OSCILLATIONS ON THE EDGE OF CHAOS VIA DISSIPATION AND DIFFUSION

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The primary purpose of this paper is to show that simple dissipation can bring about oscillations in certain kinds of asymptotically stable nonlinear dynamical systems; namely when the system is locally active where the dissipation is introduced. Furthermore, if these nonlinear dynamical systems are coupled with appropriate choice of diffusion coefficients, then the coupled system can exhibit spatio-temporal oscillations. The secondary purpose of this paper is to show that spatio-temporal oscillations can occur in spatially discrete reaction diffusion equations operating on the edge of chaos, provided the array size is sufficiently large.

Keywords: Oscillation; dissipation; diffusion; reaction-diffusion equation; local activity; edge of chaos; sharp edge of chaos; complexity; spatio-temporal oscillation; FitzHugh-Nagumo equation.

1. Introduction

Consider a spatially discrete collection of continuous nonlinear dynamical systems called cells. A cell is said to be locally active at a cell equilibrium point, iff a variational energy delivered to the cell is negative at some finite time [Chua, 1998]. It is well known that a locally active cell can exhibit complex dynamics such as limit cycles or chaos.

A set of cells coupled with each other via diffusion, namely, a discrete reaction–diffusion equation, could exhibit complex spatio-temporal phenomena such as scroll waves and spatio-temporal chaos. An uncoupled cell of a reaction–diffusion equation is said to be on the edge of chaos iff some of its cell equilibrium points are locally active but asymptotically stable [Chua, 1998, 2005]. An uncoupled cell on the edge of chaos may cause a reaction–diffusion equation to exhibit nonhomogeneous patterns and other spatio-temporal phenomena under appropriate choice of diffusion coefficients. However, it does not imply that it is always possible to find some diffusion coefficients to destabilize a homogeneous solution. In fact, it is possible to prove that in general, such a set of destabilizing diffusion coefficients exist only for a proper subset of the edge of chaos parameter domain, which is called the sharp edge of chaos domain [Chua, 2005]. By definition, a cell equilibrium point on the sharp edge of chaos can be destabilized by diffusive coupling, however, it is not always possible to find some diffusion coefficients to bring about oscillation. For example, Turing’s equation [Turing, 1952] cannot exhibit oscillation even if the cells are operated
on the sharp edge of chaos. Furthermore, in the case of large arrays, it is not easy to find diffusion coefficients to destabilize a homogeneous equilibrium.

*Hopf bifurcation* is a local bifurcation phenomenon in which a fixed point of a dynamical system loses stability, and a small amplitude limit cycle emerges from the fixed point [Alligood et al., 1997; Marsden & McCracken, 1976]. If a dissipation term gives rise to a Hopf bifurcation in an uncoupled asymptotically stable cell, then the cell oscillates with a small amplitude in a small neighborhood of the bifurcation point. Furthermore, if such oscillating cells are coupled with diffusion coefficients, they could exhibit spatio-temporal phenomena. Although the Hopf bifurcation is not the only bifurcation which can bring about oscillations, the conditions for the Hopf bifurcation are mathematically testable for low-dimensional dynamical systems [Guckenheimer & Holmes, 1983; Mees & Chua, 1979].

In this paper, we present several examples showing the addition of *dissipation* can bring about oscillation in certain kinds of asymptotically stable nonlinear dynamical systems. That is, the dissipation can give rise to a Hopf bifurcation. This phenomenon is counter intuitive because our experience suggests that dissipation would usually *damp* out any onset of oscillations, thereby suppressing it; yet the opposite behavior emerges! Furthermore, if these nonlinear dynamical systems are coupled with appropriate choice of diffusion and dissipation coefficients, then the coupled system can exhibit spatio-temporal patterns. We also show that if two nonidentical asymptotically stable but locally active dynamical systems are coupled by diffusion, then periodic oscillations can occur. Finally, we show that spatio-temporal oscillations can occur in spatially discrete reaction–diffusion equations on the edge of chaos, provided the array size is sufficiently large.

### 2. Discrete Reaction–Diffusion Equations

Let us consider an $N \times N$ array with $N^2$ identical cells, each one defined by “$n$” state variables $V_1(j,k), V_2(j,k), \ldots, V_n(j,k)$, evolving according to a system of ODE:

\[
\begin{align*}
\frac{dV_1(j,k)}{dt} &= f_1\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
\frac{dV_2(j,k)}{dt} &= f_2\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
&\vdots \\
\frac{dV_m(j,k)}{dt} &= f_m\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
\frac{dV_{m+1}(j,k)}{dt} &= f_{m+1}\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
&\vdots \\
\frac{dV_{m+2}(j,k)}{dt} &= f_{m+2}\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
&\vdots \\
\frac{dV_n(j,k)}{dt} &= f_n\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right).
\end{align*}
\]

(1)

where $j = 1, 2, \ldots, N, k = 1, 2, \ldots, N$. Assuming the cells are coupled by standard diffusions, we obtain the following system of discrete reaction–diffusion equations [Chua, 1998, 2005]:

\[
\begin{align*}
\frac{dV_1(j,k)}{dt} &= f_1\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right) + D_1 \nabla^2 V_1(j,k), \\
\frac{dV_2(j,k)}{dt} &= f_2\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right) + D_2 \nabla^2 V_2(j,k), \\
&\vdots \\
\frac{dV_m(j,k)}{dt} &= f_m\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right) + D_m \nabla^2 V_m(j,k), \\
\frac{dV_{m+1}(j,k)}{dt} &= f_{m+1}\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
&\vdots \\
\frac{dV_{m+2}(j,k)}{dt} &= f_{m+2}\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right), \\
&\vdots \\
\frac{dV_n(j,k)}{dt} &= f_n\left(V_1(j,k), V_2(j,k), \ldots, V_n(j,k)\right).
\end{align*}
\]

(2)
where $V_\sigma(j, k)$ denotes the $\sigma$th state variable, $\sigma = 1, 2, \ldots, n$ of a "reaction" cell located at the grid point $r \triangleq (j, k)$ (with integer coordinates) of a spatial array in $\mathbb{R}^2$, $D_\sigma, \sigma = 1, 2, \ldots, m$, denotes the positive diffusion coefficient associated with the state variable $V_\sigma(j, k)$, and $\nabla^2 V_\sigma(j, k)$ denotes the discretized Laplacian operator in $\mathbb{R}^2$:

$$\nabla^2 V_\sigma(j, k) \triangleq V_\sigma(j + 1, k) + V_\sigma(j - 1, k) + V_\sigma(j, k + 1) + V_\sigma(j, k - 1) - 4V_\sigma(j, k),$$  \hspace{1cm} (3)

where $\sigma = 1, 2, \ldots, m$. The discretized Laplacian operator (3) is described by the following template \cite{Chua1998, ChuaRoska2002}:

$$
\begin{array}{ccc}
0 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & 0 \\
\end{array}
$$
\hspace{1cm} (4)

Equation (2) is a discretized version of the following system of continuous reaction–diffusion partial differential equations \cite{Chua1998, Chua2005}:

$$\begin{align*}
\frac{\partial v_1(r)}{\partial t} &= f_1 \left( V_1(r), V_2(r), \ldots, V_n(r) \right) + D_1 \nabla^2 V_1(r), \\
\frac{\partial v_2(r)}{\partial t} &= f_2 \left( V_1(r), V_2(r), \ldots, V_n(r) \right) + D_2 \nabla^2 V_2(r), \\
&\vdots \\
\frac{\partial v_m(r)}{\partial t} &= f_m \left( V_1(r), V_2(r), \ldots, V_n(r) \right) + D_m \nabla^2 V_m(r), \\
\frac{\partial v_{m+1}(r)}{\partial t} &= f_{m+1} \left( V_1(r), V_2(r), \ldots, V_n(r) \right), \\
\frac{\partial v_{m+2}(r)}{\partial t} &= f_{m+2} \left( V_1(r), V_2(r), \ldots, V_n(r) \right), \\
&\vdots \\
\frac{\partial v_n(r)}{\partial t} &= f_n \left( V_1(r), V_2(r), \ldots, V_n(r) \right),
\end{align*}$$
\hspace{1cm} (5)

where $r \in \mathbb{R}^2$. Since the "n" state variables $V_\sigma(r)$ pertain to the same array point $r \triangleq (j, k)$ in the discretized case, or $r \in \mathbb{R}^2$ in the continuous case, they pertain to a "reaction" cell lumped at one point $r$. Note that for the sake of generality, we assume $D_{m+1} = D_{m+2} = \cdots = D_n = 0$.

The discrete reaction–diffusion equation (2) is not yet completely defined mathematically because it usually contains two kinds of undefined "end" variables; namely,

$$\begin{align*}
V_i(0, k), &\quad k = 1, 2, \ldots, N \\
V_i(j, 0), &\quad j = 1, 2, \ldots, N
\end{align*}$$
\hspace{1cm} (6)

and

$$\begin{align*}
V_i(N + 1, k), &\quad k = 1, 2, \ldots, N \\
V_i(j, N + 1), &\quad j = 1, 2, \ldots, N
\end{align*}$$
\hspace{1cm} (7)

These two kinds of "end" variables must be defined by specifying appropriate boundary conditions. The following three boundary conditions are typical.

1. Fixed (Dirichlet) Boundary Condition:

$$\begin{align*}
V_i(0, k) &= v_1, & k = 1, 2, \ldots, N \\
V_i(j, 0) &= v_2, & j = 1, 2, \ldots, N
\end{align*}$$
\hspace{1cm} (8)

and

$$\begin{align*}
V_i(N + 1, k) &= v_3, & k = 1, 2, \ldots, N \\
V_i(j, N + 1) &= v_4, & j = 1, 2, \ldots, N
\end{align*}$$
\hspace{1cm} (9)

where $v_1, v_2, v_3, v_4$ are real constants.

