RTD-BASED CELLULAR NEURAL NETWORKS
WITH MULTIPLE STEADY STATES

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In this paper, we study the relationship between the standard cellular neural network (CNN) and the resonant tunneling diode (RTD)-based CNN. We investigate the functional and advanced capabilities of a new generation of CNNs that exploit the multiplicity of steady states. We also include in the analysis higher order CNNs. Furthermore, some methods for designing RTD-based CNNs with multiple steady states are presented.

1. Introduction

The resonant tunneling diode (RTD), a class of quantum effect device, is an excellent candidate for both analog and digital nanoelectronics applications because of its structural simplicity, relative ease of fabrication, inherent high speed and design flexibility.

The use of RTDs in applications to cellular neural networks (CNN) has been previously reported in [Dogaru et al., 1999; Hänggi & Chua, 2000]. Bistable RTD-based CNN exhibits good performance for a number of interesting image processing applications because of its high-speed processing and high cell density. Thus, it is possible that a new generation of low power, high-speed, and large array-size CNNs appears with the introduction of the RTD-based CNN.

It is well known that series connected RTDs have several stable operating points [Tang & Lin, 1996]. As a consequence, RTD-based CNNs inherit this property and exhibit multiple equilibrium points, as shown in [Hänggi & Chua, 2000]. In order to develop the new generation of RTD-based CNNs it is essential to produce adequate tools for fully utilizing the multiple equilibrium point characteristics. Since the research on RTD-based CNNs has recently begun, there are only a few design and analysis tools available. In addition, as the CNN possesses the pattern formation property, the new generation RTD-based CNN may be designed to utilize this property in graphic and memory system design.

It is our intention to devote this paper to the development of a new generation of RTD-based CNNs. The organization of the paper is as follows. In Sec. 2, we analyze the relationship between the standard and the RTD-based CNN. In Sec. 3, we study the functional capabilities of the CNN with...

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multiple steady states. In Sec. 4, we synthesize the CNN using an existing set of desired equilibrium points. In Sec. 5, we study the advanced pattern formation property of second-order and third-order CNNs. Finally, in Sec. 6, we propose several methods for designing RTD-based CNNs exploiting the multiplicity of steady states.

2. The Relationship Between the Standard CNN and the RTD-Based CNN

The dynamics of a standard cellular neural network with a neighborhood of radius \( r \) are governed by a system of \( n = MN \) differential equations [Chua, 1997]

\[
\frac{dx_{ij}}{dt} = -x_{ij} + \sum_{k,l \in N_{ij}} (a_{k-i,l-j}y_{kl} + b_{k-i,l-j}u_{kl}) + z_{ij}, \quad (i, j) \in \{1, \ldots, M\} \times \{1, \ldots, N\}
\]

(1)

where \( N_{ij} \) denotes the \( r \)-neighborhood of cell \( C_{ij} \) and \( a_{kl} \, b_{kl} \) and \( z_{ij} \) denote the feedback, control and threshold template parameters, respectively.\(^1\)

The output \( y_{ij} \) and the state \( x_{ij} \) of each cell are related through the piecewise-linear saturation function

\[ y_{ij} = f(x_{ij}) = \frac{1}{2}(|x_{ij} + 1| - |x_{ij} - 1|), \quad (2) \]

which is illustrated in Fig. 1. If we restrict the neighborhood radius of every cell to 1 and assume that \( z_{ij} \) is the same for the whole network, the template \( \{A, B, z\} \) is fully specified by 19 parameters, which are the elements of the \( 3 \times 3 \) matrices \( A \) and \( B \), namely

\[
A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix},
\]

(3)

and element \( z \). The corresponding CNN gene is given by

\[ \tau = [a_1, a_2, \ldots, a_9, b_1, b_2, \ldots, b_9, z]. \]

The circuit implementation of the standard CNN is illustrated in Fig. 2.

The dynamics of the RTD-based CNN are governed by a system of \( n = MN \) differential equations [Hänggi & Chua, 2000]

\[
\frac{dx_{ij}}{dt} = -g(x_{ij}) + \sum_{k,l \in N_{ij}} (\tilde{a}_{k-i,l-j}x_{kl} + \tilde{b}_{k-i,l-j}u_{kl}) + \tilde{z}_{ij}, \quad (i, j) \in \{1, \ldots, M\} \times \{1, \ldots, N\}
\]

(5)

where \( N_{ij} \) denotes the \( r \)-neighborhood of cell \( C_{ij} \), and \( \tilde{a}_{kl} \, \tilde{b}_{kl} \) and \( \tilde{z}_{ij} \), denote the feedback, control and threshold template parameters, respectively.

\(^1\)The matrices \( A = \{a_{kl}\} \) and \( B = \{b_{kl}\} \) are referred to as the feedback template \( A \) and the feedforward (input) template \( B \), respectively.
The $v$–$i$ characteristic of the RTD is given by

$$g(x_{ij})$$

$$= \alpha x_{ij} + \gamma(|x_{ij} - V_p| - |x_{ij} - V_v|)$$

$$- \gamma(|x_{ij} + V_p| - |x_{ij} + V_v|),$$

where $\alpha = \gamma(V_v - V_p)$, $\alpha$, $\gamma$ are constants, and $V_p$, $V_v$ are the peak and valley voltages of the RTD for the positive region of $x_{ij}$, respectively. For the sake of simplicity, the $v$–$i$ characteristic of the RTD is modeled by a piecewise-linear function symmetric with respect to the origin as shown in Fig. 3. Furthermore, the slopes in the regions $\{|x_{ij}| > V_v\}$ and $\{|x_{ij}| < V_p\}$ are chosen to be the same. A circuit implementation of the RTD-based CNN is illustrated in Fig. 4.

![Fig. 3. The v–i characteristic of the RTD.](image)

![Fig. 4. Circuit implementation of an RTD-based CNN.](image)

### 2.1. Isolated RTD-based CNN cell

The state equation of the isolated RTD-based cell for $x_{ij} \geq 0$ is defined as

$$\frac{dx_{ij}}{dt} = -\alpha x_{ij} - \beta - \gamma(|x_{ij} - V_p| - |x_{ij} - V_v|)$$

$$+ \tilde{a}_{00} x_{ij} + \tilde{b}_{00} u_{ij} + \tilde{z}_{ij}.$$  

(7)

If we introduce new variables

$$\hat{x}_{ij} = \frac{x_{ij} - V_o}{V_d},$$

$$V_a = \frac{V_o + V_p}{2},$$

$$V_d = \frac{V_o - V_p}{2},$$

we obtain the equation

$$\frac{d\hat{x}_{ij}}{dt} = -\alpha \hat{x}_{ij} - \alpha \hat{V}_a - \hat{\beta} - \gamma(|\hat{x}_{ij} + 1| - |\hat{x}_{ij} - 1|)$$

$$+ \tilde{a}_{00} \hat{x}_{ij} + \tilde{a}_{00} \hat{V}_a + \tilde{b}_{00} \hat{u}_{ij} + \hat{z}_{ij}$$

$$= -(\alpha - \tilde{a}_{00}) \hat{x}_{ij} - \gamma(|\hat{x}_{ij} + 1| - |\hat{x}_{ij} - 1|)$$

$$+ \tilde{b}_{00} \hat{u}_{ij} + \hat{z}_{ij},$$  

(9)

where

$$\hat{V}_a = \frac{V_a}{V_d}, \quad \tilde{b}_{00} = \frac{\tilde{b}_{00}}{V_d}, \quad \hat{\beta} = \frac{\beta}{V_d},$$

$$\hat{z}_{ij} = \frac{\tilde{z}_{ij}}{V_d}, \quad \tilde{z}_{ij} = \tilde{z}_{ij} - \hat{\beta} - (\alpha - \tilde{a}_{00}) \hat{V}_a.$$

(10)

Thus, Eq. (9) is transformed into the state equation of the standard isolated CNN cell

$$\frac{d\tau x_{ij}}{d\tau} = -\tau_{ij} + \tau_{00} f(\tau_{ij}) + \tau_{00} u_{ij} + \tau_{ij},$$

(11)

where

$$\tau = (\alpha - \tilde{a}_{00}) t, \quad \tau_{ij} = \hat{x}_{ij}, \quad \tau_{00} = \frac{-2\gamma}{\alpha - \tilde{a}_{00}},$$

$$\tilde{b}_{00} = \frac{\tilde{b}_{00}}{\alpha - \tilde{a}_{00}}, \quad \tilde{z}_{ij} = \frac{\hat{z}_{ij}}{\alpha - \tilde{a}_{00}},$$

and

$$f(\tau_{ij}) = \frac{1}{2}(|\tau_{ij} + 1| - |\tau_{ij} - 1|).$$

(12)

(13)

It follows that if we use the positive operating region of the RTD (more rigorously, the region characterized by $x_{ij} \geq -V_p$), the dynamics of the RTD-based isolated CNN cell are equivalent to those of the standard isolated CNN cell [Chua, 1997].
Accordingly, the equivalent parameters of the RTD-based isolated CNN cell can be obtained from the parameters of the standard isolated CNN cell as follows

$$\tilde{a}_{00} = \alpha + \frac{2\gamma}{a_{00}} \Rightarrow \tilde{b}_{00} = (\alpha - \tilde{a}_{00})\tilde{b}_{00}V_d$$

$$\Rightarrow \tilde{z}_{00} = V(\alpha - \tilde{a}_{00})\tilde{z}_{00}V_d + \beta + (\alpha - \tilde{a}_{00})\tilde{V}_a$$  \hspace{1cm} (14)

where

$$V_a = \frac{V_o + V_p}{2}, \quad V_d = \frac{V_o - V_p}{2}, \quad t = \frac{\alpha - \tilde{a}_{00}}{\tau}.$$  \hspace{1cm} (15)

### 2.2. Uncoupled RTD-based CNN cell

A CNN is said to be uncoupled if all parameters in $A = \{a_{kl}\}$ are zero except the center element $a_{00}$. The state equation of the uncoupled RTD-based CNN is given by

$$\frac{d\tilde{x}_{ij}}{dt} = -g(x_{ij}) + \tilde{a}_{00}x_{ij} + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \tilde{b}_{k-i,l-j}u_{kl} + \tilde{z}_{ij}.$$  \hspace{1cm} (16)

We can easily transform this equation into the form

$$\frac{d\tilde{x}_{ij}}{d\tau} = -\tilde{x}_{ij} + \tilde{a}_{00}f(\tilde{x}_{ij}) + \sum_{k,l \in N_{ij}} \tilde{b}_{k-i,l-j}u_{kl} + \tilde{z}_{ij}.$$ \hspace{1cm} (17)

where

$$\tilde{\tau}_{k-i,l-j} = \frac{\tilde{b}_{k-i,l-j}}{(\alpha - \tilde{a}_{00})\tilde{V}_d}.$$ \hspace{1cm} (18)

Therefore, the dynamics of the uncoupled RTD-based CNN are also equivalent to those of the uncoupled standard CNN.

### 2.3. Coupled RTD-based CNN cell

In the case of the coupled RTD-based CNN, the state equation is transformed into the form

$$\frac{d\bar{\tau}_{k-i,l-j}}{d\tau} = \frac{\bar{\tau}_{k-i,l-j}}{(\alpha - a_{00})V_d},$$ \hspace{1cm} (19)

where

$$\bar{\tau}_{k-i,l-j} = \frac{a_{k-i,l-j}}{(\alpha - a_{00})V_d},$$ \hspace{1cm} (20)

Note that in this case the state $\tau_{k-i,l-j}$ is used as the output, instead of the function $f(\tau_{ij})$.

In the next subsection, we study the relationship between the standard CNN and the RTD-based CNN in the coupled case. In order to do this, we modify the characteristic equation and propose a generalized CNN equation. This generalized equation plays a major role in clarifying the relationship between both coupled state equations.

