) for all \( j \). Note that the required frames of types \( 1^{37}, 1^{30}\_7^1 \) and \( 1^{35}\_7^2 \) are given in Examples 3.11, 3.12 and 3.13, respectively.

**Theorem 5.29.** Suppose there exists a sequence \( \{g_i\}_{i=1}^{\infty} \) such that:

(a) TD\((37,g_i)\) exists for all \( i = 1, \ldots, n \);
(b) \((3,6)\) GWhD\((g_i + 1)\) exists for all \( i = 1, \ldots, n \);
(c) \(37g_{i+1} \leq 49g_i\) for all \( i = 1, \ldots, n - 1 \);
(d) for each \( i \) (fixed) and for each \( v \) such that \( v \equiv 5 \pmod{6} \), \( 37g_i \leq v \leq 49g_i \), there exist integers \( s,u \) such that:
   (i) \( 0 \leq s,u \leq g_i \);
   (ii) \( v = 37g_i + 6s + 6u \);
   (iii) there exist \((3,6)\) GWhD\((g_i + 6s + 1)\) and \((3,6)\) GWhD\((g_i + 6u + 1)\);
(e) \( g_n \geq 37g_1 \).

Then there exist \((3,6)\) GWhD\((v)\) for all \( v \geq 37g_1 + 1, v \equiv 0 \pmod{6} \).

**Proof.** Certainly (a)–(e) coupled with Theorem 5.27 cover all such \( v \) for which \( 37g_1 + 1 \leq v \leq 49g_n + 1 \). Note, in particular, that (d) implies that if \( 37g_i \leq v \leq 49g_i \) then there exists a \((3,6)\) GWhD\((v)\). Consider the sequence \( \{h_i\}_{i=1}^{\infty} \) defined by \( h_{i+n+s} = 7^2/ g_i \).
\( \ell = 0, 1, \ldots \); \( s = 1, \ldots, n \). Note that each \( h_1 \equiv 5 \pmod{6} \), there exist TD(37, \( h_1 \)) and

\[
37h_{\ell+n+s+1} = 7^{2\ell}(37g_s+1) \leq 7^{2\ell}(49g_s) = 49h_{\ell+n+s}.
\]

We proceed now by strong induction. Let \( m \) be such that \( m = \ell + n + s \) with \( \ell \geq 1 \) and assume that for all \( w \leq m \), there exists a \((3, 6)\) GWhD(\( h_w + 1 \)) and for all \( v \equiv 5 \pmod{6} \), \( 37h_w \leq v \leq 49h_w \) there exists a \((3, 6)\) GWhD(v + 1). Referral to Theorem 5.27 indicates that \( h_{m+1} \) will guarantee existence of \((3, 6)\) GWhD(\( v+1 \)) for all \( v \equiv (\pmod{6}) \), \( 37h_{m+1} \leq v \leq 49h_{m+1} \) provided that (a) a \((3, 6)\) GWhD(\( h_{m+1} + 1 \)) exists and (b) a \((3, 6)\) GWhD(\( z+1 \)) exists for all \( z \equiv 5 \pmod{6} \), \( h_{m+1} \leq z \leq 7h_{m+1} \). That (a) and (b) are true follows from the induction hypothesis. Indeed, for (a) observe that

\[
h_{m+1} = 49h_{(\ell-1)n+s+1} = 49h'_{m'},
\]

where \( m' < m \). As for (b) we have

\[
7h_{m+1} = 49(7^{2\ell-2})(7g_{s+1}) < 49(7^{2\ell-2})(49g_s) = 49h_m.
\]

The proof is now complete via strong induction.

**Theorem 5.30.** There exist \((3, 6)\) GWhD(\( 6n \)) for all \( n \geq 1104 \).

**Proof.** A sequence \( g_i \) that works for an application of Theorem 5.29 is the following:

\[
179, 227, 251, 311, 359, 467, 569, 719, 863, 1109, 1289, 1523, 1721, 2009, 2591, 3407, 4001, 5021, 6287, 7643.
\]

It turns out that for \( g = 227, 251, 311, 359 \) and 863, some of the smaller \( v \) in the associated ranges are not covered. However all such exceptions are covered by preceding values of \( g \) in the sequence.

Combining all of the materials of this section we can formulate the following theorem concerning the existence of \((3, 6)\) GWhD(\( 6n \)).

**Theorem 5.31.** There does not exist a \((3, 6)\) GWhD(\( 6 \)). \((3, 6)\) GWhD(\( 6n \)) exist for all \( n > 1 \) except possibly for the 73 values of \( 6n \) given in Table 1.

<table>
<thead>
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<td>Values of ( 6n ) for which ((3, 6)) GWhD(( 6n )) is unknown</td>
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References