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1In image processing, a discretized Laplacian template is often used for detection of edges \cite{Chua1998}.
2. Zero-Flux (Neumann) Boundary Condition:
\[
V_i(0, k) = V_i(1, k), \quad k = 1, 2, \ldots, N \\
V_i(j, 0) = V_i(j, 1), \quad j = 1, 2, \ldots, N
\]  
(10)

and
\[
V_i(N + 1, k) = V_i(N, k), \quad k = 1, 2, \ldots, N \\
V_i(j, N + 1) = V_i(j, N), \quad j = 1, 2, \ldots, N
\]  
(11)

3. Periodic (Toroidal) Boundary Condition:
\[
V_i(0, k) = V_i(N, k), \quad k = 1, 2, \ldots, N \\
V_i(j, 0) = V_i(j, N), \quad j = 1, 2, \ldots, N
\]  
(12)

and
\[
V_i(N + 1, k) = V_i(1, k), \quad k = 1, 2, \ldots, N \\
V_i(j, N + 1) = V_i(j, 1), \quad j = 1, 2, \ldots, N
\]  
(13)

Let us define \( \mathbf{V} = (V_1, V_2, \ldots, V_m, V_{m+1}, \ldots, V_n) \) and recast Eq. (2) into the following compact vector form:
\[
\frac{d\mathbf{V}}{dt} = \mathbf{f}(\mathbf{V}, \mu) + \mathbf{D}\nabla^2 \mathbf{V},
\]  
(14)

where \( \mathbf{f}(\mathbf{V}) \in \mathbb{R}^n \) denotes the vector function defined by
\[
\mathbf{f}(\mathbf{V}) = (f_1(\mathbf{V}), f_2(\mathbf{V}), \ldots, f_m(\mathbf{V}), f_{m+1}(\mathbf{V}), \ldots, f_n(\mathbf{V})),
\]  
(15)

\( \mathbf{D} \) denotes an \( n \times n \) diagonal matrix defined by
\[
D_{\sigma \sigma} = \begin{cases} 
D_\sigma & \text{for } \sigma = 1, 2, \ldots, m \\
0 & \text{for } \sigma = m + 1, m + 2, \ldots, n
\end{cases}
\]  
(16)

and \( \nabla^2 \mathbf{V} \) denotes an \( n \times 1 \) vector defined by the “n” discrete Laplacian operators
\[
\nabla^2 \mathbf{V} = (\nabla^2 V_1, \nabla^2 V_2, \ldots, \nabla^2 V_m, \nabla^2 V_{m+1}, \ldots, \nabla^2 V_n).
\]  
(17)

Moreover, in general any reaction–diffusion equation associated with a real physical system has some tunable parameters
\[
\mu = (\mu_1, \mu_2, \ldots, \mu_\rho).
\]  
(18)

Hence, the isolated cell equation (1) and the reaction–diffusion equation (2) for each cell at location \( r = (j, k) \) are described by:
\[
\frac{dV_i(r)}{dt} = f(V_i(r), \mu),
\]  
(19)

and
\[
\frac{dV(r)}{dt} = \mathbf{f}(\mathbf{V}(r), \mu) + \mathbf{D}\nabla^2 \mathbf{V}(r),
\]  
(20)

respectively.

3. Hopf Bifurcation

Hopf bifurcation is a local bifurcation phenomenon in which a fixed point of a dynamical system loses stability (when a pair of complex conjugate eigenvalues of the fixed point cross the imaginary axis in the complex plane), and a small-amplitude limit cycle emerges from the fixed point [Marsden & McCracken, 1976] as one tunes some parameter \( \mu \) just beyond the bifurcation point.

To describe Hopf bifurcation more precisely, consider a differential equation with a parameter \( \mu \):
\[
\frac{d\mathbf{V}}{dt} = \mathbf{f}(\mathbf{V}, \mu),
\]  
(21)

where \( \mathbf{V} = (V_1, V_2, \ldots, V_n) \in \mathbb{R}^n \) denotes the \( n \) state variables, \( \mathbf{f}(\mathbf{V}) \in \mathbb{R}^n \) denotes the vector function defined by
\[
\mathbf{f}(\mathbf{V}) = (f_1(\mathbf{V}), f_2(\mathbf{V}), \ldots, f_n(\mathbf{V}, \mu)),
\]  
(22)

and \( \mu \in I \), where the symbol \( I \) denotes an interval \( I = (M_1, M_2) \). Let the state equation (21) have an equilibrium point \( \overline{\mathbf{V}}(\mu) \) and assume its Jacobian matrix
\[
\frac{\partial \mathbf{f}(\overline{\mathbf{V}}(\mu), \mu)}{\partial \mathbf{V}}
\]  
(23)

have the following properties:

- a pair of complex eigenvalues \( \lambda_{1,2}(\mu) = \epsilon(\mu) \pm i\omega(\mu) \) with eigenvectors \( \xi_{1,2}(\mu) \) such that \( d\epsilon(\mu)/d\mu > 0 \) and \( \omega(\mu) > 0 \). The real part \( \epsilon(\mu) \) of the eigenvalues \( \lambda_{1,2}(\mu) \) changes its sign at some \( \mu^* \in I \), where \( \epsilon(\mu^*) = 0 \).
- \( n - 2 \) eigenvalues with negative real parts for all \( \mu \in I \).

Then, for any \( \mu \) sufficiently close to \( \mu^* \), the solutions sufficiently close to the equilibrium point \( \overline{\mathbf{V}}(\mu) \) is topologically equivalent to the solution of the normal form of a Hopf bifurcation defined by
\[
\frac{da}{dt} = (\epsilon + i\omega^*)a + (-\alpha + i\beta)|a|^2 a,
\]  
(24)

where \( a(t) \in \mathbb{C}, \epsilon = \epsilon(\mu), \omega^* = \omega(\mu^*) \), and \( \alpha \) and \( \beta \) depend on the nonlinear terms in the Taylor expansion of \( \mathbf{f}(\overline{\mathbf{V}}(\mu), \mu) \) around \( \overline{\mathbf{V}}(\mu^*) \) and \( \mu^* \).
If the equilibrium point $\mathbf{\bar{V}}(\mu)$ is asymptotically stable for $\mu = \mu^*$, then $\alpha > 0$. Furthermore, if $\alpha > 0$, there is a limit cycle

- of amplitude $\approx \sqrt{\epsilon/\alpha}$,
- of frequency $\approx \omega^*$,
- stable if $\alpha > 0$ and unstable if $\alpha < 0$.

## 4. Oscillation via Simple Dissipation

In this section, we show simple dissipation can bring about oscillation in some asymptotically stable non-linear dynamical systems.

Let us consider a differential equation with a globally asymptotically stable equilibrium point $\mathbf{\bar{V}}$

$$\frac{d\mathbf{V}}{dt} = \mathbf{g}(\mathbf{V}),$$

(25)

where $\mathbf{V} = (V_1, V_2, \ldots, V_n) \in \mathbb{R}^n$ denotes the state variables and $\mathbf{g}(\mathbf{V}) \in \mathbb{R}^n$ denotes the vector function defined by

$$\mathbf{g}(\mathbf{V}) = \left(g_1(\mathbf{V}), g_2(\mathbf{V}), \ldots, g_n(\mathbf{V})\right).$$

(26)

Define a new differential equation with a parameter $\mu$ and a vector function $k(\mathbf{V})$:

$$\frac{d\mathbf{V}}{dt} = \mathbf{g}(\mathbf{V}) - \mu k(\mathbf{V}),$$

(27)

where $\mu > 0$ and $k(\mathbf{V}) \in \mathbb{R}^n$ is a vector function

$$k(\mathbf{V}) = \left(k_1(\mathbf{V}), k_2(\mathbf{V}), \ldots, k_n(\mathbf{V})\right),$$

(28)

satisfying the inequality

$$\langle \mathbf{V}, k(\mathbf{V}) \rangle > 0,$$

(29)

for nonzero $\mathbf{V}$, where the notation $\langle x, y \rangle$ denotes the scalar "dot" product between the two vectors $x$ and $y$. The function $-\mu k(\mathbf{V})$ in Eq. (27) can be interpreted as energy dissipation in the sense that this term comes from resisters in an electrical circuit, or "dampings" in a mechanical system, described by Eq. (27).

Let the system (27) have an equilibrium point $\mathbf{\bar{V}}(\mu)$, and assume its Jacobian matrix

$$\frac{\partial \mathbf{g}(\mathbf{\bar{V}}(\mu), \mu)}{\partial \mathbf{V}} - \mu \frac{\partial k(\mathbf{\bar{V}}(\mu), \mu)}{\partial \mathbf{V}}$$

(30)

has a pair of complex eigenvalues $\lambda_{1,2}(\mu) = \epsilon(\mu) \pm i\omega(\mu)$ with eigenvectors $\xi_{1,2}(\mu)$ such that $d\epsilon/d\mu > 0$. The real part $\epsilon(\mu)$ of the eigenvalues $\lambda_{1,2}(\mu)$ changes its sign in the interval $I = (0, M)$ and for some $\mu^* \in I$, $\epsilon(\mu^*) = 0$. Assume also the rest of eigenvalues of the Jacobian matrix (30) have negative real parts for all $\mu \in I$. If the stable equilibrium point $\mathbf{\bar{V}}(\mu)$ is asymptotically stable for all $\mu \leq \mu^*$, and changes its stability as $\mu$ passes through $\mu^*$, then a unique stable limit cycle generically bifurcates from it under the condition of $\epsilon\alpha > 0$.

If we can find such vector functions $\mathbf{g}(\mathbf{V})$ and $k(\mathbf{V})$, then the uncoupled cell at location $\mathbf{r} = (j, k)$ with the dynamics

$$\frac{d\mathbf{V}(\mathbf{r})}{dt} = \mathbf{g}(\mathbf{V}(\mathbf{r})), $$

(31)

can oscillate by adding a dissipation term $-\mu k(\mathbf{V}(\mathbf{r}))$ to each cell. Then, the dynamics of the uncoupled cell at location $\mathbf{r} = (j, k)$ is described by

$$\frac{d\mathbf{V}(\mathbf{r})}{dt} = \mathbf{g}(\mathbf{V}(\mathbf{r})) - \mu k(\mathbf{V}(\mathbf{r})), $$

(32)

where $\mu \geq \mu^*$.

We next show some examples of dynamical systems in which simple dissipation can bring about oscillation.

### 4.1. FitzHugh–Nagumo equation

Consider the FitzHugh–Nagumo equation [Chua, 1998]

$$\begin{align*}
\frac{dx}{dt} &= -y - f(x), \\
\frac{dy}{dt} &= \xi(x - \beta y + \gamma),
\end{align*}$$

(33)

where $f(x) = (x^3/3) - x$, $\beta = 1.28$, $\gamma = 0.12$, and $\xi = 0.1$. Equation (33) has an equilibrium point $(x_e, y_e)$ satisfying

$$\begin{align*}
-x_e - f(x_e) &= -y_e - \frac{x_e^3}{3} + x_e = 0, \\
x_e - \beta y_e + \gamma &= x_e - 1.28y_e + 0.12 = 0,
\end{align*}$$

(34)

and its Jacobian matrix

$$\begin{bmatrix}
-df(x_e) & -1 \\
\xi & -\beta\xi
\end{bmatrix} = \begin{bmatrix}
-x_e^2 + 1 & -1 \\
0.1 & -0.128
\end{bmatrix},$$

(35)

has a pair of complex eigenvalues with a negative real part. Denoting "approximately equal" by $\approx$, \ldots
the equilibrium point and its eigenvalues can be described by
\[
\begin{align*}
\{ (x_e, y_e) &\approx (-0.9724, -0.6659) \\
\lambda_{1,2} &\approx -0.03675 \pm i0.03028.
\}
\end{align*}
\tag{36}
\]