#### 2.3.1. Generalized CNN cell

In order to generalize the standard CNN state equation, we first need to modify it as follows:

$$\frac{d\bar{x}_{ij}}{d\tau} = -\bar{x}_{ij} + \bar{a}_{00}f(\bar{x}_{ij}) + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \bar{a}_{k-i,l-j}k(\bar{x}_{k-i,l-j}) + \sum_{k,l \in N_{ij}} \bar{b}_{k-i,l-j}u_{kl} + \bar{z}_{ij},$$ \hspace{1cm} (21)

where $k(\bar{x}_{k-i,l-j})$ is a continuous function of $\tau_{k-i,l-j}$.

Next, let us consider the difference between the dynamics of a cell in the generalized standard CNN.
and a cell in an RTD-based CNN. Accordingly, let us define the function \( \overline{y}_{k-i,l-j} = k(\overline{x}_{k-i,l-j}) \) as follows:

\[
\overline{y}_{k-i,l-j} = k(\overline{x}_{k-i,l-j}) = (1 - \varepsilon)f(\overline{x}_{k-i,l-j}) + \varepsilon \overline{x}_{k-i,l-j},
\]

\((0 \leq \varepsilon \leq 1, \ (k, l) \neq (i, j)) \) . (22)

Then, it is clear that \( k(\overline{x}_{k-i,l-j}) \) satisfies

\[
k(\overline{x}_{k-i,l-j}) = \begin{cases} f(\overline{x}_{k-i,l-j}) & \text{if } \varepsilon = 0, \\ \overline{x}_{k-i,l-j} & \text{if } \varepsilon = 1. \end{cases}
\] (23)

and

\[
k(\overline{x}_{k-i,l-j}) = \overline{x}_{k-i,l-j}, \quad \text{if } |\overline{x}_{k-i,l-j}| \leq 1 \text{ and } \varepsilon \in [0, 1] .
\] (24)

Thus, if \( \varepsilon = 1 \), Eq. (21) is equivalent to that of the coupled standard CNN. If \( \varepsilon = 0 \), then Eq. (21) is equivalent to that of the coupled RTD-based CNN. In other words, if we increase parameter \( \varepsilon \) from 0 to 1, the state equation varies from that of the coupled RTD-based CNN to that of the coupled standard CNN. We note that in the region \( |\overline{x}_{k-i,l-j}| \leq 1 \), the dynamics are not changed even if \( \varepsilon \) is varied from 0 to 1. However, in the region \( |\overline{x}_{k-i,l-j}| \geq 1 \), the two dynamics are different.

In order for the RTD-based CNN to have the same dynamics as the standard CNN, we must modify the output function of the RTD-based CNN. This is due to the reason that the output function of the RTD-based CNN is a linear function of \( x_{ij} \), whereas that of the standard CNN is a saturated function of \( x_{ij} \). Accordingly, if we replace the output function \( x_{kl} \) in Eq. (5) with the saturated function \( g(x_{kl}) - \alpha x_{kl} (= \beta + \gamma(|x_{ij} - V_p| - |x_{ij} - V_v|)) \), then we get the following modified state equation of the RTD-based CNN

\[
\frac{dx_{ij}}{dt} = -g(x_{ij}) + \tilde{a}_{00}x_{ij} + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \tilde{a}_{k-i,l-j}(g(x_{kl}) - \alpha x_{kl}) + \sum_{k,l \in N_{ij}} \tilde{b}_{k-i,l-j} u_{kl} + \tilde{z}_{ij}
\]

\[
= a_{00}x_{ij} - \alpha x_{ij} - \beta - \gamma(|x_{ij} - V_p| - |x_{ij} - V_v|))
\]

\[
+ \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \tilde{a}_{k-i,l-j}(\beta + \gamma(|x_{ij} - V_p| - |x_{ij} - V_v|)) + \sum_{k,l \in N_{ij}} \tilde{b}_{k-i,l-j} u_{kl} + \tilde{z}_{ij} .
\] (25)

Thus, Eq. (25) is transformed into

\[
\frac{d\overline{x}_{ij}}{d\tau} = -\overline{x}_{ij} + \overline{a}_{00}f(\overline{x}_{ij}) + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \overline{a}_{k-i,l-j} f(\overline{x}_{k-i,l-j}) + \sum_{k,l \in N_{ij}} \overline{b}_{k-i,l-j} u_{kl} + \overline{z}_{ij}
\]

\[
= -\overline{x}_{ij} + \overline{a}_{00}\overline{y}_{ij} + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} \overline{a}_{k-i,l-j} \overline{y}_{k-i,l-j} + \sum_{k,l \in N_{ij}} \overline{b}_{k-i,l-j} u_{kl} + \overline{z}_{ij}
\]

\[
= -\overline{x}_{ij} + \sum_{k,l \in N_{ij}} (\overline{a}_{k-i,l-j}\overline{y}_{k-i,l-j} + \overline{b}_{k-i,l-j} u_{kl}) + \overline{z}_{ij} ,
\] (26)

where \( \overline{y}_{k-i,l-j} = f(\overline{x}_{k-i,l-j}) \). It follows then that both dynamics are equivalent. However, note that the original RTD-based CNN and the modified RTD-based CNN are different in the output function.

Finally, we arrive to the generalized CNN state equation

\[
\frac{dx_{ij}}{dt} = -h(x_{ij}) + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} a_{k-i,l-j} k(x_{k-i,l-j}) + \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij} .
\] (27)

In this model, each cell consists of the function \( h(x_{ij}) \) for generating nonlinear dynamics and of the output
function \( k(x_{ij}) \) for coupling the cells. In this case, we have the following relationships:

<table>
<thead>
<tr>
<th>Generalized CNN Model</th>
<th>( h(x_{ij}) )</th>
<th>( k(x_{ij}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>original standard CNN model</td>
<td>( x_{ij} - a_{00} f(x_{ij}) )</td>
<td>( f(x_{ij}) )</td>
</tr>
<tr>
<td>modified standard CNN model</td>
<td>( x_{ij} - a_{00} f(x_{ij}) )</td>
<td>( k(x_{ij}) )</td>
</tr>
<tr>
<td>original RTD-based CNN model</td>
<td>( g(x_{ij}) - \tilde{a}<em>{00} x</em>{ij} )</td>
<td>( x_{ij} )</td>
</tr>
<tr>
<td>modified RTD-based CNN model</td>
<td>( g(x_{ij}) - \tilde{a}<em>{00} x</em>{ij} )</td>
<td>( g(x_{ij}) - \alpha x_{ij} - \beta )</td>
</tr>
</tbody>
</table>

Therefore, these four CNN models are a special case of the generalized CNN model (27).

2.3.2. Higher-order CNN models

Higher-order autonomous CNN models were proposed by [Chua, 1997]. They possess the pattern formation and active wave propagation properties. In this paper, we define the second- and third-order CNN models by using the generalized CNN model

\[
\frac{\zeta}{dt} x_{ij} = v_{ij} - h(x_{ij}) + \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} a_{k-l} k(x_{k-l}) + \sum_{k,l \in N_{ij}} b_{k-l} u_{kl} + z_{ij},
\]

\[
\frac{dv_{ij}}{dt} = x_{ij} - v_{ij},
\]

where \( \delta, \rho, \zeta \) and \( \eta \) are some constants. The circuit implementations of the second- and third-order CNNs are illustrated in Figs. 5 and 6. In the case of the standard CNN model, Eq. (28) has the form

\[
\frac{\zeta}{dt} x_{ij} = v_{ij} - x_{ij} + \sum_{k,l \in N_{ij}} a_{k-l} f(x_{k-l}) + \sum_{k,l \in N_{ij}} b_{k-l} u_{kl} + z_{ij},
\]

\[
\frac{dv_{ij}}{dt} = -x_{ij} - \eta v_{ij},
\]
and
\[
\frac{dx_{ij}}{dt} = v_{ij} - x_{ij} + \sum_{k,l \in N_{ij}} a_{k-i,l-j} f(x_{k-i,l-j}) + \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij},
\]
(31)
\[
\frac{dv_{ij}}{dt} = v_{ij} - d_{ij} + w_{ij},
\]
\[
\frac{dw_{ij}}{dt} = -\zeta v_{ij} - \eta w_{ij}.
\]

Note that \(h(x_{ij}) = x_{ij} - a_{00} f(x_{ij})\) and \(k(x_{ij}) = x_{ij}\). We can also obtain the RTD-based Chua’s reaction-diffusion CNN equations by modifying Eqs. (28) and (29) slightly. In this case, we must use the linear output function \(x_{k-i,l-j}\) instead of the non-linear output function \(k(x_{k-i,l-j})\) [Chua, 1997].

3. Multiple Output Levels

The \(v-i\) characteristic of series connected RTDs can be designed to have multiple peaks. In the case of two series connected RTDs, the \(v-i\) characteristic for the positive voltage region can be modeled by

\[
g(x_{ij}) = \alpha x_{ij} + \gamma_1(|x_{ij} - V_{p1}| - |x_{ij} - V_{c1}|)
+ \gamma_2(|x_{ij} - V_{p2}| - |x_{ij} - V_{c2}|)
- \gamma_1(|x_{ij} + V_{p1}| - |x_{ij} + V_{c1}|)
- \gamma_2(|x_{ij} + V_{p2}| - |x_{ij} + V_{c2}|),
\]
(32)

where \(V_{p1} < V_{p2}\) and \(V_{c1} < V_{c2}\) as illustrated in Fig. 7. In order to simplify the discussion, the slopes \(m_1, m_2\) and \(m_3\) are assumed to be equal, however, note that this is not necessarily true for the slopes \(m_4\) and \(m_5\). If we choose the output function as \(k(x_{ij}) = g(x_{ij}) - \alpha x_{ij}\), then we have

\[
k(x_{ij}) = \gamma_1(|x_{ij} - V_{p1}| - |x_{ij} - V_{c1}|)
+ \gamma_2(|x_{ij} - V_{p2}| - |x_{ij} - V_{c2}|)
- \gamma_1(|x_{ij} + V_{p1}| - |x_{ij} + V_{c1}|)
- \gamma_2(|x_{ij} + V_{p2}| - |x_{ij} + V_{c2}|).
\]
(33)

This function has five output levels: \(\{0, \pm c_1, \pm (c_1 + c_2)\}\), as shown in Fig. 8. Here, \(c_1 = 2\gamma(V_{c1} - V_{p1})\), \(c_2 = 2\gamma(V_{c2} - V_{p2})\), \(d_1 = V_{p1}\), \(d_2 = V_{c1}\), \(d_3 = V_{c2}\), \(d_2 = V_{p2}\). Therefore, if the compound (stacked) RTD has \(m\) peaks, \(2m + 1\) output levels are obtained, and multivalued logic problems can be addressed.