Consider next the FitzHugh–Nagumo equation with a dissipation term
\[
\begin{align*}
\frac{dx}{dt} &= -y - f(x) - \mu x, \\
\frac{dy}{dt} &= \xi(x - \beta y + \gamma),
\end{align*}
\tag{37}
\]
where \( f(x) = (x^3/3) - x, \beta = 1.28, \gamma = 0.12, \)
\( \xi = 0.1, \) and \( \mu > 0 \) is a dissipation coefficient. The equilibrium point \((x_d, y_d)\) of Eq. (37) satisfies the following relation:
\[
\begin{align*}
- y_d - \frac{x_d^3}{3} + x_d - \mu x_d &= 0, \\
x_d - \beta y_d + \gamma &= 0.
\end{align*}
\tag{38}
\]
It follows that
\[
\frac{-x_d + \gamma}{\beta} - \frac{x_d^3}{3} + (1 - \mu)x_d = 0. \tag{39}
\]
The Jacobian matrix of Eq. (37) at \((x_d, y_d)\) is
\[
\begin{bmatrix}
- \frac{df(x_d)}{dx} - \mu & -1 \\
\xi & -\beta \xi
\end{bmatrix}
= \begin{bmatrix}
- x_d^2 + 1 - \mu & -1 \\
\xi & -\beta \xi
\end{bmatrix}.
\tag{40}
\]
Thus, we get the following characteristic equation:
\[
\lambda^2 - (-x_d^2 + 1 - \mu - \beta \xi) \lambda
+ \beta \xi(x_d^2 - 1 + \mu) + \xi = 0. \tag{41}
\]
If \( x_d \) and \( \mu^* \) satisfy the relation
\[
\begin{align*}
\lambda_1 + \lambda_2 &= -x_d^2 + 1 - \mu^* - \beta \xi = 0, \\
\lambda_1 \lambda_2 &= \beta \xi(x_d^2 - 1 + \mu^*) + \xi > 0,
\end{align*}
\tag{42}
\]
\((\lambda_i: \text{pure imaginary}) \) and \( \alpha \) and \( \epsilon(\mu) \) satisfy
\[
\frac{d\epsilon(\mu)}{d\mu} > 0, \tag{43}
\]
and
\[
\alpha \epsilon(\mu) > 0 \tag{44}
\]
for \( \mu > \mu^* \), then a unique stable limit cycle generically bifurcates from the equilibrium point \((x_d, y_d)\). That is, a Hopf bifurcation occurs at \( \mu^* \). From Eqs. (38) and (42), we obtain
\[
\begin{align*}
(x_d, y_d) &\approx (-0.9083, -0.6159), \\
\mu^* &\approx 0.04694, \\
\lambda_1 \lambda_2 &\approx 0.08362 > 0,
\end{align*}
\tag{45}
\]
in the neighborhood of \( \mu = 0 \).

We next show that Eqs. (43) and (44) are satisfied in the neighborhood of \( \mu^* \). Differentiating Eq. (39) with respect to \( \mu \), we obtain
\[
\frac{dx_d}{d\mu} = -\frac{x_d}{x_d^2 + \frac{1}{\beta} + \mu - 1}. \tag{46}
\]
Differentiating
\[
\epsilon(\mu) = \frac{(\lambda_1 + \lambda_2)}{2} = -\frac{x_d^2 + 1 - \mu - \beta \xi}{2} \tag{47}
\]
with respect to \( \mu \), we get
\[
\frac{d\epsilon(\mu)}{d\mu} = -x_d \frac{dx_d}{d\mu} - \frac{1}{2}. \tag{48}
\]
Substituting Eq. (46) into Eq. (48), we have
\[
\frac{d\epsilon(\mu)}{d\mu} = \frac{x_d^2}{x_d^2 + \frac{1}{\beta} + \mu - 1} - \frac{1}{2} \tag{49}
\frac{d\epsilon(\mu)}{d\mu} \approx \frac{d\epsilon(\mu^*)}{d\mu} = \frac{x_d^2}{x_d^2 + \frac{1}{\beta} + \mu^* - 1} - \frac{1}{2}
\approx 0.7630 > 0. \tag{50}
\]
Thus, Eq. (43) is satisfied. Since \( \epsilon(\mu)/d\mu > 0 \) and \( \epsilon(\mu^*) = 0, \epsilon(\mu) \) satisfies
\[
\epsilon(\mu) > 0 \quad \text{for } 0 < \mu - \mu^* \ll 1. \tag{51}
\]
Furthermore, \( \alpha > 0 \), since the equilibrium point for \( \mu = \mu^* \) is asymptotically stable. The sign of \( \alpha \) can be obtained from
\[
\text{sign}(\alpha) = -\text{sign} \left[ \frac{d^3 F(0)}{du^3} + \frac{\left(\frac{d^2 F(0)}{du^2}\right)^2}{a - b} \right] \tag{52}
\]
where \( a = \beta \xi, \ b = 1/\beta, \) and
\[
\begin{align*}
F(u) &= -f(u + x_d) - \mu x_d - y_d \\
&= -\frac{(u + x_d)^3}{3} + (1 - \mu)(u + x_d) - y_d \tag{53}
\end{align*}
\]
we conclude that $\alpha > 0$. It follows that $\alpha \epsilon(\mu) > 0$
for $0 < \mu - \mu^* \ll 1$. Thus, we can find the
dissipation coefficient $\mu$, which can destabilize the
equilibrium point, and furthermore brings about
oscillation. The Hopf bifurcation phenomena of
the FitzHugh–Nagumo equation are illustrated in
Fig. 1. Observe that a unique stable limit cycle

\[ \frac{d^3 F(0)}{du^3} + \left( \frac{d^2 F(0)}{du^2} \right)^2 \frac{a - b}{a - b} \approx -7.052, \quad (54) \]

Fig. 1. Hopf bifurcation of the FitzHugh–Nagumo equation. A unique stable limit cycle bifurcates from an equilibrium point $(x_d, y_d)$. Initial value $(x(0), y(0)) = (0.1, 0.1)$ is used to plot the trajectories. (a) $\mu = 0$; $(x_e, y_e) \approx (-0.9724, -0.6659)$. (b) $\mu = 0.05$; $(x_d, y_d) \approx (-0.9083, -0.6159)$. 
in Fig. 1(b) bifurcates from the equilibrium point \((x_e, y_e)\) in Fig. 1(a). We next examine the “local activity” and “edge of chaos” by using the definition of [Chua, 2005]. Let \(\delta x\) and \(\delta y\) denote infinitesimal variables in the neighborhood of the equilibrium point \(x_d(\mu)\) and \(y_d(\mu)\), namely,

\[
\begin{align*}
  x(t) &= x_d(\mu) + \delta x(t), \\
  y(t) &= y_d(\mu) + \delta y(t).
\end{align*}
\]

Let us define

\[i(t) = -\mu x_d(\mu) - \delta i(t).\]

where \(i(t)\) can be interpreted as an input signal. From Eq. (37), we obtain the variational equation

\[
\begin{align*}
  \frac{d(\delta x)}{dt} &= -\delta y - \frac{df(x_d(\mu))}{dx}\delta x + \delta i, \\
  \frac{d(\delta y)}{dt} &= \xi(\delta x - \beta \delta y).
\end{align*}
\]

where

\[
\frac{df(x_d(\mu))}{dx} = x_d(\mu)^2 - 1.
\]

Let us define the Laplace transform of \(\delta x\), \(\delta y\) and \(\delta i\) by

\[
\begin{align*}
  \hat{x}(s) &= \int_0^\infty \delta x(t)e^{-st}dt, \\
  \hat{y}(s) &= \int_0^\infty \delta y(t)e^{-st}dt, \\
  \hat{i}(s) &= \int_0^\infty \delta i(t)e^{-st}dt,
\end{align*}
\]

respectively. Applying the Laplace transform to each term in Eq. (57), we obtain

\[
\begin{align*}
  s\hat{x}(s) &= -\hat{y}(s) - (x_d(\mu)^2 - 1)\hat{x}(s) + \hat{i}(s), \\
  s\hat{y}(s) &= \xi(\hat{x}(s) - \beta \hat{y}(s)).
\end{align*}
\]

Solving the second equation in Eq. (60) for \(\hat{y}(s)\), we obtain

\[
\hat{y}(s) = \frac{\xi \hat{x}(s)}{s + \xi\beta},
\]

Substituting this equation into the first equation in Eq. (60), we obtain

\[
Y(s)\hat{x}(s) = \hat{i}(s),
\]

where

\[
Y(s) \triangleq s + \frac{\xi}{s + \xi\beta} + x_d(\mu)^2 - 1
\]

\[
= s^2 + (\xi\beta + x_d(\mu)^2 - 1)s + \xi\beta(x_d(\mu)^2 - 1) + \xi
\]

\[
= \frac{s^2 + (\xi\beta + x_d(\mu)^2 - 1)s + \xi\beta(x_d(\mu)^2 - 1) + \xi}{s + \xi\beta}
\]

(63)

is the complexity function with respect to the “input-signal-port” [Chua, 2005]. Thus, \(Y(s)\) has two zeros: \(z_1, z_2\) and one pole: \(p_1\). For \(\mu = 0\), we obtain

\[
\begin{align*}
p_1 &= -0.128, \\
z_i &\approx -0.03678 \pm i0.3028.
\end{align*}
\]

For \(\mu = \mu^*\), we obtain

\[
\begin{align*}
p_1 &= -0.128, \\
z_i &\approx 0.02350 \pm i0.2776.
\end{align*}
\]

Let us examine the real part of \(Y(s)\) for \(s = i\omega\), \(\omega \in (-\infty, \infty)\):

\[
Y(i\omega) = \text{Re}[Y(i\omega)] + i \text{Im}[Y(i\omega)]
\]

where

\[
Y(i\omega) = i\omega + \frac{\xi}{i\omega + \xi\beta} + x_d(\mu)^2 - 1
\]

\[
= \left[ x_d(\mu)^2 - 1 + \frac{\xi^2\beta}{\xi^2\beta^2 + \omega^2} \right]
\]

\[
= \left[ 1 - \frac{\xi}{\xi^2\beta^2 + \omega^2} \right],
\]

(67)

and

\[
\begin{align*}
\text{Re}[Y(i\omega)] &= \left[ x_d(\mu)^2 - 1 + \frac{\xi^2\beta}{\xi^2\beta^2 + \omega^2} \right], \\
\text{Im}[Y(i\omega)] &= \omega \left[ 1 - \frac{\xi}{\xi^2\beta^2 + \omega^2} \right].
\end{align*}
\]

(68)

If \(\mu = 0\), we found that \(\text{Re}[Y(i\omega)] < 0\) for \(|\omega| > 0.4678\). Thus, Eq. (37) is locally active and on edge of chaos in the neighborhood of \(\mu = 0\). The behavior of the complexity function \(Y(i\omega)\) as a function of frequency \(\omega\) is depicted in Fig. 2.