3.1. Multiple steady states of the standard CNN

The dynamics of the uncoupled CNN are described by

\[
\frac{dx_{ij}}{dt} = -x_{ij} + a_{00} f(x_{ij}) + w_{ij},
\]
(34)
where \(w_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij}\), and the
output function is given by

\[
f(x_{ij}) = \frac{1}{2} \left\{ \frac{c_1}{d_2 - d_1} (|x_{ij} - d_1| - |x_{ij} - d_2|) + \frac{c_2}{d_4 - d_3} (|x_{ij} - d_3| - |x_{ij} - d_4|) \right\} \\
- \frac{1}{2} \left\{ \frac{c_1}{d_2 - d_1} (|x_{ij} + d_1| - |x_{ij} + d_2|) + \frac{c_2}{d_4 - d_3} (|x_{ij} + d_3| - |x_{ij} + d_4|) \right\},
\]

where \(c_i > 0\) and \(d_j\) are some constants (0 < \(d_j < d_{j+1}\)). This function has five output levels \(\{0, \pm c_1, \pm (c_1 + c_2)\}\) as shown in Fig. 8. From Fig. 9, we can calculate the steady state \(x_e = x_{ij}(\infty) = \lim_{t \to \infty} x_{ij}(t)\) and the steady state output \(f(x_e) = \lim_{t \to \infty} f(x_{ij}(t))\) of (34) as follows:

<table>
<thead>
<tr>
<th>(x_{ij}(0))</th>
<th>(w_{ij})</th>
<th>(x_e)</th>
<th>(f(x_e))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_{ij}(0) &lt; -d_4)</td>
<td>(w_{ij} &lt; -d_4 + a_{00}(c_1 + c_2))</td>
<td>(-a_{00}(c_1 + c_2) + w_{ij})</td>
<td>(-c_1 - c_2)</td>
</tr>
<tr>
<td>(-d_3 &lt; x_{ij}(0) &lt; -d_2)</td>
<td>(-d_3 + a_{00} c_1 &lt; w_{ij} &lt; -d_2 + a_{00} c_1)</td>
<td>(-c_1 a_{00} + w_{ij})</td>
<td>(-c_1)</td>
</tr>
<tr>
<td>(-d_1 &lt; x_{ij}(0) &lt; d_1)</td>
<td>(-d_1 &lt; w_{ij} &lt; d_1)</td>
<td>(w_{ij})</td>
<td>(0)</td>
</tr>
<tr>
<td>(d_2 &lt; x_{ij}(0) &lt; d_3)</td>
<td>(d_2 - a_{00} c_1 &lt; w_{ij} &lt; d_3 - a_{00} c_1)</td>
<td>(c_1 a_{00} + w_{ij})</td>
<td>(c_1)</td>
</tr>
<tr>
<td>(d_4 &lt; x_{ij}(0))</td>
<td>(d_4 - a_{00} (c_1 + c_2) &lt; w_{ij})</td>
<td>(a_{00} (c_1 + c_2) + w_{ij})</td>
<td>(c_1 + c_2)</td>
</tr>
</tbody>
</table>

Therefore, the output \(y_{ij}(\infty) = \lim_{t \to \infty} y_{ij}(t) = \lim_{t \to \infty} f(x_{ij}(t))\) depends on both the initial condition \(x_{ij}(0)\) and the variable \(w_{ij}\). This property sometimes causes difficulty in designing the templates, but at the same time allows more complicated dynamics in the CNN.

Assume that \(-x_{ij} + a_{00} f(x_{ij})\) is an increasing function of \(x_{ij}\), that is, the inequalities \(d_1 + a_{00} c_1 < d_2\) and \(d_3 + a_{00} c_2 < d_4\) are satisfied. Then, the system has only one steady state for an arbitrarily given \(w_{ij}\). That is, the output \(y_{ij}(\infty) = f(x_{ij}(\infty))\) does not depend on the initial condition \(x_{ij}(0)\).
Then, we can easily design the CNN state equations. For example, if we set \( d_1 = 1, d_2 = 3, d_3 = 4, d_4 = 6, c_1 = c_2 = 1, a_{00} = 1 \) (see Fig. 10), then these two inequalities are satisfied, and we obtain the following table

<table>
<thead>
<tr>
<th>( w_{ij} )</th>
<th>( x_e )</th>
<th>( f(x_e) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{ij} &lt; -4 )</td>
<td>( -2 + w_{ij} )</td>
<td>( -2 )</td>
</tr>
<tr>
<td>( -4 \leq w_{ij} &lt; -3 )</td>
<td>( 2(1 + w_{ij}) )</td>
<td>( w_{ij} + 2 )</td>
</tr>
<tr>
<td>( -3 &lt; w_{ij} &lt; -2 )</td>
<td>( -1 + w_{ij} )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>( -2 \leq w_{ij} &lt; -1 )</td>
<td>( 2w_{ij} + 1 )</td>
<td>( w_{ij} + 1 )</td>
</tr>
<tr>
<td>( -1 &lt; w_{ij} &lt; 1 )</td>
<td>( w_{ij} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( 1 \leq w_{ij} &lt; 2 )</td>
<td>( 2w_{ij} - 1 )</td>
<td>( w_{ij} - 1 )</td>
</tr>
<tr>
<td>( 2 &lt; w_{ij} &lt; 3 )</td>
<td>( 1 + w_{ij} )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( 3 \leq w_{ij} \leq 4 )</td>
<td>( 2(w_{ij} - 1) )</td>
<td>( w_{ij} - 2 )</td>
</tr>
<tr>
<td>( 4 &lt; w_{ij} )</td>
<td>( 2 + w_{ij} )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

In this case, the output state \( f(x_e) \) does not depend on the initial condition \( x_{ij}(0) \), but on the input \( w_{ij} \). Such an input–output characteristic can be also realized by a piecewise-linear function with multiple output levels

\[
y_{ij} = \sigma(w_{ij})
\]

\[
= \frac{1}{2} \left\{ (|w_{ij} - 1| - |w_{ij} - 2|) + (|w_{ij} - 3| - |w_{ij} - 4|) \\
- (|w_{ij} + 1| - |w_{ij} + 2|) - (|w_{ij} + 3| - |w_{ij} + 4|) \right\}.
\]  

### 3.2. Multiple steady states of the RTD-based CNN

In this section, we study the multiple steady states of the uncoupled RTD-based CNN. The dynamics of the uncoupled RTD-based CNN are given by

\[
\frac{dx_{ij}}{dt} = -g(x_{ij}) + a_{00}x_{ij} + w_{ij},
\]

where \( w_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j}u_{kl} + z_{ij} \). If we consider two series connected RTDs with two wells, operating in the positive voltage region, then the \( v-i \) characteristic is given by

\[
g(x_{ij}) = \alpha x_{ij} + \gamma_1(|x_{ij} - V_{p1}| - |x_{ij} - V_{e1}|) \\
+ \gamma_2(|x_{ij} - V_{p2}| - |x_{ij} - V_{e2}|) \\
- \gamma_1(|x_{ij} + V_{p1}| - |x_{ij} + V_{e1}|) \\
- \gamma_2(|x_{ij} + V_{p2}| - |x_{ij} + V_{e2}|),
\]

and the state equation has the form

\[
\frac{dx_{ij}}{dt} = - (\alpha - \tilde{a}_{00})x_{ij} - \gamma_1(|x_{ij} - V_{p1}| - |x_{ij} - V_{e1}|) \\
- \gamma_2(|x_{ij} - V_{p2}| - |x_{ij} - V_{e2}|) \\
+ \gamma_1(|x_{ij} + V_{p1}| - |x_{ij} + V_{e1}|) \\
+ \gamma_2(|x_{ij} + V_{p2}| - |x_{ij} + V_{e2}|) + \tilde{w}_{ij}.
\]
Introducing the new parameters
\[
\tau = (\alpha - a_{00}) t, \quad c_1 = \frac{-2\gamma_1(V_{v1} - V_{p1})}{\alpha - a_{00}},
\]
\[
c_2 = \frac{-2\gamma_2(V_{v2} - V_{p2})}{\alpha - a_{00}}, \quad \overline{w}_{ij} = \frac{\overline{w}_{ij}}{\alpha - a_{00}},
\]

(40)

\[
\frac{dx_{ij}}{dt} = -x_{ij} + f(x_{ij}) + \overline{w}_{ij},
\]

(41)

where

\[
f(x_{ij}) = \frac{1}{2} \left\{ \frac{c_1}{d_2 - d_1} (|x_{ij} - d_1| - |x_{ij} - d_2|) + \frac{c_2}{d_4 - d_3} (|x_{ij} - d_3| - |x_{ij} - d_4|) \right\}
\]

\[-\frac{1}{2} \left\{ \frac{c_1}{d_2 - d_1} (|x_{ij} + d_1| - |x_{ij} + d_2|) + \frac{c_2}{d_4 - d_3} (|x_{ij} + d_3| - |x_{ij} + d_4|) \right\}.
\]

(42)

Therefore, the dynamics of the uncoupled RTD-based CNN are equivalent to those of the uncoupled standard CNN. However, the output function is equal to \(x_{ij}\), so it does not possess saturated levels. The multiple-valued output function quite frequently simplifies the design of CNNs. If we wanted to design a coupled RTD-based CNN, we should assume that \(|x_{ij}| \leq 1\), or that the output function \(x_{ij}\) has the saturated property [Hänggi & Chua, 2000].

The steady state \(x_e\) of the RTD-based CNN is given as follows (for reference, \(f(x_e)\) is also given)

\[
\begin{align*}
(1) & \text{ if } x_{ij}(0) < -V_{v2} \quad \text{and} \quad \frac{2(\gamma_1(V_{v1} - V_{p1}) + \gamma_2(V_{v2} - V_{p2})) + w_{ij}}{\alpha - a_{00}} < -V_{v2}, \\
& \text{then } x_e = \frac{2(\gamma_1(V_{v1} - V_{p1}) + \gamma_2(V_{v2} - V_{p2})) + w_{ij}}{\alpha - a_{00}}, \quad f(x_e) = \frac{2\gamma_1(V_{v1} - V_{p1}) + \gamma_2(V_{v2} - V_{p2})}{\alpha - a_{00}}.
\end{align*}
\]

\[
\begin{align*}
(2) & \text{ if } -V_{p2} < x_{ij}(0) < -V_{v1} \quad \text{and} \quad -V_{p2} < \frac{2\gamma_1(V_{v1} - V_{p1}) + w_{ij}}{\alpha - a_{00}} < -V_{v1}, \\
& \text{then } x_e = \frac{2\gamma_1(V_{v1} - V_{p1}) + w_{ij}}{\alpha - a_{00}}, \quad f(x_e) = \frac{2\gamma_1(V_{v1} - V_{p1})}{\alpha - a_{00}}.
\end{align*}
\]

\[
\begin{align*}
(3) & \text{ if } -V_{p1} < x_{ij}(0) < V_{p1} \quad \text{and} \quad -V_{p2} < \frac{w_{ij}}{\alpha - a_{00}} < V_{p1}, \\
& \text{then } x_e = \frac{w_{ij}}{\alpha - a_{00}}, \quad f(x_e) = 0.
\end{align*}
\]

\[
\begin{align*}
(4) & \text{ if } V_{p2} < x_{ij}(0) < V_{v1} \quad \text{and} \quad V_{p2} < \frac{-2\gamma_1(V_{v1} - V_{p1}) + w_{ij}}{\alpha - a_{00}} < V_{v1}, \\
& \text{then } x_e = \frac{-2\gamma_1(V_{v1} - V_{p1}) + w_{ij}}{\alpha - a_{00}}, \quad f(x_e) = -\frac{2\gamma_1(V_{v1} - V_{p1})}{\alpha - a_{00}}.
\end{align*}
\]
Therefore, the RTD-based CNN equation has only one steady state $x_e$, for a given $w_{ij}$, if the following conditions are satisfied:

$$
\begin{align*}
-\alpha - 2\gamma_1 + a_{00} &< 0, \\
-\alpha - 2\gamma_2 + a_{00} &< 0.
\end{align*}
$$

(43)

In this case, the output states do not depend on the initial conditions $x_{ij}(0)$, but on the input $w_{ij}$. Thus, the dynamics are equivalent to those of the standard CNN for $a_{00} = 0$, that is,

$$
\frac{dx_{ij}}{dt} = -x_{ij} + w_{ij},
$$

(44)

where $w_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij}$. Note that the steady state of this equation is equal to $w_{ij}$, however the steady state of the RTD-based CNN is not equal to $w_{ij}$.