4.2. FitzHugh–Nagumo equation with quintic nonlinearity

In this section, we show that simple dissipation can generate two limit cycles without changing the stability of the equilibrium point.
Consider the FitzHugh–Nagumo equation with where $\xi = 0.1$, and a quintic nonlinearity defined by
\[
\begin{align*}
\frac{dx}{dt} &= -y - g(x), \\
\frac{dy}{dt} &= \xi(x - y + 0.1),
\end{align*}
\]
\begin{align*}
\text{(69)} & \quad g(x) = \frac{1}{5}x^5 - \frac{17}{48}x^3 - \frac{15}{16}x.
\end{align*}
\begin{align*}
\text{(70)}
\end{align*}

Fig. 2. Complexity function $Y(i\omega)$. (a) Pole-zero configuration of $Y(i\omega)$. (b) Plot of $\text{Im}[Y(i\omega)]$ versus $\text{Re}[Y(i\omega)]$. (c) Plot of $\text{Re}[Y(i\omega)]$ versus $\omega$. (d) Plot of $\text{Im}[Y(i\omega)]$ versus $\omega$. 
Equation (69) has an equilibrium point \((x_e, y_e)\) satisfying
\[
\begin{align*}
-ye - \frac{1}{5} x_e^5 + \frac{17}{48} x_e^3 + \frac{15}{16} x_e = 0, \\
x_e - y_e + 0.1 = 0.
\end{align*}
\] (71)

Its Jacobian matrix
\[
\begin{bmatrix}
-x_e^4 + \frac{17}{16} x_e^2 + \frac{15}{16} & -1 \\
0.1 & -0.1
\end{bmatrix}
\] (72)
has a pair of complex eigenvalues with a negative real part. The equilibrium point and its eigenvalues are given by
\[ (x_e, y_e) \approx (-1.343, -1.243), \quad \lambda_{1,2} \approx -0.25 \pm i0.2784. \] (73)

Consider next the above FitzHugh-Nagumo equation with a *dissipation term*
\[
\begin{align*}
\frac{dx}{dt} &= -y - g(x) - \mu x, \\
\frac{dy}{dt} &= \xi(x - y + 0.1),
\end{align*}
\] (74)

where \(\mu > 0\) is a dissipation coefficient, \(\xi = 0.1\), and
\[ g(x) = \frac{1}{5} x^5 - \frac{17}{48} x^3 - \frac{15}{16} x. \] (75)

The equilibrium point \((x_d, y_d)\) of Eq. (74) satisfies the following relation:
\[
\begin{align*}
-y_d - \frac{x_d^5}{5} + \frac{17}{48} x_d^3 + \frac{15}{16} x_d - \mu x_d &= 0, \\
x_d - y_d + 0.1 &= 0.
\end{align*}
\] (76)

It follows that
\[
\frac{1}{5} x_d^5 - \frac{17}{48} x_d^3 - \frac{15}{16} x_d + (\mu + 1) x_d + 0.1 = 0. \quad (77)
\]

The Jacobian matrix of Eq. (74) at \((x_d, y_d)\) is given by
\[
\begin{bmatrix}
-x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} - \mu & -1 \\
0.1 & -0.1
\end{bmatrix}.
\] (78)

Thus, we get the following characteristic equation:
\[
\begin{align*}
\lambda^2 - \left( -x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} - \mu - 0.1 \right) \lambda \\
+ 0.1 \left( x_d^4 - \frac{17}{16} x_d^2 - \frac{15}{16} + \mu + 1 \right) &= 0.
\end{align*}
\] (79)

If \(x_d\) and \(\mu^*\) satisfy the relation
\[
\begin{align*}
\lambda_1 + \lambda_2 &= -x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} - \mu^* - 0.1 = 0, \\
\lambda_1 \lambda_2 &= x_d^4 - \frac{17}{16} x_d^2 - \frac{15}{16} + \mu^* + 1 > 0,
\end{align*}
\] (80)

(\(\lambda_i\): pure imaginary) and if \(\alpha\) and \(\epsilon(\mu)\) satisfy the two inequalities
\[
\frac{d\epsilon(\mu)}{d\mu} > 0, \quad (81)
\]

and
\[
\alpha \epsilon(\mu) > 0 \quad (82)
\]

for \(\mu > \mu^*\), then a unique stable limit cycle generically bifurcates from the equilibrium point \((x_d, y_d)\). From Eqs. (76) and (80), we obtain the following two solutions:
\[
\begin{align*}
(x_d^1, y_d^1) &\approx (-1.244, -1.1439), \quad \mu_1^* \approx 0.08704, \\
(x_d^2, y_d^2) &\approx (-0.1101, -0.01008), \quad \mu_2^* \approx 0.8502.
\end{align*}
\] (83)

We next show that Eqs. (81) and (82) are satisfied in a neighborhood of \(\mu_1^*\) and \(\mu_2^*\). Differentiating Eq. (77) with respect to \(\mu\), we obtain
\[
\frac{dx_d}{d\mu} = -\frac{x_d}{\left( -x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} + \mu + 1 \right)}.
\] (84)

Differentiating
\[
\epsilon(\mu) = \frac{\lambda_1 + \lambda_2}{2} = -x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} - \mu - 0.1
\] (85)

with respect to \(\mu\), we get
\[
\frac{d\epsilon(\mu)}{d\mu} = \left( -2x_d^3 + \frac{17}{16} x_d \right) \frac{dx_d}{d\mu} - \frac{1}{2}.
\] (86)

Substituting Eq. (84) into Eq. (86), we obtain
\[
\frac{d\epsilon(\mu)}{d\mu} = -\frac{x_d \left( -2x_d^3 + \frac{17}{16} x_d \right)}{\left( -x_d^4 + \frac{17}{16} x_d^2 + \frac{15}{16} + \mu + 1 \right)} - \frac{1}{2}.
\] (87)
It follows from Eqs. (80) and (87) that

\[
\frac{d\epsilon(\mu^*)}{d\mu} = -\frac{x_d \left(-2x_d^3 + \frac{17}{16}x_d\right)}{\left(x_d^4 - \frac{17}{16}x_d^2 - \frac{15}{16} + \mu^* + 1\right)^{\frac{1}{2}}} - \frac{1}{2}.
\]  

Substituting Eq. (83) into Eq. (88), we obtain

\[
\begin{align*}
\frac{d\epsilon(\mu^*_1)}{d\mu} &= 2.995, \\
\frac{d\epsilon(\mu^*_2)}{d\mu} &= -0.514.
\end{align*}
\]  

(89)

4.2.1. \( \mu^* = \mu_1^* \)

In the case of \( \mu^* = \mu_1^* \), we get the relations:

\[
\frac{d\epsilon(\mu_1^*)}{d\mu} > 0, \quad \epsilon(\mu_1^*) = 0,
\]  

(90)

and therefore

\[
\epsilon(\mu) > 0,
\]  

(91)

for \( 0 < \mu - \mu_1^* \ll 1 \). Furthermore, \( \alpha > 0 \), since the equilibrium point with \( \mu = \mu^* \) is asymptotically stable. The sign of \( \alpha \) can also be obtained from Eq. (52) for \( a = 0.1, b = 1.0 \), and \( F(u) = -g(u + x_d) - \mu x_d - y_d \). Since

\[
\left[ \frac{d^3 F(0)}{du^3} + \left( \frac{d^2 F(0)}{du^2} \right)^2 \right] \approx -44.86,
\]  

(92)

we obtain \( \alpha > 0 \). Thus, we conclude that \( \alpha \epsilon(\mu) > 0 \) for \( 0 < \mu - \mu_1^* \ll 1 \). This implies that a stable limit cycle generically bifurcates from the asymptotically-stable equilibrium point, and this equilibrium point becomes unstable.

4.2.2. \( \mu^* = \mu_2^* \)

In the case of \( \mu^* = \mu_2^* \), let us introduce the following new parameters \( \rho, \rho_1^* \) and \( \rho_2^* \):

\[
\begin{align*}
\rho &= -\mu < 0, \\
\rho_1^* &= -\mu_1^* < 0, \\
\rho_2^* &= -\mu_2^* < 0.
\end{align*}
\]  

(93)

Using these parameters, we obtain

\[
\frac{d\epsilon(\rho_2^*)}{d\rho} = -\frac{d\epsilon(\mu_2^*)}{d\rho} > 0, \quad \epsilon(\rho_2^*) = 0
\]  

(94)

and

\[
\epsilon(\rho) < 0,
\]  

(95)

for \( 0 < \rho_2^* - \rho \ll 1 \). Thus,

\[
\alpha \epsilon(\rho) < 0.
\]  

(96)

It follows that an unstable limit cycle generically bifurcates from the unstable equilibrium point, and this equilibrium point becomes asymptotically stable. That is, in the neighborhood of \( \rho = \rho_2^* \), there are two limit cycles, one is an inner unstable limit cycle, and the other is an outer stable limit cycle, which bifurcated at \( \rho = \rho_1^* \). In contrast to the supercritical Hopf bifurcation in the \( \mu^* = \mu_1^* \) case, the above bifurcation phenomenon \( \mu^* = \mu_2^* \) is called a subcritical Hopf bifurcation [Alligood et al., 1997].

Note that the dissipation \( \rho x \) satisfies the inequalities

\[
0 < \rho_2^* - \rho \ll 1, \quad \rho < 0,
\]  

(97)

and hence does not change the stability of the equilibrium point of Eq. (69), however, oscillation occurs in this system. Hence, dissipation can bring about the oscillation even if the stable equilibrium point is not destabilized.

The Hopf bifurcation phenomena associated with the FitzHugh–Nagumo with quintic nonlinearity are illustrated in Fig. 3. Observe that a unique stable limit cycle bifurcates from an equilibrium point. Furthermore, an unstable limit cycle bifurcates from the unstable equilibrium point.

4.3. Rössler’s equation

In this section, we show the simple dissipation can bring about chaotic oscillations in Rössler’s equation [Rössler, 1977], which is defined by

\[
\begin{align*}
\frac{dx}{dt} &= x - xy - z, \\
\frac{dy}{dt} &= z^2 - \mu y, \\
\frac{dz}{dt} &= bx - cz + d,
\end{align*}
\]  

(98)

where \( b = 0.08, c = 0.38, d = 0.0015 \), and \( \mu > 0 \) denotes the dissipation coefficient.
Fig. 3. Hopf bifurcation of the FitzHugh–Nagumo equation with quintic nonlinearity. (a) A stable equilibrium point for $\mu = 0$. (b) A stable limit cycle for $\mu = 0.1$. (c) An outer stable limit cycle for $\mu = 0.9$. (d), (e) An inner unstable limit cycle for $\mu = 0.9$. (f) Coexistence of two limit cycles for $\mu = 0.9$. In the neighborhood of $\mu \approx 0.98704$, a unique stable limit cycle bifurcates from a stable equilibrium point. In the neighborhood of $\mu \approx 0.8502$, there are two limit cycles, one is an outer stable limit cycle of (c), and the other is an inner unstable limit cycle of (d). Initial value $(x(0), y(0)) = (0.1, 0.1)$ is used to plot the trajectories of (a) and (b). Different initial values are used to plot the limit cycles of (c) and (d).
Fig. 3. (Continued)
For $\mu = 0.01$, the equilibrium points and their eigenvalues are as follow:

<table>
<thead>
<tr>
<th>Type</th>
<th>Equilibrium Points and Eigenvalues for $\mu = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>stable focus</td>
<td>$(x_0, y_0, z_0) \approx (-0.09125, 0.8327, -0.01526)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^0 \approx (-0.2053, -0.008735 \pm i0.1776)$</td>
</tr>
<tr>
<td>saddle</td>
<td>$(x_3, y_3, z_3) \approx (0.005016, 0.002516, 0.005003)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^3 \approx (0.9367, -0.319251, -0.009936)$</td>
</tr>
<tr>
<td>saddle-focus</td>
<td>$(x_\gamma, y_\gamma, z_\gamma) \approx (0.08624, 0.7437, 0.02210)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^\gamma \approx (-0.2368, 0.05155 \pm i0.1431)$</td>
</tr>
</tbody>
</table>

In this case, almost all trajectories tend to the stable focus at $(x_0, y_0, z_0)$.