### 3.3. Second-order isolated CNN cell

The second-order CNN may have oscillating solutions. In this section, we study the steady states of the second-order isolated CNN cell with multiple output levels. The dynamics are given by

$$
\begin{align*}
\varepsilon \frac{dx_{ij}}{dt} &= v_{ij} - x_{ij} + a_{00} f(x_{ij}) + w_{ij}, \\
\frac{dv_{ij}}{dt} &= -x_{ij} - \eta v_{ij},
\end{align*}
$$

(45)

where

$$
w_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij}.
$$

(46)

Here, we assume that

$$
\begin{align*}
0 < \varepsilon \ll 1, \\
a_{00} = 2, \quad \eta = 0.1, \\
f(x_{ij}) = \frac{1}{6}\left\{(|x_{ij} - 1| - |x_{ij} - 4|) - (|x_{ij} + 1| - |x_{ij} + 4|)\right\}.
\end{align*}
$$

Then, we get the following relations from Figs. 11 and 12

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>$x_e$</th>
<th>$f(x_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{ij} \leq -38$</td>
<td>$\frac{w_{ij} - 6}{11}$</td>
<td>1</td>
</tr>
<tr>
<td>$-38 &lt; w_{ij} \leq -11$</td>
<td>limit cycle</td>
<td>oscillation</td>
</tr>
<tr>
<td>$-11 &lt; w_{ij} &lt; 11$</td>
<td>limit cycle or $\frac{w_{ij}}{11}$</td>
<td>oscillation or 0</td>
</tr>
<tr>
<td>$11 \leq w_{ij} &lt; 38$</td>
<td>limit cycle</td>
<td>oscillation</td>
</tr>
<tr>
<td>$38 \leq w_{ij}$</td>
<td>$\frac{w_{ij} + 6}{11}$</td>
<td>1</td>
</tr>
</tbody>
</table>

As we can easily see from this table, the steady state depends on the initial conditions if $-11 < w_{ij} < 11$. Furthermore, the parameter $w_{ij}$ serves as a bifurcation parameter. We can easily incorporate chaotic attractors as one of the steady states by using the third-order CNN equation (29).
Fig. 11. Characteristic of the nonlinear function \( h(x_{ij}) = x_{ij} - a_{00}f(x_{ij}) \).
3.4. Second-order isolated RTD-based CNN cell

The dynamics of the second-order RTD-based CNN are governed by the equations
\[
\begin{align*}
\varepsilon \frac{dx_{ij}}{dt} &= v_{ij} - h(x_{ij}) + w_{ij}, \\
\frac{dv_{ij}}{dt} &= -x_{ij} - \eta v_{ij},
\end{align*}
\]
where
\[
\begin{align*}
h(x_{ij}) &= g(x_{ij}) - a_{90}x_{ij}, \\
w_{ij} &= \sum_{k,l \in N_{ij}} b_{k-l,i-l} u_{kl} + z_{ij}.
\end{align*}
\]
Here, \(\varepsilon\) and \(\eta\) are positive constants. It is easy to see that these equations are equivalent to those of the second-order CNN, and therefore the same conclusions apply.

4. Synthesis of Desired Equilibrium Points

A method for the synthesis of equilibrium points of CNN was proposed in [Liu & Michel, 1993]. This method was applied later to the case of the CNN with multiple output levels in [Kanagawa et al., 1996]. In those references, the asymptotically stable equilibrium points are considered as vectors to be memorized. In this section, we illustrate the design in the case of the standard CNN, and then we show how it can be extended to the RTD-based CNN.

4.1. Equilibrium points of the standard CNN

In this section, we explore the potential of the standard CNN with multiple steady states. First, we rewrite the state equation of the standard CNN into the form
\[
\begin{align*}
\frac{dx}{dt} &= -x + T f(x) + S u + z,
\end{align*}
\]
where
\[
\begin{align*}
x &= (x_{11}, x_{12}, \ldots, x_{MN})^T, \\
u &= (u_{11}, u_{12}, \ldots, u_{MN})^T, \\
z &= (z_{11}, z_{12}, \ldots, z_{MN})^T, \\
f(x) &= (f(x_{11}), f(x_{12}), \ldots, f(x_{MN}))^T, \\
T &= [T_{ij}], \quad S = [S_{ij}] \in R^{n \times n} \quad (n = M \times N).
\end{align*}
\]

Here, we assume that
\[
\begin{align*}
f(x_{ij}) &= \frac{1}{2} \left\{ \frac{c_1}{d_2 - d_1} (|x_{ij} - d_1| - |x_{ij} - d_2|) \\
&+ \frac{c_2}{d_4 - d_3} (|x_{ij} - d_3| - |x_{ij} - d_4|) \right\} \\
&- \frac{1}{2} \left\{ \frac{c_3}{d_2 - d_1} (|x_{ij} + d_1| - |x_{ij} + d_2|) \\
&+ \frac{c_4}{d_4 - d_3} (|x_{ij} + d_3| - |x_{ij} + d_4|) \right\},
\end{align*}
\]
where \(c_1 = c_2 = 1\). Furthermore, for the sake of simplicity, we study the case where \(Su + z = 0\). We now define \(n(=MN)\) linearly independent memory vectors
\[
p_i = (p_{i1}, p_{i2}, \ldots, p_{in}), \quad i = 1, 2, \ldots, n
\]
where \(p_{ij} \in \{0, \pm 1, \pm 2\}\). If \(x_e = r_i p_i\) \((r_i > 0)\) is an equilibrium point it satisfies
\[
-r_i p_i + T f(r_i p_i) = 0.
\]
Let us choose the parameters \(r_i\) and \(d_k\) so that for any \(p_i\),
\[
f(r_i p_i) = p_i,
\]
or equivalently,
\[
f(r_i p_{ij}) = p_{ij}.
\]

In this case, the parameters \(r_i\) and \(d_k\) satisfy the following conditions
\[
\begin{align*}
r_i &\geq 0, \\
d_2 &\leq r_i \leq d_3, \\
d_4 &\leq 2r_i.
\end{align*}
\]
These conditions are easily satisfied if we set \(3 \leq r_i \leq 4, d_1 = 1, d_2 = 2, d_3 = 4, d_4 = 5\). Furthermore, let us suppose that \(r_i p_{ij} \neq d_k\). Then, we get the relation
\[
T p_i = r_i p_i,
\]
or
\[
T(p_1, p_2, \ldots, p_m)
= \text{diag}(r_1, r_2, \ldots, r_m)(p_1, p_2, \ldots, p_m),
\]

\[\text{Note the implicit dependency of } f \text{ on } d_k.\]

\[\text{These conditions can be obtained from (56).}\]
where the symbol diag denotes a diagonal matrix. Thus, if \( m = n \), then matrix \( T \) can be written as

\[
T = PRP^{-1}, \tag{60}
\]

where \( R = \text{diag}(r_1, r_2, \ldots, r_n) \). In the case where \( m < n \) we can use the singular value decomposition. For more details, see [Liu & Michel, 1993]. All the equilibrium points \( r_jp_j \) are asymptotically stable since

\[
\frac{\partial F(x)}{\partial x} \bigg|_{x=r_jp_j} = \text{diag}(-1, -1, \ldots, -1), \tag{61}
\]

where \( F(x) = -x + Tf(x) \). Note that the origin is also an asymptotically stable equilibrium point. Furthermore, if the vector \( q = \sum a_i p_i \) satisfies the condition \( q = f(r_jq) \), then \( r_jq \) is also an equilibrium point, i.e.

\[
F(r_jq) = -r_j \sum a_i p_i + Tf(r_j \sum a_i p_i) = -r_j \sum a_i p_i + T \sum a_i p_i = -r_j \sum a_i p_i + \sum a_i r_j p_i = 0. \tag{62}
\]

For example, if \( q = p_1 + p_2 \), then we have the following relation

<table>
<thead>
<tr>
<th>( p_{1j} )</th>
<th>( p_{2j} )</th>
<th>( q_j = p_{1j} + p_{2j} )</th>
<th>( f(r_jq_j) )</th>
<th>( \Rightarrow )</th>
<th>( q_j = f(r_jq_j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>|</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>|</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>|</td>
<td>T</td>
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<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>|</td>
<td>T</td>
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<tr>
<td>2</td>
<td>-2</td>
<td>0</td>
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<td>F</td>
</tr>
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<td>-2</td>
<td>-2</td>
<td>-4</td>
<td>-2</td>
<td>|</td>
<td>F</td>
</tr>
</tbody>
</table>
where the symbols T and F indicate “true” and “false”, respectively. As we can see, some cases are false. However, if all the images are given by binary data, i.e. values in the set \{-1, 1\}, then we obtain the following table:

<table>
<thead>
<tr>
<th>$p_{ij}$</th>
<th>$f(r_{ij}p_{ij})$</th>
<th>$p_j = f(r_{ij}p_{ij})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>T</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>T</td>
</tr>
</tbody>
</table>

Then the patterns stored as $p_i$ are colored red, new patterns generated by a linear combination of $p_i$ are colored white, and the intersection points of two images are colored yellow. Thus, the CNN with multiple output levels is able to perform the complicated task. Note that we are not imposing restrictions on matrix $T$. More generalized designs are proposed in [Liu & Michel, 1993].

It is also possible to specify a design using the following methodology that introduces two different types of memory patterns

$$p_i = (p_{i1}, p_{i2}, \ldots, p_{in}), \ (p_{ij} \in \{0, 1\}),$$

$$q_k = (q_{k1}, q_{k2}, \ldots, q_{kn}), \ (q_{kl} \in \{0, -1\}),$$

which are used as the inputs $w_{ij}$. Let us choose the following parameters and palette

$$T = \text{diag}(1, 1, \ldots, 1), \quad a_{00} = 1, \quad b_{00} = 1,$$

$$c_1 = 1, \quad c_2 = 1, \quad d_1 = 0.4, \quad d_2 = 0.8.$$  \hspace{2cm} (64)

<table>
<thead>
<tr>
<th>$y_{ij}$</th>
<th>color</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>yellow</td>
</tr>
<tr>
<td>1</td>
<td>red</td>
</tr>
<tr>
<td>0</td>
<td>white</td>
</tr>
<tr>
<td>-1</td>
<td>green</td>
</tr>
<tr>
<td>-2</td>
<td>green</td>
</tr>
</tbody>
</table>

Fig. 13. The driving point plot. $h(x_{ij}) = -x_{ij} + a_{00}f(x_{ij})$. 

\hspace{2cm} $0.8 < x_{ij} \Rightarrow$ red 
\hspace{2cm} $-0.4 < x_{ij} < 0.4 \Rightarrow$ white (background) 
\hspace{2cm} $x_{ij} < -0.8 \Rightarrow$ green
From the driving point plot \( h(x_{ij}) = -x_{ij} + a_{00} f(x_{ij}) \) shown in Fig. 13, we have the following table:

<table>
<thead>
<tr>
<th>( u_{ij} )</th>
<th>( x_{ij}(0) )</th>
<th>( x_{e} = x_{ij}(\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

where the symbols G, R and W denote green, red and white, respectively. The output patterns for several inputs and initial conditions are illustrated in Fig. 14. From Figs. 14(b)–14(d) we see that it is possible to use three kinds of patterns. Furthermore, if two memory patterns belong to the same group \( (p_i \) or \( q_{ki} \) and have no crossing points, then their combination is a new pattern. If two memory patterns belong to different groups (one belongs to \( p_i \) and the other to \( q_{ki} \)), then their combination is also a new pattern, and a crossing point of two patterns becomes white (background color), that is, it disappears [see Figs. 14(e)–14(h)]. If the pattern \( o_j \) is obtained from a linear combination of several memory patterns, then the coloring of the crossing points is decided by the majority.

### 4.2. Equilibrium points of RTD-based CNN

In case of the RTD-based CNN, the dynamics can be written as

\[
\frac{dx}{dt} = -g(x) + T x + Su + z,
\]

where

\[
x = (x_{11}, x_{12}, \ldots, x_{MN})^T,
\]

\[
u = (u_{11}, u_{12}, \ldots, u_{MN})^T,
\]

\[
z = (z_{11}, z_{12}, \ldots, z_{MN})^T,
\]

\[
g(x) = (g(x_{11}), g(x_{12}), \ldots, g(x_{MN}))^T,
\]

\[
T = [T_{ij}], \quad S = [S_{ij}] \in R^{n \times n} \quad (n = M \times N).
\]

For the sake of simplicity, we study the case where \( Su + z = 0 \) and

\[
g(x_{ij}) = \alpha x_{ij} + \gamma_1 (|x_{ij} - V p_1| - |x_{ij} - V v_1|) + \gamma_2 (|x_{ij} - V p_2| - |x_{ij} - V v_2|) - \gamma_1 (|x_{ij} - V p_1| - |x_{ij} + V v_1|) - \gamma_2 (|x_{ij} + V p_2| - |x_{ij} + V v_2|).
\]

Then the state equation can be written as

\[
\frac{dx}{dt} = -h(x) + T' x,
\]

where \( h(x) = g(x) - \alpha x \) and \( T' = T - \alpha \). Let \( x_e = r_i p_i \) \((r_i < 0)\) be an equilibrium point. Then, we have the equation

\[
-h(r_i p_i) + T' r_i p_i = 0.
\]

Assume that \( r_i \) satisfies

\[
h(r_i p_i) = p_i,
\]

in which case, we obtain

\[
-r_i^{-1} (r_i p_i) + T' (r_i p_i) = 0
\]

and then

\[
T' = P'R'P'^{-1},
\]

where \( P' = (r_1 p_1, r_2 p_2, \ldots, r_n p_n) \) and \( R' = \text{diag}(r_1^{-1}, r_2^{-1}, \ldots, r_n^{-1}) \). The equilibrium point \( x_e = r_i p_i \) is asymptotically stable, since \( r_i < 0 \). Once more, we are in the same situation as in the case of the standard CNN, and similar conclusions apply.
Fig. 14. Output patterns. (a) Initial condition $u_{ij}$, (b) output $y_{ij}$ for $x_{ij}(0) = -1$, (c) output $y_{ij}$ for $x_{ij}(0) = 1$, (d) output $y_{ij}$ for $x_{ij}(0) = 0$, (e) input pattern $s_1$, (f) input pattern $s_2$, (g) input $s_1 + s_2$, (h) output $y_{ij}$ for $s_1 + s_2$. 
5. Pattern Formation of Second-Order CNN

The standard CNN possesses the spatial pattern formation property, i.e. it is able to generate patches, checkerboards, stripes, etc. Chua’s reaction–diffusion CNN can generate spatiotemporal patterns, such as target patterns, spiral waves, and scroll waves [Chua, 1997]. In this section, we study the pattern formation properties of the second-order CNN proposed in Sec. 2. Let us consider the second-order CNN

\[ \frac{d^2 x_{ij}}{dt^2} = v_{ij} - x_{ij} + \sum_{k,l \in N_{ij}} a_{ij} f(x_{k-l,l-j}) + \sum_{k,l \in N_{ij}} b_{k-l,l-j} u_{kl} + z_{ij}, \]  

(73)

\[ \frac{dv_{ij}}{dt} = -x_{ij} - \eta v_{ij}, \]

where

\[ f(x_{k-l,l-j}) = \frac{1}{2}(|x_{k-l,l-j} + 1| - |x_{k-l,l-j} - 1|). \]  

(74)

For the analytical investigation of Eq. (73), it is convenient to transform this equation into the coupled oscillators form. By defining the variables

\[ a_{ij} = A^+_0 + 2, \quad \text{if } (i, j) = (0, 0), \]

\[ a_{ij} = A^+_i, \quad \text{if } (i, j) \neq (0, 0), \]  

(75)

we obtain

\[ \frac{d^2 x_{ij}}{dt^2} = v_{ij} - g(x_{ij}) + \sum_{k,l \in N_{ij}} A^+_k f(x_{k-l,l-j}) + \sum_{k,l \in N_{ij}} b_{k-l,l-j} u_{kl} + z_{ij}, \]  

(76)

\[ \frac{dv_{ij}}{dt} = -x_{ij} - \eta v_{ij}, \]

where

\[ g(x_{ij}) = x_{ij} - 2f(x_{ij}) = x_{ij} - |x_{k-l,l-j} + 1| + |x_{k-l,l-j} - 1|. \]  

(77)

In this system, each cell consists of a piecewise-linear Bonhoeffer–Van der Pol oscillator, which has a limit cycle for \( \eta \ll 1 \) and \( \zeta > 0 \). In this case, we will set \( \eta = 0.1 \) and \( \zeta = 0.2 \).

In Chua’s reaction–diffusion CNNs, the following one- and two-dimensional discrete Laplacian templates are used as the A feedback template:

\[ A^+ = [a^+_0, a^+_1], \quad A = [a_0, a_1] = [a^+_0 + 2, a^+_1], \]

(79)

\[ B = [b_0, b_1], \]

Let us study the pattern formation of the one-dimensional CNN, which consists of N cells. Here, we change notation from \( x_{ij}, a_{ij}, b_{ij}, z_{ij} \) to \( x_i, a_i, b_i, z_i \), respectively. In this case, the templates \( A^+, A, B \) and \( z_i \) have the form

\[ A^+ = [1 -2 1], \quad A = [1 0 1], \quad B = [0 0 0], \quad z_i = [0], \]  

(80)

\[ A^+ = [1 -1 0], \quad A = [1 1 0], \quad B = [0 0 0], \quad z_i = [0], \]  

(81)

\[ A^+ = [0 -1 1], \quad A = [0 1 1], \quad B = [0 0 0], \quad z_i = [0], \]  

(82)

and the fixed boundary condition

\[ x_0 = -1, \quad x_{N+1} = -1, \]

(83)

under the initial condition \( x_i(0) = -1, v_i(0) = 0 \) \( (i = 1, 2, \ldots, N) \). Furthermore, let us define the following palette for the state \( x_{ij} \):

- \( 1 < x_{ij} \) \( \Rightarrow \) red
- \( -1 \leq x_{ij} \leq 1 \) \( \Rightarrow \) gray scale
- \( x_{ij} < -1 \) \( \Rightarrow \) green
Then, we can observe traveling waves, i.e. wave propagation, as illustrated in Figs. 15 and 16. In the case where $A^\dagger = [1, -2, 1]$, a red strip decreases gradually, and a new one of green color increases gradually. Then, the green strip starts to decrease, and a new red strip increases (See Fig. 15). In the case where $A^\dagger = [1, -1, 0]$, a red strip and a green strip move from the left to the right as shown in Fig. 16. In the case where $A^\dagger = [0, -1, 1]$, the strips move from the right to the left.

If we choose the following templates:

\[
A^\dagger = [-1 \quad 0 \quad -1], \quad A = [-1 \quad 2 \quad -1], \quad B = [0 \quad 0 \quad 0], \quad z_i = [0],
\]

(84)

A forward Euler algorithm with a time step size of 0.05 was used for all the numerical integrations. It has an error proportional to the time step size. Therefore, if other integration methods or time step sizes are used for the numerical integration, the simulation results may be slightly different from the results obtained in this paper.
then we can observe the traveling waves on the striped patterns as shown in Figs. 17 and 18. In the case where $A^t = [-1, 0, 1]$, black and gray pixels appear at both ends, move to the center and disappear after a collision. Then, new black pixels appear at the end, and have a collision again (see Fig. 17). In the case where $A^t = [-1, -1, 0]$, black and gray pixels appear at the left end, and move from the left to the right on the stripe pattern shown in Fig. 18. In the case where $A^t = [0, -1, -1]$, the patterns move from the right to the left.

If we choose the following templates:

$$A^t = [-1, 1, 0], \quad A = [-1, 1, 0], \quad B = [0, 0, 0], \quad z_i = [0],$$  \hspace{1cm} (85)

$$A^t = [0, -1, 1], \quad A = [0, 1, -1], \quad B = [0, 0, 0], \quad z_i = [0],$$  \hspace{1cm} (86)

then we observe the patterns illustrated in Fig. 19. In this case, groups of red pixels move from the left to the right.

Some interesting effects can be observed if we introduce nonidentical cells. Let us assume that some cells do no have limit cycles, but have asymptotically stable equilibrium points at $(x_{ij}, v_{ij}) = (3, 30)$. Then, these cells are barriers for the wave
propagation. If we choose the following template:

\[
A^\dagger = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad z_i = [0],
\]

then we observe the behavior shown in Fig. 20. A red strip moves from the left to the right tunneling through the barriers (red cells at center). Furthermore, it is combined with a new strip generated by the barrier when it passes through the barrier.

The pattern formation phenomena can also be found in two-dimensional CNN. In this case we will use the parameter \(a_{00}\) and the function \(h(x_{ij})\) defined as

\[
a_{00} = a_{00}^\dagger + 2q, \quad h(x_{ij}) = x_{ij} - 2q f(x_{i,j}),
\]

where \(q\) is a constant. Let us choose the following feedback template

\[
A^\dagger = \frac{1}{2} \begin{bmatrix} 0 & q & 0 \\ q & -4q & q \\ 0 & q & 0 \end{bmatrix}, \quad A = \frac{1}{2} \begin{bmatrix} 0 & q & 0 \\ q & 0 & q \\ 0 & q & 0 \end{bmatrix},
\]

or

\[
A^\dagger = \frac{1}{4} \begin{bmatrix} q & q & q \\ q & -8q & q \\ q & q & q \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} q & q & q \\ q & 0 & q \\ q & q & q \end{bmatrix},
\]

and the fixed boundary condition

\[
x_{i,0} = -1, \quad i = 0, 1, \ldots, N + 1
\]
\[
x_{i,N+1} = -1, \quad i = 0, 1, \ldots, N + 1
\]
\[
x_{0,j} = -1, \quad j = 0, 1, \ldots, N + 1
\]
\[
x_{N+1,j} = -1, \quad i = 0, 1, \ldots, N + 1
\]

Furthermore, let us use uniformly distributed noise in the range \([-0.1, 0.1]\) as the initial condition for
Fig. 21. Target patterns. (a) Initial condition $x_{ij}(0)$, (b) $t = 7.0$, (c) $t = 7.25$, (d) $t = 7.75$, (e) $t = 8.15$, (f) $t = 8.4$, (g) $t = 8.8$, (h) $t = 9.6$. 
Then, we can observe the target pattern shown in Fig. 21. We can also observe the spiral waves shown in Fig. 22, if we use the zero-flux boundary condition

\[
\begin{align*}
  x_{i,0} &= x_{i,1} & i &= 0, 1, \ldots, N + 1 \\
  x_{i,N+1} &= x_{i,N} & i &= 0, 1, \ldots, N + 1 \\
  x_{0,j} &= x_{1,j} & j &= 0, 1, \ldots, N + 1 \\
  x_{N+1,j} &= x_{N,j} & i &= 0, 1, \ldots, N + 1
\end{align*}
\]  

(94)

By introducing modifications on the templates we can observe several behaviors. If we choose the template

\[
\begin{align*}
  A^\dagger &= \begin{bmatrix} 0 & -q & 0 \\ q & 2q & q \\ 0 & -q & 0 \end{bmatrix}, & A &= \begin{bmatrix} 0 & -q & 0 \\ q & 4q & q \\ 0 & -q & 0 \end{bmatrix}, \\
  B &= [0], & z_{ij} &= [0], & q &= 1,
\end{align*}
\]

then we see spiral waves on the horizontal stripe pattern.

If we choose the template

\[
\begin{align*}
  A^\dagger &= \begin{bmatrix} 0 & q & 0 \\ -q & 2q & -q \\ 0 & q & 0 \end{bmatrix}, & A &= \begin{bmatrix} 0 & q & 0 \\ -q & 4q & -q \\ 0 & q & 0 \end{bmatrix}, \\
  B &= [0], & z_{ij} &= [0], & q &= 1,
\end{align*}
\]

we observe spiral waves on the vertical stripe pattern as illustrated in Fig. 23.