If $\mu$ is increased from 0.01 to 0.02, then a stable limit cycle bifurcates from the stable focus, and this equilibrium point becomes a saddle-focus. That is, a Hopf bifurcation occurs in the neighborhood of $\mu = 0.02$. The equilibrium points of Eq. (98) and their eigenvalues are as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>Equilibrium Points and Eigenvalues for $\mu = 0.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>saddle-focus</td>
<td>$(x_0', y_0', z_0') \approx (-0.02302, 0.8203, -0.1281)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^0 \approx (-0.2393, 0.00950972 \pm i0.230195)$</td>
</tr>
<tr>
<td>saddle</td>
<td>$(x_3', y_3', z_3') \approx (0.005008, 0.001254, 0.005001)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^3 \approx (0.9380, -0.3193, -0.01994)$</td>
</tr>
<tr>
<td>saddle-focus</td>
<td>$(x_\gamma', y_\gamma', z_\gamma') \approx (0.1231, 0.7574, 0.02986)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{1,2,3}^\gamma \approx (-0.2520, 0.04732 \pm i0.2061)$</td>
</tr>
</tbody>
</table>

If $\mu$ is increased further, for example, if $\mu = 0.08$, then a chaotic attractor bifurcates from the stable limit cycle. Thus, simple dissipation can bring about periodic or chaotic oscillations in Rössler's equation. The above bifurcation phenomena of Rössler's equation are illustrated in Fig. 4.

### 4.4. Lorenz equation

In this section, we show that simple dissipation can bring about chaotic oscillations in Lorenz equation [Lorenz, 1963], which is defined by

$$
\begin{align*}
\frac{dx}{dt} &= 10(y - x), \\
\frac{dy}{dt} &= -xz + 28x - y, \\
\frac{dz}{dt} &= xy - \mu z,
\end{align*}
$$

From Eq. (99), we obtain

$$
\frac{dw}{dt} = 10x(y - x) + y(-xz + 28x - y) + (z - 38)(xy - \mu z) = -10x^2 - y^2 - \mu z^2 + 361\mu^2.
$$

where

$$
\begin{align*}
w &= x^2 + y^2 + (z - 38)^2 \\
\frac{dw}{dt} &= -10x^2 - y^2 \leq 0.
\end{align*}
$$

Hence, all trajectories of Eq. (99) enter and stay in some sufficiently small ball in the three-dimensional space for $\mu > 0$. It follows that Eq. (99) must have at least one attractor in a bounded ball. If $\mu = 0$, we would obtain

$$
\frac{dw}{dt} = -10x^2 - y^2 \leq 0.
$$
Fig. 4. Bifurcation of Rössler's equation. (a) Almost all trajectories tend to the stable focus at \((x_\alpha, y_\alpha, z_\alpha) \approx (-0.09125, 0.8327, -0.01526)\) for \(\mu = 0.01\). Initial value \((x(0), y(0), z(0)) = (0.1, 0.1, 0.1)\) is used to plot the trajectory.

(b) A stable limit cycle emerges from the stable focus \((x_\alpha, y_\alpha, z_\alpha)\) for \(\mu = 0.02\). This stable focus becomes a saddle-focus \((x'_\alpha, y'_\alpha, z'_\alpha) \approx (-0.02302, 0.8203, -0.1281)\).

(c) A chaotic attractor bifurcates from a limit cycle for \(\mu = 0.05\).
In this case, since \( \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0 \) on this axis, all trajectories converge to the z-axis, which is an invariant set of Eq. (99). Thus, the invariant set, namely, the z-axis is globally stable.

Consider next the case where \( \mu = 0.1 \). The equilibrium points of Eq. (99) and their eigenvalues are as follow:

<table>
<thead>
<tr>
<th>Type</th>
<th>Equilibrium Points and Eigenvalues for ( \mu = 0.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>saddle</td>
<td>((x_\alpha, y_\alpha, z_\alpha) = (0, 0, 0))</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{1,2,3}^\alpha \approx \left( -0.1, \frac{-11 \pm \sqrt{1201}}{2} \right))</td>
</tr>
<tr>
<td>saddle-focus</td>
<td>((x_\alpha, y_\alpha, z_\alpha) = (\sqrt{2.7}, \sqrt{2.7}, 27))</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{1,2,3}^\alpha \approx (-11.19, 0.04580 \pm i2.196))</td>
</tr>
<tr>
<td>saddle-focus</td>
<td>((x_\alpha, y_\alpha, z_\alpha) = (-\sqrt{2.7}, -\sqrt{2.7}, 27))</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{1,2,3}^\alpha \approx (-11.19, 0.04580 \pm i2.196))</td>
</tr>
</tbody>
</table>

Since all three equilibrium points are unstable at \( \mu = 0.1 \), there must be at least one attractor. In this case, there exists a stable limit cycle in a neighborhood of each saddle-focus. If \( \mu \) is increased to \( 8/3 \), then a chaotic attractor appears. Thus, we conclude that simple dissipation can bring about periodic or chaotic oscillations in Lorenz equation. The above bifurcation phenomena of Lorenz equation are illustrated in Fig. 5.

5. Oscillation of One Cell under Fixed Boundary Conditions

In this section, we study a degenerate array consisting of “only one cell” with a “fixed boundary condition”. In this case, Eq. (2) reduces to

\[
\begin{align*}
\frac{dV_1(1,1)}{dt} &= f_1 \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) + D_1 \nabla^2 V_1(1,1), \\
\frac{dV_2(1,1)}{dt} &= f_2 \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) + D_2 \nabla^2 V_2(1,1), \\
&\vdots \\
\frac{dV_m(1,1)}{dt} &= f_m \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) + D_m \nabla^2 V_m(1,1), \\
\frac{dV_{m+1}(1,1)}{dt} &= f_{m+1} \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right), \\
\frac{dV_{m+2}(1,1)}{dt} &= f_{m+2} \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right), \\
&\vdots \\
\frac{dV_n(1,1)}{dt} &= f_n \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right),
\end{align*}
\]

(103)

where \( V_\sigma(1,1) \) denotes the \( \sigma \)th state variable, \( \sigma = 1, 2, \ldots, n \) of the cell located at the grid point \((1,1)\), and \( \nabla^2 V_\sigma(1,1) \) denotes the discretized Laplacian operator in \( \mathbb{R}^2 \):

\[
\nabla^2 V_\sigma(1,1) \triangleq V_\sigma(2,1) + V_\sigma(0,1) + V_\sigma(1,2) \\
+ V_\sigma(1,0) - 4V_\sigma(1,1).
\]

(104)
Fig. 5. Bifurcation of Lorenz equation. (a) Almost all trajectories converge to the z-axis for $\mu = 0$. Initial value $(x(0), y(0), z(0)) = (0.5, 0.0, 10.0)$ is used to plot the trajectory. (b), (c) Two asymptotically stable limit cycles appear around the saddle $(x_0, y_0, z_0) = (0, 0, 0)$ for $\mu = 0.1$. (d) A chaotic attractor bifurcates from a limit cycle for $\mu = 8/3$. 
The fixed boundary condition is defined by

\begin{align}
V_i(0,1) &= v_1, \\
V_i(1,0) &= v_2,
\end{align}

(105)

and

\begin{align}
V_i(2,1) &= v_1, \\
V_i(1,2) &= v_2.
\end{align}

(106)

Substituting Eqs. (105)-(106) into Eq. (104), we obtain

\begin{align}
\nabla^2 V_\sigma(1,1) &= 2(v_1 + v_2) - 4V_\sigma(1,1). \\
\text{If we set } v_1 &= v_2 = 0 \text{ and } 4D_\sigma = \mu, \text{ then the discrete Laplacian defined by Eq. (107) is equivalent to the dissipation term introduced in the previous sections:}
\end{align}

\begin{align}
D_\sigma \nabla^2 V_\sigma(1,1) &= 2D_\sigma (v_1 + v_2) - 4D_\sigma V_\sigma(1,1) \\
&= -\mu V_\sigma(1,1).
\end{align}

(108)

In this case, Eq. (103) reduces to the following:

\begin{align}
\frac{dV_1(1,1)}{dt} &= f_1 \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) - \mu V_1(1,1), \\
\frac{dV_2(1,1)}{dt} &= f_2 \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) - \mu V_2(1,1), \\
\vdots \\
\frac{dV_m(1,1)}{dt} &= f_m \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right) - \mu V_m(1,1), \\
\frac{dV_{m+1}(1,1)}{dt} &= f_{m+1} \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right), \\
\frac{dV_{m+2}(1,1)}{dt} &= f_{m+2} \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right), \\
\vdots \\
\frac{dV_n(1,1)}{dt} &= f_n \left( V_1(1,1), V_2(1,1), \ldots, V_n(1,1) \right).
\end{align}

(109)
We will now demonstrate that Eq. (103) can exhibit oscillations under appropriate choice of diffusion coefficient and boundary values. We now show that some dynamical systems can exhibit oscillation under “nonzero” fixed boundary conditions.

- FitzHugh–Nagumo equation
  Consider the following FitzHugh–Nagumo equation for $\mu = 0$ under fixed boundary conditions:

\[
\begin{align*}
\frac{dx}{dt} &= -y - f(x) + D\left(2(v_1 + v_2) - 4x\right), \\
\frac{dy}{dt} &= \xi(x - \beta y + \gamma),
\end{align*}
\]  

(110)

where $f(x) = (x^3/3) - x$, $\beta = 1.28$, $\gamma = 0.12$, $\xi = 0.1$, and $D \geq 0$. If $D = 0$, Eq. (110) has a globally asymptotically-stable equilibrium point. However, if we set $v_1 = v_2 = 0.25$ and $D = 0.0125$, then Eq. (110) has a stable limit cycle as shown in Fig. 6.

- Rössler’s equation
  Consider the following Rössler’s equation for $\mu = 0$ under fixed boundary conditions:

\[
\begin{align*}
\frac{dx}{dt} &= x - xy - z, \\
\frac{dy}{dt} &= z^2 + D\left(2(v_1 + v_2) - 4y\right), \\
\frac{dz}{dt} &= bx - cz + d,
\end{align*}
\]  

(111)

where $b = 0.08, c = 0.38, d = 0.0015$ and $D \geq 0$. If $D = 0$, Eq. (111) is globally asymptotically stable. However, if we set $v_1 = v_2 = -0.2$ and $D = 0.02$, then Eq. (110) has a chaotic attractor as shown in Fig. 7.

- Lorenz equation
  Consider the following Lorenz equation for $\mu = 0$ under fixed boundary conditions

\[
\begin{align*}
\frac{dx}{dt} &= 10(y - x), \\
\frac{dy}{dt} &= -xz + 28x - y, \\
\frac{dz}{dt} &= xy + D\left(2(v_1 + v_2) - 4z\right),
\end{align*}
\]  

(112)

where $D \geq 0$. If $D = 0$, Eq. (112) is globally asymptotically stable. However, if we set $v_1 = v_2 = -6.75$ and $D = 2/3$, then Eq. (110) has a chaotic attractor as shown in Fig. 8.