If we choose the template

\[
\begin{align*}
  A^\dagger &= \begin{bmatrix} 0 & -q & 0 \\ -q & -2q & -q \\ 0 & -q & 0 \end{bmatrix}, & A &= \begin{bmatrix} 0 & -q & 0 \\ -q & 0 & -q \\ 0 & -q & 0 \end{bmatrix}, \\
  B &= [0], & z_{ij} &= [0], & q &= 1.3,
\end{align*}
\]

we observe the spiral waves on the checkerboard as shown in Fig. 24.

If we choose the template

\[
\begin{align*}
  A^\dagger &= \begin{bmatrix} -q & q & -q \\ q & p^\dagger & q \\ -q & q & -q \end{bmatrix}, & A &= \begin{bmatrix} -q & q & -q \\ q & p & q \\ -q & q & -q \end{bmatrix}, \\
  B &= [0], & z_{ij} &= [0], & q &= 1.2,
\end{align*}
\]

we obtain the continuously changing maze patterns shown in Fig. 25. Here, \( p^\dagger \geq 0 \) and \( p = p^\dagger + 2q \) (e.g. \( p^\dagger = 6q, \ p = 8q \)).

Finally, we simulated a second-order CNN with the following multiple output levels

\[
\begin{align*}
  f(x_{k-i,l-j}) &= \frac{5}{4}(|x_{k-i,l-j} - 0.4| - |x_{k-i,l-j} - 0.8|) \\
  &= \frac{5}{4}(|x_{k-i,l-j} + 0.4| - |x_{k-i,l-j} + 0.8|).
\end{align*}
\]

(99)

In this case, the spatial patterns are not activated whenever \(|x_{k-i,l-j}(0)| \leq 0.4\). That is, \(|x_{k-i,l-j}(t)| \to 0\) for \( t \to \infty \). In order to activate the spatial patterns, it is necessary to choose initial conditions satisfying \(|x_{k-i,l-j}(0)| > 0.4\). Then, it is possible to observe a number of spatial patterns similar to those generated by the standard output function. Furthermore, if we choose the following output function

\[
\begin{align*}
  f(x_{k-i,l-j}) &= \frac{5}{4}(|x_{k-i,l-j} - 0.4| - |x_{k-i,l-j} - 0.8|) \\
  &= \frac{5}{4}(|x_{k-i,l-j} + 0.4| - |x_{k-i,l-j} + 0.8|) \\
  &+ \frac{5}{4}(|x_{k-i,l-j} + 1.0| - |x_{k-i,l-j} + 1.4|)
\end{align*}
\]

(100)

then we can observe the rectangular spiral waves shown in Fig. 26. In this case, the output function \( f(x_{k-i,l-j}) \) has nonzero values in the local region \( 0.4 < |x_{k-i,l-j}| < 1.4 \) as illustrated in Fig. 27.

6. Design of the RTD-Based CNN

In this section, we present a method for designing the RTD-based CNN with multiple output levels. We assume that the parameters of the RTD are given by

\[
\begin{align*}
  \alpha &= \frac{15}{4}, & \gamma &= -\frac{5}{2}, & V_{p1} &= 0.4, & V_{o1} &= 0.8,
\end{align*}
\]

(101)

and so

\[
\begin{align*}
  g(x_{ij}) &= \frac{15}{4}x_{ij} - \frac{5}{2}(|x_{ij} - 0.4| - |x_{ij} - 0.8|) \\
  &+ \frac{5}{2}(|x_{ij} + 0.4| - |x_{ij} + 0.8|)
\end{align*}
\]

(102)
Fig. 22. Spiral waves. (a) Initial condition $x_{ij}(0)$, (b) $t = 40$, (c) $t = 45$, (d) $t = 50$, (e) $t = 55$, (f) $t = 60$. 
Fig. 23. Spiral waves on the vertical stripe pattern. (a) Initial condition of $x_{ij}$, (b-f) spiral patterns.
Fig. 24. Spiral waves on the checker pattern. (a) Initial condition of $x_{ij}$, (b–f) spiral waves.
Fig. 25. Maze patterns. (a) Initial condition of $x_{ij}$, (b–f) maze patterns.
Fig. 26. Rectangular spiral waves. (a) Initial condition of $x_{ij}$, (b–f) spiral waves.
The $v-i$ characteristic $g(x_{ij})$ is illustrated in Fig. 28. Then, the state equation is given by

$$\frac{dx_{ij}}{dt} = -\frac{15}{4} x_{ij} + \frac{5}{2} (|x_{ij} - 0.4| - |x_{ij} - 0.8|) - \frac{5}{2} (|x_{ij} + 0.4| - |x_{ij} + 0.8|) + a_{00} x_{ij} + w_{ij},$$

(103)

where $w_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij}$. The steady states for $a_{00} = 0$ are described in the following table.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Steady State $x_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $1 &lt; w_{ij}$ and $0.8 &lt; x_{ij}(0)$,</td>
<td>$x_e = \frac{4(w_{ij} + 2)}{15}$</td>
</tr>
<tr>
<td>If $-1.5 &lt; w_{ij} &lt; 1.5$ and $-0.4 &lt; x_{ij}(0) &lt; 0.4$,</td>
<td>$x_e = \frac{4w_{ij}}{15}$</td>
</tr>
<tr>
<td>If $w_{ij} &lt; -1$ and $x_{ij}(0) &lt; -0.8$,</td>
<td>$x_e = \frac{4(w_{ij} - 2)}{15}$</td>
</tr>
</tbody>
</table>
and the number of the steady states is:

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>Number of Stable States</th>
<th>Number of Unstable States</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5 &lt; w_{ij}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$1 &lt; w_{ij} &lt; 1.5$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$-1 &lt; w_{ij} &lt; 1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-1.5 &lt; w_{ij} &lt; -1$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$w_{ij} &lt; -1.5$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

If we use the following three regions:

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>Number of Stable States</th>
<th>Number of Unstable States</th>
<th>$x_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5 &lt; w_{ij}$</td>
<td>1</td>
<td>0</td>
<td>$x_e = \frac{4(w_{ij} + 2)}{15}$</td>
</tr>
<tr>
<td>$-1 &lt; w_{ij} &lt; 1$</td>
<td>1</td>
<td>0</td>
<td>$x_e = \frac{4w_{ij}}{15}$</td>
</tr>
<tr>
<td>$w_{ij} &lt; -1.5$</td>
<td>1</td>
<td>0</td>
<td>$x_e = \frac{4(w_{ij} - 2)}{15}$</td>
</tr>
</tbody>
</table>

then only three different steady states are available. In this case, we can apply the following palette:

- $1 < x_{ij}$    ⇒ red
- $-0.4 < x_{ij} < 0.4$ ⇒ white
- $x_{ij} < -1$  ⇒ green

That is, red pixels are used for values greater than 1, green pixels are used for values smaller than $-1$, and white pixels are used for values between $-0.4$ and $0.4$. A number of interesting examples in the case of the RTD-based CNN have been given in [Hänggi & Chua, 2000]. In this paper, we examine them from the viewpoint of the multiple valued CNN. Furthermore, some new examples are presented.

**6.1. Example 1 (Change of color)**

This operation changes red pixels into green pixels, and green pixels into red pixels. However, white pixels, that is, background pixels, are not changed:

- red ⇒ green
- white ⇒ white
- green ⇒ red

That is, for a given input $u_{ij}$, the system is required to have the following property

- $u_{ij} = 1$ ⇒ $x_{ij}(\infty) \leq -1$
- $u_{ij} = -1$ ⇒ $x_{ij}(\infty) \geq 1$
- $u_{ij} = 0$ ⇒ $|x_{ij}(\infty)| \leq 0.4$

This can be realized by an amplifier with negative gain. Here, for the sake of simplicity, we assume that $z_{ij} = 0$. Then, the parameter $b_{ij}$ should be smaller than $-1.5$, since the system has a single steady state if $|w_{ij}| > 1.5$. Accordingly, we set $b_{ij} = -2$ ($< -1.5$). The templates for this task are given by

$$A = [0 \ 0 \ 0], \ B = [0 \ -2 \ 0], \ z_{ij} = [0]. \quad (104)$$

Figure 29 shows an example of the change of colors.

**6.2. Example 2 (Horizontal line and ending point detection)**

This operation deletes horizontally isolated points. Horizontal lines (red pixels) consisting of at least two points remain and the ending points of
horizontal lines are marked white. The background color is green and we assume that the initial state of \( x_{ij} \) is arbitrarily given. The task is shown in the following table:

\[
\begin{array}{c|c|c|c|c}
  u_{i,j-1} & u_{i,j} & u_{i,j+1} & \Rightarrow & x_e = x_{i,j}(\infty) \\
  \hline
  R & R & R & \Rightarrow & x_e = R \\
  R & R & G & \Rightarrow & x_e = W \\
  R & G & R & \Rightarrow & x_e = G \\
  R & G & G & \Rightarrow & x_e = G \\
  G & R & R & \Rightarrow & x_e = W \\
  G & R & G & \Rightarrow & x_e = G \\
  G & G & R & \Rightarrow & x_e = G \\
  G & G & G & \Rightarrow & x_e = G \\
\end{array}
\]

where R, G and W indicate red, green and white, respectively. In this case, we examine two templates \( B_1 \) and \( B_2 \), namely, \( B_1 = [1 1 1] \) and \( B_2 = [1 2 1] \).
where $u = [u_{i,j-1}, u_{i,j}, u_{i,j+1}]^T$, and the symbol $^T$ indicates the transpose of the vector. If we use $B_1 = [1 1 1]$ as the B-template, then we cannot distinguish between the two input conditions \{RRG\} and \{RGR\}, which will cause two different steady states. On the other hand, the perturbed template $B_2$ permits to distinguish these two conditions with the inclusion of a negative threshold $z_{ij} = -2$ in order to adjust the steady state. The final templates are given by

$$A = [0 \ 0 \ 0], \quad B = [1 \ 2 \ 1],$$

$$z_{ij} = [-2].$$

Fig. 30. Horizontal line and ending point detection.

Fig. 31. The nonlinear function $h(x_{ij}) = g(x_{ij}) - a_{00}x_{ij}$. 
Figure 30 shows an example of horizontal line and ending point detection.

Note that these two examples are trivial, since we can easily realize the same dynamics by using the standard CNN with $a_{00} < 1$. In this system, function $g(x_{ij}) = -x_{ij} + a_{00} f(x_{ij})$ is a monotone decreasing function of $x_{ij}$. Therefore, there is only one steady state for an arbitrarily given $w_{ij}$, and so the three required steady states can be realized by the standard CNN.

Next, we design the same dynamics by using the second-order CNN equations. In this case, the ending points are marked as oscillating points. The dynamics of the second-order CNN are governed by

$$
\varepsilon \frac{dx_{ij}}{dt} = v_{ij} - h(x_{ij}) + w_{ij},
\frac{dv_{ij}}{dt} = -x_{ij} - \eta v_{ij},
$$

(106)

where

$$
h(x_{ij}) = g(x_{ij}) - a_{00} x_{ij},
$$$$
w_{ij} = \sum_{k,l \in N_{ij}} b_{k-l, l-j} u_{kl} + z_{ij},
$$

(107)

$$\varepsilon \approx 0, \quad \eta = 0.1, \quad a_{00} = \frac{5}{2},
\quad x_{ij}(0) = 1, \quad v_{ij}(0) = 0.$$

Then, from Figs. 31 and 32, we have the relation

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>$x_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{ij} \leq -7$</td>
<td>$\Rightarrow \frac{4(w_{ij} - 2)}{45}$</td>
</tr>
<tr>
<td>$-7 &lt; w_{ij} \leq -4.5$</td>
<td>$\Rightarrow$ limit cycle</td>
</tr>
<tr>
<td>$-4.5 &lt; w_{ij} &lt; 4.5$</td>
<td>$\Rightarrow$ limit cycle or $-\frac{4w_{ij}}{45}$</td>
</tr>
<tr>
<td>$4.5 \leq w_{ij} &lt; 7$</td>
<td>$\Rightarrow$ limit cycle</td>
</tr>
<tr>
<td>$7 \leq w_{ij}$</td>
<td>$\Rightarrow \frac{4(w_{ij} + 2)}{45}$</td>
</tr>
</tbody>
</table>

Here, we use the following pallete:

<table>
<thead>
<tr>
<th>$x_{ij}$</th>
<th>color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.4 \leq x_{ij}$</td>
<td>red</td>
</tr>
<tr>
<td>$x_{ij} &lt; -2.4$</td>
<td>green</td>
</tr>
</tbody>
</table>

Considering that $x_e = \pm 2.4$ for $w_{ij} = \pm 25$, we set $B = [16 \ 32 \ 16]$ and $z_{ij} = [-26]$. Then, we obtain
Thus, the ending points of the line are marked as the oscillating points.