6. Oscillation via Diffusion and Dissipation

An uncoupled cell on the edge of chaos may cause a reaction–diffusion equation to oscillate under appropriate choice of diffusion coefficients. However, it does not imply that it is always possible to find some diffusion coefficients to destabilize a homogeneous solution. Such a set of destabilizing diffusion coefficients exist only for a proper subset of the edge of chaos parameter domain, which is called the sharp edge of-chaos domain. By definition, a cell on the sharp edge of chaos can be destabilized by some locally passive coupling. However, this may lead to a nonhomogeneous static pattern. For example, Turing’s equation can exhibit static complex patterns, but it cannot exhibit oscillation even if the cells are operated on the sharp edge of chaos.

To overcome this objection, Smale considered the same two-cell example but defined the kinetic equation of each cell by a nonlinear vector field in $\mathbb{R}^4$ [Smale, 1974]. He then proceeded to prove rigorously that each uncoupled cell is globally asymptotically stable. But upon coupling the two cells by diffusion, Smale then proved that the resulting
reaction–diffusion equation has a global limit cycle attractor. However, he had difficulty in reducing the number of cell state variables to two or even three [Smale, 1974].

In this section, we show that simple dissipation may cause the coupled cells to oscillate, even if the number of cell state variables is two. Let us consider a differential equation with a globally-asymptotically stable equilibrium point $\mathbf{V}$

$$\frac{d\mathbf{V}(r)}{dt} = \mathbf{g}(\mathbf{V}(r)).$$  \hfill (113)
Fig. 7. Oscillation of Rössler’s equation under fixed boundary conditions. Initial value \((x(0), y(0), z(0)) = (0.1, 0.1, 0.1)\) is used to plot the trajectory. (a) Almost all trajectories tend to the stable focus located at \((x_0, y_0, z_0) \approx (-0.99125, 0.8327, 0.01526)\) for \(v_1 = v_2 = 0\) and \(D = 0\). (b) A chaotic attractor emerges for \(v_1 = v_2 = -0.2\) and \(D = 0.02\).

where \(r = (j, k)\) denotes the array point, \(V(r) = (V_1(r), V_2(r), \ldots, V_n(r)) \in \mathbb{R}^n\) denotes the state variable, and \(g(V) \in \mathbb{R}^n\) denotes the vector function defined by

\[
g(V) = \left( g_1(V), g_2(V), \ldots, g_n(V) \right)
\]

where

\[
g(\overline{V}) = 0.
\]

Assume the cell oscillates upon adding simple dissipations. Let the cell with dissipation be described by

\[
\frac{dV(r)}{dt} = g\left( V(r) \right) - \mu k\left( V(r) \right),
\]

where \(\mu > 0\) and \(k(V) \in \mathbb{R}^n\) is a “dissipative” vector function

\[
k(V) = \left( k_1(V), k_2(V), \ldots, k_n(V) \right),
\]
Fig. 8. Oscillation of Lorenz equation under fixed boundary conditions. Initial value \((x(0), y(0), z(0)) = (0.5, 0.0, 10.0)\) is used to plot the trajectory. (a) Almost all trajectories converge to the z-axis for \(v_1 = v_2 = 0\) and \(D = 0\). (b) A chaotic attractor emerges for \(v_1 = v_2 = -6.75\) and \(D = 2/3\).

satisfying the inequality

\[
(V, k(V)) > 0, \quad (118)
\]

for nonzero \(V\). Equation (118) is the mathematical definition of "dissipation" in the sense that it corresponds to electrical or mechanical dissipations in real physical systems. Replacing the above dissipative function with the diffusion, we obtain

\[
\frac{dV(r)}{dt} = g(V(r)) + D \nabla^2 V(r), \quad (119)
\]

where \(D\) denotes an \(n \times n\) diagonal matrix defined by

\[
D_{\sigma \sigma} = \begin{cases} 
D_\sigma & \text{for } \sigma = 1, 2, \ldots, m \\
0 & \text{for } \sigma = m + 1, m + 2, \ldots, n
\end{cases} \quad (120)
\]

and \(\nabla^2 V\) denote an \(n \times 1\) vector defined by "\(n\)" discrete Laplacian operators; namely,

\[
\nabla^2 V = (\nabla^2 V_1, \nabla^2 V_2, \ldots, \nabla^2 V_m, \nabla^2 V_{m+1}, \ldots, \nabla^2 V_n). \quad (121)
\]
Since $\nabla^2 V$ (evaluated at $V = \bar{V}$) = 0, the reaction-diffusion equation (119) has also $V(r) = \bar{V}$ as its equilibrium point. It is not always possible to find some diffusion coefficients $D_x$ to destabilize this homogeneous solution. However, the discrete reaction-diffusion equation defined by
\[
\frac{dV(r)}{dt} = g(V(r)) - \mu k(V(r)) + D \nabla^2 V(r), \quad (122)
\]
can exhibit spatio-temporal patterns under appropriate choices of diffusion coefficients $D_x$ because Eq. (122) has the form of coupled oscillators and the equilibrium point of the uncoupled cell can be destabilized by dissipation. If we set $k(V) = V$ and $\mu = D_x = D$, then the combined diffusion and dissipation template\(^2\) is given by:
\[
D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -5 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (123)
\]
In the case of one-dimensional arrays, it is described by:
\[
D = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}. \quad (124)
\]
Following are some examples of dynamical systems which can oscillate under appropriate choice of the coefficient $D$ of the template (124).

6.1. Oscillation of FitzHugh–Nagumo equations

Consider the following system of FitzHugh–Nagumo equations with diffusive coupling:
\[
\begin{align*}
\frac{dx_1}{dt} &= -y_1 - f(x_1) + D(x_2 - x_1), \\
\frac{dy_1}{dt} &= \xi(x_1 - \beta y_1 + \gamma), \\
\frac{dx_2}{dt} &= -y_2 - f(x_2) + D(x_1 - x_2), \\
\frac{dy_2}{dt} &= \xi(x_2 - \beta y_2 + \gamma),
\end{align*} \quad (125)
\]
where $f(x) = (x^3/3) - x$, $\beta = 1.28$, $\gamma = 0.12$ and $\xi = 0.1$. Since each uncoupled cell is globally asymptotically stable, the two cells cannot be destabilized by only a diffusive coupling. That is, the composite Eq. (125) has only one asymptotically-stable equilibrium point, whose location and eigenvalues are described by
\[
(x_1, y_1, x_2, y_2) \approx (-0.9724, -0.6659, -0.9724, -0.6659). \quad (126)
\]
and
\[
\lambda_{1,2} \approx -0.08675 \pm i0.3135, \\
\lambda_{3,4} \approx -0.03675 \pm i0.3028. \quad (127)
\]
respectively.

If we add also dissipation to each uncoupled cell, we would obtain the equation:
\[
\begin{align*}
\frac{dx_1}{dt} &= -y_1 - f(x_1) - \mu x_1 + D(x_2 - x_1), \\
\frac{dy_1}{dt} &= \xi(x_1 - \beta y_1 + \gamma), \\
\frac{dx_2}{dt} &= -y_2 - f(x_2) - \mu x_2 + D(x_1 - x_2), \\
\frac{dy_2}{dt} &= \xi(x_2 - \beta y_2 + \gamma),
\end{align*} \quad (128)
\]
where $\mu = D = 0.05$. The combined diffusion and dissipation template is given by Eq. (124). Equation (128) has a stable limit cycle, since the two cells are destabilized by dissipation. It has only one unstable equilibrium point, whose location and eigenvalues are described by
\[
(x_1, y_1, x_2, y_2) \approx (-0.9041, -0.6126, -0.9041, -0.6126). \quad (129)
\]
and
\[
\lambda_{1,2} \approx -0.03310 \pm i0.3017, \\
\lambda_{3,4} \approx 0.03726 \pm i0.2696. \quad (130)
\]
respectively. Computer simulations of oscillation and synchronization phenomena of Eq. (128) are given in Fig. 9. Observe that two cells are synchronized by the diffusion coupling. Similarly, arrays of FitzHugh–Nagumo equations with both dissipation and diffusion can exhibit spatio-temporal oscillation. In Sec. 8, we show a detailed study of spatio-temporal oscillations of coupled FitzHugh–Nagumo equations.

\(^2\)In image processing, the diffusion and dissipation template is used for sharpening filters by changing the sign of $D$ [Itoh & Chua, 2007].
Fig. 9. Oscillation of FitzHugh–Nagumo equation with both dissipation and diffusion. (a) Stability of cell #1 ($D = 0.05$, $\mu = 0$). (b) Stability of cell #2 ($D = 0.05$, $\mu = 0$). (c) Oscillation of cell #1 ($D = \mu = 0.05$). (d) Oscillation of cell #2 ($D = \mu = 0.05$). (e) Synchronization of $x_1$ and $x_2$. (f) Synchronization of $y_1$ and $y_2$. 
Fig. 9. (Continued)
6.2. Oscillation of Rössler's equations

Consider a system of Rössler's equations with diffusive coupling:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 - x_1 y_1 - z_1, \\
\frac{dy_1}{dt} &= z_1^2 - ay_1 + D(y_2 - y_1), \\
\frac{dz_1}{dt} &= bx_1 - cz_1 + d, \\
\frac{dx_2}{dt} &= x_2 - x_2 y_2 - z_2, \\
\frac{dy_2}{dt} &= z_2^2 - ay_2 + D(y_1 - y_2), \\
\frac{dz_2}{dt} &= bx_2 - cz_2 + d,
\end{align*}
\]

where \(a = 0.01, b = 0.08, c = 0.38, d = 0.0015\). Since each uncoupled cell is globally asymptotically stable, the two coupled cells cannot be destabilized by the diffusive coupling alone.

If we add also the dissipation to each cell, we would obtain the following combined dissipative-diffusive system of equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 - x_1 y_1 - z_1, \\
\frac{dy_1}{dt} &= z_1^2 - \mu y_1 + D(y_2 - y_1), \\
\frac{dz_1}{dt} &= bx_1 - cz_1 + d, \\
\frac{dx_2}{dt} &= x_2 - x_2 y_2 - z_2, \\
\frac{dy_2}{dt} &= z_2^2 - \mu y_2 + D(y_1 - y_2), \\
\frac{dz_2}{dt} &= bx_2 - cz_2 + d,
\end{align*}
\]

where \(\mu = D = 0.08\). This system of equations exhibits chaotic oscillations, since it has the form of coupled chaotic oscillators. Computer simulations of oscillation and synchronization phenomena of Eq. (132) are given in Fig. 10. Observe that the two chaotic cells are synchronized by the diffusion coupling. Similarly, arrays of Rössler's equations with both dissipation and diffusions term can also exhibit spatio-temporal oscillations.

6.3. Oscillation of Lorenz equations

Consider the following system of Lorenz equations with diffusive coupling:

\[
\begin{align*}
\frac{dx_1}{dt} &= 10(y_1 - x_1), \\
\frac{dy_1}{dt} &= -x_1 z_1 + 28x_1 - y_1, \\
\frac{dz_1}{dt} &= x_1 y_1 + D(z_2 - z_1), \\
\frac{dx}{dt} &= 10(y_1 - x_1), \\
\frac{dy}{dt} &= -x_2 z_2 + 28x_2 - y_2, \\
\frac{dz}{dt} &= x_2 y_2 + D(z_1 - z_2),
\end{align*}
\]

Since each uncoupled cell is globally asymptotically stable, the two cells can be destabilized by diffusive coupling alone.