### 6.3. Example 3 (Wrong coloring detection)

This operation detects wrong coloring. We use red and green pixels for drawing and blue pixels for the background. The teacher provides the correct coloring as an input image. A student colors a picture with red and green, and that image is taken as the initial condition of $x_{ij}$. The task of detecting and correcting the wrong coloring is shown in the following table.

<table>
<thead>
<tr>
<th>$u_{i,j-1}$</th>
<th>$u_{i,j}$</th>
<th>$u_{i,j+1}$</th>
<th>$w_{ij} = Bu + z_{ij}$</th>
<th>$x_e$</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>38</td>
<td>32/9</td>
<td>R</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-26</td>
<td>-112/45</td>
<td>G</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-58</td>
<td>-16/3</td>
<td>G</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-26</td>
<td>-112/45</td>
<td>G</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-58</td>
<td>-16/3</td>
<td>G</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-90</td>
<td>-368/45</td>
<td>G</td>
</tr>
</tbody>
</table>

The wrong coloring is marked by P or L. When a child colors the pixels in the background, they are changed into blue, automatically. In this example, the color of $x_{ij}(\infty)$ is defined as

\[
\begin{array}{l}
0.8 < x_{ij}(\infty) \quad \Rightarrow \quad R \text{ (red)} \\
0.2 < x_{ij}(\infty) < 0.4 \quad \Rightarrow \quad P \text{ (purple)} \\
-0.1 < x_{ij}(\infty) < 0.1 \quad \Rightarrow \quad B \text{ (blue)} \\
-0.4 < x_{ij}(\infty) < -0.2 \quad \Rightarrow \quad L \text{ (light blue)} \\
x_{ij}(\infty) < -0.8 \quad \Rightarrow \quad G \text{ (green)}
\end{array}
\]

The CNN design that performs the desired operation can be synthesized in the following table:

<table>
<thead>
<tr>
<th>$u_{i,j}$: Teacher’s Coloring</th>
<th>$x_{i,j}(0)$: Student’s Coloring</th>
<th>$x_{i,j}(\infty)$: Correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>R</td>
<td>G</td>
<td>P</td>
</tr>
<tr>
<td>R</td>
<td>B</td>
<td>P</td>
</tr>
<tr>
<td>G</td>
<td>R</td>
<td>L</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>B</td>
<td>L</td>
</tr>
<tr>
<td>B</td>
<td>R</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>G</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

where the symbols R, G, P, B and L indicate red, green, purple, blue and light blue, respectively.

Since the steady state $x_e$, for a given $w_{ij}$, is
given as

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>Number of Steady States</th>
<th>$x_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5 &lt; w_{ij}$</td>
<td>1</td>
<td>$\frac{4(w_{ij} + 2)}{15}$</td>
</tr>
<tr>
<td>$1 &lt; w_{ij} &lt; 1.5$</td>
<td>2</td>
<td>$\frac{4(w_{ij} + 2) - 4w_{ij}}{15}$</td>
</tr>
<tr>
<td>$-1 &lt; w_{ij} &lt; 1$</td>
<td>1</td>
<td>$\frac{4w_{ij}}{15}$</td>
</tr>
<tr>
<td>$-1.5 &lt; w_{ij} &lt; -1$</td>
<td>2</td>
<td>$\frac{4(w_{ij} - 2) - 4w_{ij}}{15}$</td>
</tr>
<tr>
<td>$w_{ij} &lt; -1.5$</td>
<td>1</td>
<td>$x_e = \frac{4(w_{ij} - 2)}{15}$</td>
</tr>
</tbody>
</table>

we only have to set $B = [0 1.25 0]$, in order to achieve the desired behavior. The final CNN design is detailed in the following table:

<table>
<thead>
<tr>
<th>$u_{ij}$</th>
<th>$w_{ij}$</th>
<th>$x_{ij}(0)$</th>
<th>$x_e = x_{ij}(\infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.25</td>
<td>1</td>
<td>13/15</td>
</tr>
<tr>
<td>1</td>
<td>1.25</td>
<td>-1, 0</td>
<td>1/3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$-1 \leq x_{ij}(0) \leq 1$</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1.25</td>
<td>0, 1</td>
<td>$1/3$</td>
</tr>
<tr>
<td>-1</td>
<td>-1.25</td>
<td>-1</td>
<td>$-13/15$</td>
</tr>
</tbody>
</table>

Thus, we have designed the CNN with the required dynamics. The templates are given by

$$A = [0 \ 0 \ 0], \quad B = [0 \ 1.25 \ 0], \quad z_{ij} = [0], \quad (108)$$

and a computer simulation illustrating this task is shown in Fig. 33.

![Fig. 33. Wrong coloring detection. (a) Teacher, (b) student, (c) teacher, (d) wrong coloring detection.](image-url)
6.4. Example 4 (Shadow marking)

This operation takes the left shadow of an object. The object and the background are colored with black and white, respectively. The image is given by the initial state \( x_{ij}(0) \). The shadow is marked gray. The objects in the shadow are colored with red. The task is shown in the following table:

<table>
<thead>
<tr>
<th>( x_{i,j} )</th>
<th>( x_{i,j+1} )</th>
<th>( x_{i,j}(\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>W</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>G</td>
<td>R</td>
</tr>
<tr>
<td>W</td>
<td>B</td>
<td>G</td>
</tr>
<tr>
<td>W</td>
<td>W</td>
<td>G</td>
</tr>
<tr>
<td>W</td>
<td>G</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>B</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>W</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
<td>G</td>
</tr>
</tbody>
</table>

Here, the symbols B, G, R and W indicate black, gray, red and white, respectively. Furthermore, we use the following palette:

| \( 1 < x_{ij} \) | \( \Rightarrow \) | \( R \) (red) |
| \( 0.8 < x_{ij} < 1 \) | \( \Rightarrow \) | \( B \) (black) |
| \( 0 < x_{ij} < 0.4 \) | \( \Rightarrow \) | \( G \) (gray) |
| \( x_{ij} < -0.8 \) | \( \Rightarrow \) | \( W \) (white) |

and realize the task by using the coupled RTD-based CNN with the dynamics:

\[
\frac{dx_{ij}}{dt} = -\frac{15}{4}x_{ij} + \frac{5}{2}(|x_{ij} - 0.4| - |x_{ij} - 0.8|) - \frac{5}{2}(|x_{ij} + 0.4| - |x_{ij} + 0.8|) + a_{00}x_{ij} + w_{ij},
\]

where

\[
w_{ij} = \sum_{k,l \in N(i,j), (k,l) \neq (i,j)} a_{ij}x_{k-i,l-j}
+ \sum_{k,l \in N(i,j)} b_{k-l,j}u_{kl} + z_{ij}.
\]

If we assume that \( a_{00} = 25/16, a_{ij} = 0 \) (i, j \( \neq \) 0), then the driving point plot is illustrated in Fig. 34. The steady states are given by the

![Fig. 34. The driving point plot: \( h(x_{ij}) = g(x_{ij}) - a_{00}x_{ij} \).](image)
The following table

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>Steady States</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.875 &lt; w_{ij}$</td>
<td>$\frac{16(w_{ij} + 2)}{35} \in B$ or $R$</td>
</tr>
<tr>
<td>$0.25 &lt; w_{ij} &lt; 0.875$</td>
<td>$\frac{16(w_{ij} + 2)}{35} \in B$ or $R$, $\frac{16w_{ij}}{35} \in G$</td>
</tr>
<tr>
<td>$-0.25 &lt; w_{ij} &lt; 0.25$</td>
<td>$\frac{16(w_{ij} + 2)}{35} \in B$ or $R$, $\frac{16(w_{ij} - 2)}{35} \in W$</td>
</tr>
<tr>
<td>$-0.875 &lt; w_{ij} &lt; -0.25$</td>
<td>$\frac{16w_{ij}}{35} \in G$, $\frac{16(w_{ij} - 2)}{35} \in W$</td>
</tr>
<tr>
<td>$w_{ij} &lt; -0.875$</td>
<td>$\frac{16(w_{ij} - 2)}{35} \in W$</td>
</tr>
</tbody>
</table>

The transition of the state $x_{ij}(t)$ is given by

<table>
<thead>
<tr>
<th>$w_{ij}$</th>
<th>Transition of the State $x_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.875 &lt; w_{ij}$</td>
<td>$B, W, G \Rightarrow R$</td>
</tr>
<tr>
<td>$0.25 &lt; w_{ij} &lt; 0.875$</td>
<td>$G \Rightarrow G$, $B \Rightarrow R$, $W \Rightarrow G$</td>
</tr>
<tr>
<td>$-0.25 &lt; w_{ij} &lt; 0.25$</td>
<td>$G \Rightarrow G$, $B \Rightarrow B$ or $R$, $W \Rightarrow W$</td>
</tr>
<tr>
<td>$-0.875 &lt; w_{ij} &lt; -0.25$</td>
<td>$G \Rightarrow G$, $B \Rightarrow G$, $W \Rightarrow W$</td>
</tr>
<tr>
<td>$w_{ij} &lt; -0.875$</td>
<td>$B, W, G \Rightarrow W$</td>
</tr>
</tbody>
</table>

Thus, if we set

$$A = \begin{bmatrix} 0 & \frac{25}{16} & 0.3 \end{bmatrix}, \quad B = [0 \quad 0 \quad 0], \quad z_{ij} = [0.3],$$  \hspace{1cm} (111)$$

and apply the transition table, we obtain the following:

<table>
<thead>
<tr>
<th>Color</th>
<th>Range of $x_{ij}$</th>
<th>Color</th>
<th>Range of $x_{ij+1}$</th>
<th>$w = 0.3(x_{ij+1} + 1)$</th>
<th>Transition of $x_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$0 &lt; x_{ij} \leq 1$</td>
<td>B</td>
<td>$0 &lt; x_{ij+1} \leq 1$</td>
<td>$0.54 &lt; w \leq 0.6$</td>
<td>$B \Rightarrow B$</td>
</tr>
<tr>
<td>B</td>
<td>$0 &lt; x_{ij} \leq 1$</td>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>$0 \leq w &lt; 0.06$</td>
<td>$B \Rightarrow B$</td>
</tr>
<tr>
<td>B</td>
<td>$0 &lt; x_{ij} \leq 1$</td>
<td>G</td>
<td>$0 \leq x_{ij} &lt; 0.4$</td>
<td>$0.3 &lt; w &lt; 0.42$</td>
<td>$B \Rightarrow R$</td>
</tr>
<tr>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>B</td>
<td>$0 &lt; x_{ij+1} \leq 1$</td>
<td>$0.54 &lt; w \leq 0.6$</td>
<td>$W \Rightarrow G$</td>
</tr>
<tr>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>$0 \leq w &lt; 0.06$</td>
<td>$W \Rightarrow W$</td>
</tr>
<tr>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>G</td>
<td>$0 \leq x_{ij} &lt; 0.4$</td>
<td>$0.3 &lt; w &lt; 0.42$</td>
<td>$W \Rightarrow G$</td>
</tr>
<tr>
<td>G</td>
<td>$0 &lt; x_{ij} &lt; 0.4$</td>
<td>B</td>
<td>$0 &lt; x_{ij+1} \leq 1$</td>
<td>$0.54 &lt; w \leq 0.6$</td>
<td>$G \Rightarrow G$</td>
</tr>
<tr>
<td>G</td>
<td>$0 &lt; x_{ij} &lt; 0.4$</td>
<td>W</td>
<td>$-1 \leq x_{ij} &lt; -0.8$</td>
<td>$0 \leq w &lt; 0.06$</td>
<td>$G \Rightarrow G$</td>
</tr>
<tr>
<td>G</td>
<td>$0 &lt; x_{ij} &lt; 0.4$</td>
<td>G</td>
<td>$0 &lt; x_{ij} \leq 0.4$</td>
<td>$0.3 &lt; w &lt; 0.42$</td>
<td>$G \Rightarrow G$</td>
</tr>
</tbody>
</table>

Thus, this confirms that the RTD-based CNN has the expected behavior. Figure 35 shows an example.
6.5. Example 5 (Synthesis of desired synchronization patterns)

A method for synchronizing chaotic systems was proposed in [Pecora & Carroll, 1990]. In this section, we synthesize the autonomous RTD-based CNN with a desired synchronization pattern.