If we add dissipation to each cell, we would obtain the following combined dissipative-diffusive system of equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= 10(y_1 - x_1), \\
\frac{dy_1}{dt} &= -x_1 z_1 + 28x_1 - y_1, \\
\frac{dz_1}{dt} &= x_1 y_1 - \mu z_1 + D(z_2 - z_1), \\
\frac{dx}{dt} &= 10(y_1 - x_1), \\
\frac{dy}{dt} &= -x_2 z_2 + 28x_2 - y_2, \\
\frac{dz}{dt} &= x_2 y_2 - \mu z_2 + D(z_1 - z_2),
\end{align*}
\]
Fig. 10. Oscillation of Rössler’s equation with both dissipation and diffusion. (a) Stability of cell #1 \((D = 0.08, \mu = 0)\). (b) Stability of cell #2 \((D = 0.08, \mu = 0)\). (c) Oscillation of cell #1 \((D = \mu = 0.08)\). (d) Oscillation of cell #2 \((D = \mu = 0.08)\). (e) Synchronization of \(y_2\) and \(y_2\). (f) Synchronization of \(x_1\) and \(x_2\).
where \( \mu = D = 8/3 \). This system of equations exhibits chaotic oscillations, since it has the form of coupled chaotic oscillators. Computer simulations of oscillation and synchronization phenomena of Eq. (134) are given in Fig. 11. Observe that the two chaotic cells are synchronized by the diffusion coupling.

Finally, we show that a coupled system of Lorenz equations without dissipation cannot exhibit spatio-temporal oscillations. Consider the following two-dimensional array of Lorenz equations without dissipation (\( \mu = 0 \))

\[
\begin{align*}
\frac{dx(j,k)}{dt} &= 10\left(y(j,k) - x(j,k)\right) + D_1 \nabla^2 x(j,k), \\
\frac{dy(j,k)}{dt} &= -x(j,k)z(j,k) + 28x(j,k) \\
&\quad - y(j,k) + D_2 \nabla^2 y(j,k), \\
\frac{dz(j,k)}{dt} &= x(j,k)y(j,k) + D_3 \nabla^2 z(j,k),
\end{align*}
\]

(135)

where \( j = 1,2,\ldots,N, \) \( k = 1,2,\ldots,N, \) \( D_i \geq 0, \) \( x(j,k), y(j,k) \) and \( z(j,k) \) denote the states \( x, y \) and \( z \) at the grid point \( (j,k) \) of a spatial array in \( \mathbb{R}^2, \) respectively, and

\[
\begin{align*}
\nabla^2 x(j,k) &\triangleq x(j+1,k) + x(j-1,k) + x(j,k+1) \\
&\quad - x(j,k-1) - 4x(j,k), \\
\nabla^2 y(j,k) &\triangleq y(j+1,k) + y(j-1,k) + y(j,k+1) \\
&\quad - y(j,k-1) - 4y(j,k), \\
\nabla^2 z(j,k) &\triangleq z(j+1,k) + z(j-1,k) + z(j,k+1) \\
&\quad - z(j,k-1) - 4z(j,k).
\end{align*}
\]

(136)

Define

\[
W = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{x(j,k)^2 + y(j,k)^2 + z(j,k)^2}{2},
\]

(137)

where \( \tilde{z}(j,k) = z(j,k) - 38. \) From Eq. (135), we obtain

\[
\frac{dW}{dt} = \sum_{j=1}^{N} \sum_{k=1}^{N} \left(-10x(j,k)^2 - y(j,k)^2\right) + W_D,
\]

(138)

where

\[
W_D = \sum_{j=1}^{N} \sum_{k=1}^{N} \left\{ D_1 x(j,k) \nabla^2 x(j,k) \\
+ D_2 y(j,k) \nabla^2 y(j,k) + D_3 \tilde{z}(j,k) \nabla^2 \tilde{z}(j,k) \right\}.
\]

(139)

and

\[
\nabla^2 \tilde{z}(j,k) \triangleq \left[ z(j+1,k) - 38 \right] + \left[ z(j-1,k) - 38 \right] \\
+ \left[ z(j,k+1) - 38 \right] + \left[ z(j,k-1) - 38 \right] \\
- 4z(j,k) - 38.
\]

(140)

Since \( W_d \leq 0 \) (see [Itoh & Chua, 2005]), we obtain the inequality

\[
\frac{dW}{dt} \leq 0.
\]

(141)

Thus, Eq. (135) cannot exhibit spatio-temporal oscillations. However, if we add both dissipation and diffusion, then the arrays of Lorenz equations can exhibit a spatio-temporal oscillations.

7. Oscillation of Nonidentical Cells via Diffusive Couplings

In this section, we show that "nonidentical" asymptotically stable cells may exhibit oscillations, if they are coupled by diffusion alone. Assume that the uncoupled cell is defined by the following differential equation with a parameter \( \kappa \):

\[
\frac{dV}{dt} = g(V, \kappa),
\]

(142)

and assume that Eq. (142) has an asymptotically-stable equilibrium point \( \bar{V}(\kappa) \). The dynamics of the coupled cells is defined by

\[
\begin{align*}
\frac{dV_1}{dt} &= g(V_1, \kappa_1) + D(V_2 - V_1), \\
\frac{dV_2}{dt} &= g(V_2, \kappa_2) + D(V_1 - V_2),
\end{align*}
\]

(143)

where \( D \neq 0. \)

If \( \bar{V}(\kappa_1) = \bar{V}(\kappa_2) \), then Eq. (143) has \( \bar{V}(V_1, V_2) = (\bar{V}(\kappa_1), \bar{V}(\kappa_1)) \) as its equilibrium point.
Fig. 11. Oscillation of Lorenz equation with both dissipation and diffusion. (a) Stability of cell #1 \((D = 8/3, \mu = 0)\). (b) Stability of cell #2 \((D = 8/3, \mu = 8/3)\). (c) Oscillation of cell #1 \((D = \mu = 8/3)\). (d) Oscillation of cell #2 \((D = \mu = 8/3)\). (e) Synchronization of \(z_1\) and \(z_2\). (f) Synchronization of \(x_1\) and \(x_2\).
Fig. 11. (Continued)
However, if \( \overline{V}(\kappa_1) \neq \overline{V}(\kappa_2) \), we obtain the relations
\[
\begin{align*}
g \left( \overline{V}(\kappa_1), \kappa_1 \right) + D \left( \overline{V}(\kappa_2) - \overline{V}(\kappa_1) \right) &= D \left( \overline{V}(\kappa_2) - \overline{V}(\kappa_1) \right) \\
g \left( \overline{V}(\kappa_2), \kappa_2 \right) + D \left( \overline{V}(\kappa_1) - \overline{V}(\kappa_2) \right) &= D \left( \overline{V}(\kappa_1) - \overline{V}(\kappa_2) \right)
\end{align*}
\]
(144)

Hence, Eq. (143) cannot have \((V_1, V_2) = (\overline{V}(\kappa_1), \overline{V}(\kappa_2))\) as its equilibrium point. However, the following examples show that Eq. (143) may have unstable equilibrium points under appropriate choices of the parameter \( \kappa \).

7.1. Nonidentical cell coupling

Consider the following system of FitzHugh–Nagumo equations with two parameters \( \beta_1 \) and \( \beta_2 \):
\[
\begin{align*}
\frac{d{x_i}}{dt} &= -y_i - f(x_i), \\
\frac{d{y_i}}{dt} &= \xi(x_i - \beta_1 y_i + \gamma),
\end{align*}
\]
(145)

where \( i = 1, 2 \), \( f(x) = (x^3/3) - x \), \( \gamma = 0.12 \), \( \xi = 0.1 \). By coupling these cells by diffusion, we obtain the following system of equations:
\[
\begin{align*}
\frac{d{x_1}}{dt} &= -y_1 - f(x_1) + D(x_2 - x_1), \\
\frac{d{y_1}}{dt} &= \xi(x_1 - \beta_1 y_1 + \gamma), \\
\frac{d{x_2}}{dt} &= -y_2 - f(x_2) + D(x_1 - x_2), \\
\frac{d{y_2}}{dt} &= \xi(x_2 - \beta_2 y_2 + \gamma),
\end{align*}
\]
(146)

where \( D = 0.05 \) is the diffusion parameter. If we set \( \beta_1 = \beta_2 = 1.28 \), then Eq. (145) has a globally asymptotically-stable equilibrium point, and the coupled system (146) can have only one asymptotically-stable equilibrium point. Hence, it cannot have oscillations. However, if we set \( \beta_1 = 1.28, \beta_2 = 2.1 \), then Fig. 12 shows that Eq. (146) exhibits oscillations. In this case, one uncoupled cell has only one asymptotically-stable equilibrium point. The other uncoupled cell has two asymptotically-stable equilibrium points and a saddle point. The coupled system (146) has one stable equilibrium point, four unstable equilibrium points, and one stable limit cycle.

7.2. Different cell coupling

Consider the one-dimensional system defined by
\[
\frac{dx_2}{dt} = -x_2 + a,
\]
(147)

where \( a \) is a constant. All trajectories of Eq. (147) converge to the equilibrium point \( x_2 = a \). By coupling Eq. (147) with the FitzHugh–Nagumo equation by diffusion, we obtain the following equation:
\[
\begin{align*}
\frac{d{x_1}}{dt} &= -y_1 - f(x_1) + D(x_2 - x_1), \\
\frac{d{y_1}}{dt} &= \xi(x_1 - \beta_1 y_1 + \gamma), \\
\frac{d{x_2}}{dt} &= -x_2 + a + D(x_1 - x_2),
\end{align*}
\]
(148)

where \( f(x) = (x^3/3) - x, \beta = 1.28, \gamma = 0.12, \xi = 0.1, \) and \( a = 0.5 \). If \( D = 0 \), the two uncoupled cells are globally asymptotically stable. However, if we set \( D = 0.05 \), then Fig. 13 shows that Eq. (148) has a stable limit cycle. Thus, two different stable cells can exhibit oscillations by diffusive couplings.

Similarly, if Eq. (147) is coupled with the Rössler’s equation, or the Lorenz equation, then the system also can exhibit oscillations.