We define the dynamics of the second-order RTD-based CNN as follows

\[
\begin{align*}
\frac{dx}{dt} &= v - h(x) + w, \\
\frac{dv}{dt} &= -x - \eta v, \\
\end{align*}
\]

(112)

where

\[
\begin{align*}
x &= \left(x_1, x_2, \ldots, x_n\right)^T, \\
v &= \left(v_1, v_2, \ldots, v_n\right)^T, \\
h(x) &= g(x) - \alpha x \\
&= (g(x_1) - \alpha x_1, \\
g(x_2) - \alpha x_2, \ldots, g(x_n) - \alpha x_n)^T, \\
&= \left(h(x_1), h(x_2), \ldots, h(x_n)\right)^T, \\
g(x) &= \left(g(x_1), g(x_2), \ldots, g(x_n)\right)^T, \\
h(x_i) &= \frac{15}{4} x_i - \frac{5}{2} \left(|x_i - 0.4| - |x_i - 0.8|\right) \\
&\quad + \frac{5}{2} \left(|x_i + 0.4| - |x_i + 0.8|\right), \\
w &= Tx + Su + z.
\end{align*}
\]

Here, \( \alpha = 5/2, \varepsilon > 0, \) and the piecewise-linear function \( g(x_{ij}) \) is realized by the RTD. Function \( h(x_j) \) is illustrated in Fig. 36.

Let \( x_m \) and \( x_k \) (\( k = k_1, k_2, \ldots, k_l \)) be the state variables of the master system and the slave systems, respectively. Then, we can synchronize the master and slave cells by using star coupling. That is, all the elements of \( T_{ij} \) except the following

\[
\begin{align*}
T_{kk} &= -K \quad \text{(for } k = k_1, k_2, \ldots, k_l), \\
T_{km} &= K \quad \text{(for } k = k_1, k_2, \ldots, k_l),
\end{align*}
\]

(114)

are set to zero, where \( K \) is a sufficiently large number. In this case, unidirectional coupling is used. Note that we have to choose \( Su + z, \varepsilon \) and \( \eta \) such that the master and slave cells have the same limit cycles. Then, we have the following:

- Two independent synchronization patterns can coexist.
- If there is a common cell that belongs to two synchronization patterns, then the complex behavior may occur at this cell (two driving signals are injected to this cell). If the unidirectional chain connection (that is, all the cells are cascaded along the unidirectional coupling) is applied to the cell coupling, the complex behavior begins from this cell. That is, the cell behavior develops from the crossing point.
- Bidirectional coupling is also available for synchronizing chain and star cells.

One-dimensional reaction–diffusion type connection is sometimes useful for synchronizing the cells. In
Fig. 36. Nonlinear function $h(x_i) = g(x_i) - a_{00}x_i$.

In this case, bidirectional coupling is used. The connection matrix $T$ is given by

$$T = K \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}.$$  

(115)

Some spatiotemporal patterns can appear for identical or nonidentical cells. Furthermore, if we use chaotic RTD-based CNN cells (proposed in the next section), then the desired chaotic synchronization patterns can be generated.

### 6.6. Example 6 (Chaotic RTD-based CNN cell)

In this example, we propose a chaotic RTD-based CNN cell. As a standard model, Chua’s canonical oscillator is used.\(^5\) The state equation of the chaotic RTD-based CNN is given by

$$\frac{dx_{ij}}{dt} = \rho(v_{ij} - g(x_{ij}) + w_{ij}),$$  
$$\frac{dy_{ij}}{dt} = \zeta y_{ij} - \eta v_{ij},$$  
$$\frac{dv_{ij}}{dt} = -x_{ij} + y_{ij},$$

(116)

where

$$w_{ij} = \sum_{k,l \in N_{ij}, (k,l) \neq (i,j)} a_{ij}kx_{k-l-j} + \sum_{k,l \in N_{ij}} b_{k-l-j}u_{kl} + z_{ij},$$  

(117)

$$g(x_{ij}) = \frac{15}{4}x_{ij} - \frac{5}{2}(|x_{ij} - 0.4| - |x_{ij} - 0.8|) + \frac{5}{2}(|x_{ij} + 0.4| - |x_{ij} + 0.8|).$$  

(118)

We can also use Chua’s oscillator as a reference model, since Chua’s oscillator and the canonical Chua’s oscillator are topologically equivalent. In this case, we have to connect a negative resistor in parallel to the RTD.

\(^5\)We can also use Chua’s oscillator as a reference model, since Chua’s oscillator and the canonical Chua’s oscillator are topologically equivalent. In this case, we have to connect a negative resistor in parallel to the RTD.
we can observe the chaotic attractors and limit cycles shown in Figs. 38–44. The asymptotically stable equilibrium points are also observed for sufficiently large $|w_{ij}|$. The bifurcation diagram of the RTD-based CNN cell is shown in Fig. 45. From these figures, we can see that a number of multiple states are available as the cell outputs.

6.7. Example 7 (RTD-based FitzHugh–Nagumo Equation)

The FitzHugh–Nagumo Equation is the mathematical model of excitation and propagation of impulses in nerve membranes. The corresponding reaction–diffusion CNN was proposed in [Chua, 1997]. In the case of the RTD-based CNN, the FitzHugh–Nagumo type equation has the form

\begin{align}
\frac{d x_i}{dt} & = v_i - h(x_{i,j}) + w_{ij} + D(x_{i-1} - 2x_i + x_i), \\
\frac{d v_i}{dt} & = -x_i - \eta v_i,
\end{align}

(120)

where

\begin{align}
h(x_i) &= g(x_i) - a_{00} x_i, \quad w_i = u_i + z_i, \\
\varepsilon &> 0, \quad \eta = 0.1, \quad a_{00} = \frac{5}{2},
\end{align}

(121)

The RTD-based FitzHugh–Nagumo Equation is equivalent to the coupled piecewise linear Bonhoeffer–Van der Pol equations.

6.8. Example 8 (Pattern formation)

The dynamics of the second-order RTD-based CNN are given by

\begin{align}
\zeta \frac{dx_{ij}}{dt} & = v_{ij} - g(x_{ij}) + \sum_{k,l \in \mathcal{N}_{ij}} a_{ij} x_{k-l,j} + b_{k-l,j} u_{kl} + z_{ij}, \\
\frac{dv_{ij}}{dt} & = -x_{ij} - \eta v_{ij},
\end{align}

(122)

where $\zeta = 0.2$, $\eta = 0.1$.

---

The symbols $X$, $Y$ and $V$ in Figs. 38–44 indicate $x_{ij}$, $y_{ij}$ and $v_{ij}$, respectively.
Fig. 38. Chaotic attractor observed in the RTD-based CNN cell ($w_{ij} = 11/28$).

Fig. 39. Chaotic attractor observed in the RTD-based CNN cell ($w_{ij} = -11/28$).
Fig. 40. Location of two chaotic attractors.

Fig. 41. Location of two chaotic attractors.
If we choose the following template

\[ A = \begin{bmatrix} q & q & q \\ q & q & q \\ q & q & q \end{bmatrix}, \quad B = [0], \quad z_{ij} = [0], \quad q = 0.25 \]

and use a zero-flux boundary condition, then we observe spiral patterns. On the other hand, if we choose the template

\[ A = \begin{bmatrix} 0 & q & 0 \\ q & 4q & q \\ 0 & q & 0 \end{bmatrix}, \quad B = [0], \quad z_{ij} = [0], \quad q = 0.4 \]

then we observe the target pattern. The system can generate a number of patterns, which are...
Fig. 44. Locations of the four limit cycles ($w_{ij} = \pm 0.012$).

Fig. 45. Bifurcation diagram of the RTD-based CNN cell ($0 \leq w_{ij} \leq 0.7$).
Fig. 46. Associative memory pattern. (a) Initial condition $x_{ij}(0)$, (b) $t = 20$, (c) $t = 42$, (d) $t = 48$, (e) $t = 55$, (f) $t = 61$, (g) $t = 235$, (h) $t = 266$. 
Fig. 47. Associative memory pattern. (a) A motif pattern, (b–h) generated patterns.
similar to the patterns generated by the second-order CNN.

In the case of the third-order CNN equation

\[
\rho \frac{dx_{ij}}{dt} = v_{ij} - g(x_{ij}) + \sum_{k,l \in N_{ij}} a_{ij} x_{k-i,l-j} \\
+ \sum_{k,l \in N_{ij}} b_{k-i,l-j} u_{kl} + z_{ij},
\]

\[
\frac{dv_{ij}}{dt} = \zeta v_{ij} - \eta w_{ij},
\]

\[
\frac{dw_{ij}}{dt} = -x_{ij} + v_{ij},
\]

if we let \( \rho = 0.5, \zeta = 0.45, \eta = 0.32 \) and we choose the following template:

\[
A = \begin{bmatrix}
0 & -q & 0 \\
-q & -4q & -q \\
0 & -q & 0
\end{bmatrix}, \quad B = [0],
\]

\[
z_{ij} = [0], \quad q = 1.5,
\]

then the associative memory property arises. In this case, if some pattern (for example, a Chinese character) is given as the initial condition, \( x_{ij}(0) \), then the third-order CNN has the following behavior: First, the given pattern disappears; then, it is created again, and appears periodically for a while; finally, the pattern collapses gradually. This implies that the third-order CNN first learns the pattern, then remembers, and finally forgets. This behavior is illustrated in Fig. 46.

On the other hand, if we choose the template

\[
A = \begin{bmatrix}
0 & q & 0 \\
q & -4q & q \\
0 & q & 0
\end{bmatrix}, \quad B = [0],
\]

\[
z_{ij} = [0], \quad q = 1,
\]

and set the motif patterns as the initial condition, then a number of patterns appear, which are modified artistically by the CNN as shown in Fig. 47. These patterns are different from the fractal design. Finally, it can also be mentioned that spiral waves and target patterns are generated by using the parameters \( \rho = 5, \zeta = 1, \eta = 0.01 \), and the template of the second-order CNN's.

7. Conclusions

In this paper, we have studied the relationship between the standard CNN and the RTD-based CNN. In order to do this, we incorporated both models in the common framework provided by the generalized CNN cell. We also studied the second- and third-order CNN equations. We presented a method for designing RTD-based CNN with multiple steady states and exemplified it with a number of cases. We found that the CNN with multiple steady states, including the second- and third-order cases, can generate a variety of dynamic phenomena, and indeed permits to achieve advanced functional capabilities.

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References