8. Spatio-Temporal Oscillation via Diffusion

An uncoupled cell on the edge of chaos may cause a reaction–diffusion equation to exhibit spatio-temporal phenomena under appropriate choices of diffusion coefficients. However, it does not imply that it is always possible to find some diffusion coefficients to destabilize an homogeneous solution. Indeed, the coupled FitzHugh–Nagumo equations (125) cannot be destabilized by diffusion alone. However, a large array of FitzHugh–Nagumo equations can exhibit spatio-temporal oscillations [Dogaru & Chua, 1998]. In this section, we show that spatio-temporal oscillations can occur in arrays of FitzHugh–Nagumo reaction–diffusion equations on the edge of chaos provided the array size is sufficiently large.
Fig. 12. Oscillation of two coupled nonidentical FitzHugh–Nagumo equations. (a) Oscillation of the nonidentically coupled FitzHugh–Nagumo equation for $\beta_1 = 1.28$. (b) Oscillation of the nonidentically coupled FitzHugh–Nagumo equation for $\beta_2 = 2.1$. (c) Oscillation on the $(x_1, x_2)$-plane.
Fig. 13. Two different stable cells can exhibit oscillations by diffusive coupling. (a) Oscillation in the \((x_1,y_1)\)-plane. (b) Oscillation in the \((x_1,y_1,x_2)\)-space. (c) Oscillation in the \((x_1,x_2)\)-plane.
Consider the following system of discrete reaction–diffusion equations with boundary conditions:

\[
\begin{align*}
\frac{dV_1(j, k)}{dt} &= f_1(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_1 \nabla^2 V_1(j, k), \\
\frac{dV_2(j, k)}{dt} &= f_2(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_2 \nabla^2 V_2(j, k), \\
&\vdots \\
\frac{dV_m(j, k)}{dt} &= f_m(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_m \nabla^2 V_m(j, k), \\
\frac{dV_{m+1}(j, k)}{dt} &= f_{m+1}(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)), \\
\frac{dV_{m+2}(j, k)}{dt} &= f_{m+2}(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)), \\
&\vdots \\
\frac{dV_n(j, k)}{dt} &= f_n(V_1(j, k), V_2(j, k), \ldots, V_n(j, k))
\end{align*}
\] (149)

Here, \(V_\sigma(j, k)\) denotes the \(\sigma\)th state variable, \(\sigma = 1, 2, \ldots, n\) of a cell located at the grid point \((j, k)\) of a spatial array in \(\mathbb{R}^2\), \(\sigma = 1, 2, \ldots, m\), denotes a positive diffusion coefficient associated with the state variable \(V_\sigma(j, k)\), and \(\nabla^2 V_\sigma(j, k)\) denotes the discretized Laplacian operator

\[
\nabla^2 V_\sigma(j, k) \triangleq V_\sigma(j + 1, k) + V_\sigma(j - 1, k) \\
+ V_\sigma(j, k + 1) + V_\sigma(j, k - 1) \\
- 4V_\sigma(j, k),
\] (150)

in \(\mathbb{R}^2\) where \(\sigma = 1, 2, \ldots, m\).

The spatio-temporal oscillation of Eq. (149) propagates through the discretized Laplacian coupling (150). That is, a wave propagates from one cell to another while synchronizing neighboring cells via the Laplacian coupling. If the neighboring cells are almost synchronized, the Laplacian term (150) becomes almost null, and the amplitude of the oscillation tends to decrease since the uncoupled cell is asymptotically stable. Thus, each cell transfers the energy to the next in a kind of "bucket brigade" fashion. The propagation of the wave is not disturbed as long as the wave does not reach the boundary.

Let us define the distance between cells \(C(j, k)\) and \(C(l, m)\), located at the grid points \((j, k)\) and \((l, m)\), respectively. If \(C(j, k)\) and \(C(l, m)\) are connected by the Laplacian operator, then the distance is equal to zero, namely,

\[
\begin{align*}
\begin{cases}
\quad d(C(j, k), C(l, m)) = 0, & \text{if } l = j - 1, j + 1, \\
\quad m = k - 1, k + 1,
\end{cases}
\end{align*}
\] (151)

That is,

\[
\begin{align*}
\quad d(C(j, k), C(j - 1, k)) = 0, \\
\quad d(C(j, k), C(j + 1, k)) = 0, \\
\quad d(C(j, k), C(j, k - 1)) = 0, \\
\quad d(C(j, k), C(j, k + 1)) = 0.
\end{align*}
\] (152)

If three cells \(C(j, k)\), \(C(l, m)\) and \(C(n, o)\) satisfy

\[
\begin{align*}
\quad d(C(j, k), C(l, m)) = 0, & \quad d(C(l, m), C(n, o)) = 0,
\end{align*}
\] (153)

then the distance between \(C(j, k)\) and \(C(n, o)\) is defined by

\[
\begin{align*}
\quad d(C(j, k), C(n, o)) = 1,
\end{align*}
\] (154)

where \((j, k) \neq (l, m), (l, m) \neq (n, o), (j, k) \neq C(n, o)\). Furthermore, if four-cell \(C(p, q)\) satisfy

\[
\begin{align*}
\quad d(C(j, k), C(l, m)) = 0, & \quad d(C(l, m), C(n, o)) = 0,
\end{align*}
\]
\[ d\left( C(n,o), C(p,q) \right) = 0, \quad (155) \]

then

\[ d\left( C(j,k), C(p,q) \right) = 2, \quad (156) \]

where \( (p,q) \not\in \{(j,k), (l,m), (n,o)\} \). The distance which is greater than 2 is inductively defined. Note that we always choose the shortest distance between two cells.

In the case of one-dimensional arrays, the discretized Laplacian operator is defined by

\[
\nabla^2 V_\sigma(j, k) \overset{\Delta}{=} V_\sigma(j + 1) + V_\sigma(j - 1) - 2V_\sigma(j), \quad (157)
\]

where \( \sigma = 1, 2, \ldots, m \). The zero distance is defined by

\[
\begin{align*}
  d\left( C(j), C(l) \right) &= 0, \quad \text{if } l = j - 1, j + 1, \\
  d\left( C(j), C(l) \right) &\neq 0, \quad \text{otherwise}.
\end{align*}
\]

That is,

\[
\begin{align*}
  d\left( C(j), C(j - 1) \right) &= 0, \\
  d\left( C(j), C(j + 1) \right) &= 0.
\end{align*}
\]

(159)

Similarly, if three cells \( C(j), C(k) \) and \( C(l) \) satisfy

\[ d\left( C(j), C(k) \right) = 0, \quad d\left( C(k), C(l) \right) = 0, \quad (160) \]

then the distance between \( C(j) \) and \( C(l) \) is defined by

\[ d\left( C(j), C(l) \right) = 1. \quad (161) \]

where \( j \neq k, k \neq l, j \neq l \).

In order to explain spatio-temporal oscillations in the FitzHugh–Nagumo reaction–diffusion equation, where the cells are operating on the edge of chaos, let us set up the following hypotheses for one-dimensional arrays:

- No reflection occurs at a zero-flux boundary.
- A progressive wave must return to the beginning point (cell) without decay in order to sustain a spatio-temporal periodic oscillation.
- Any set of three contiguous cells is separated by at least a set of two contiguous cells (see Fig. 14), since two cells with zero distance tend to synchronize via diffusion.

Under the above hypotheses, a progressive wave can return to the beginning cell in the following array:

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Length ( N ) of array</th>
</tr>
</thead>
<tbody>
<tr>
<td>periodic boundary</td>
<td>5, 6, 7, \ldots</td>
</tr>
</tbody>
</table>

In the case of a zero-flux boundary, a progressive wave cannot return to the beginning cell, because we assumed that no reflection can occur at the boundary. The oscillation under a fixed boundary condition will be discussed in the last part of this section.

Similarly, we set up the following hypotheses for two-dimensional arrays:

- No reflection occurs at a zero-flux boundary.
- A progressive wave must return to the beginning point (cell) without decay in order to sustain a spatio-temporal periodic oscillation. Hence, the progressive wave propagates over a closed locus \( \Gamma \) made of contiguous cells.
- A closed locus \( \Gamma \) must be separated by at least a \( 2 \times 2 \) array situated inside of \( \Gamma \). In the case of a periodic boundary, \( \Gamma \) must be separated by both a \( 2 \times 2 \) array inside of \( \Gamma \), and a \( 2 \times 2 \) array outside of \( \Gamma \) (see Fig. 15). It is due to this reason why the cells with zero distance from \( \Gamma \) tend to synchronize via diffusion, and the whole cells would synchronize if \( \Gamma \) is not separated by a \( 2 \times 2 \) cell.

Under the above hypotheses, a progressive wave can return to the beginning cell in the following array:

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>( N \times N ) Array</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero-flux boundary</td>
<td>4 \times 4, 5 \times 5, 6 \times 6, \ldots</td>
</tr>
<tr>
<td>periodic boundary</td>
<td>5 \times 5, 6 \times 6, 7 \times 7, \ldots</td>
</tr>
</tbody>
</table>

Some return routes are illustrated in Fig. 16. Note that there exist many other return routes.
Consider next the fixed boundary condition
\[
V_i(0, k) = v_1, \quad k = 1, 2, \ldots, N \\
V_i(j, 0) = v_2, \quad j = 1, 2, \ldots, N
\]
(162)
and
\[
V_i(N + 1, k) = v_3, \quad k = 1, 2, \ldots, N \\
V_i(j, N + 1) = v_4, \quad j = 1, 2, \ldots, N
\]
(163)

Fig. 14. Separation of three contiguous cells. (a) A 4-array which does not satisfy the hypothesis. (b) A 5-array which satisfies the hypothesis.

Fig. 15. The closed locus \( \Gamma \) in blue is separated by a 2 \( \times \) 2 array in orange. (a) A 4 \( \times \) 4 array under zero-flux boundary conditions. (b) A 6 \( \times \) 6 array under periodic boundary conditions.

Fig. 16. Propagation of waves along \( \Gamma \) in blue. (a) A 4 \( \times \) 4 array under zero-flux boundary conditions. (b) A 5 \( \times \) 5 array under periodic boundary conditions. (c) A 6 \( \times \) 6 array under periodic boundary conditions.
(see Eqs. (8) and (9)). If we assume that the outermost cells are connected to the boundaries, and if \( v_1 = v_2 = v_3 = v_4 \), but they are not contiguous neighbor cells, then the dynamics of the outermost cells are described by:

\[
\begin{align*}
\frac{dV_1(j, k)}{dt} &= f_1(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_1(v - 4V_1(j, k)), \\
\frac{dV_2(j, k)}{dt} &= f_2(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_2(v - 4V_2(j, k)), \\
&\vdots \quad \vdots \\
\frac{dV_m(j, k)}{dt} &= f_m(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)) + D_m(v - 4V_m(j, k)), \\
\frac{dV_{m+1}(j, k)}{dt} &= f_{m+1}(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)), \\
\frac{dV_{m+2}(j, k)}{dt} &= f_{m+2}(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)), \\
&\vdots \quad \vdots \\
\frac{dV_n(j, k)}{dt} &= f_n(V_1(j, k), V_2(j, k), \ldots, V_n(j, k)),
\end{align*}
\]

(164)

where \( j = N \) or \( k = N \), and \( v = v_1 \) (except for corner cells) or \( v = 2v_1 \) (corner cells). If the term \( D_\sigma(v - 4V_\sigma(j, k)) \) \( (\sigma = 1, 2, \ldots, m) \) can bring about oscillation as shown in Sec. 5, then spatio-temporal oscillations may occur from boundary cells. In this case, the homogeneous property of the Local Activity Principle [Chua, 1998, 1999] is also broken at the boundary cell.

Local Activity Principle

A homogeneous nonconservative medium cannot exhibit complexity unless it is locally active.

8.1. **FitzHugh–Nagumo reaction–diffusion equation**

In this section, we study spatio-temporal oscillations in discrete FitzHugh–Nagumo reaction–diffusion equations using computer simulations. Consider the following one-dimensional array of FitzHugh–Nagumo reaction–diffusion equations

\[
\begin{align*}
\frac{dx(j)}{dt} &= -y(j) - f(x(j)) + D_1 \nabla^2 x(j), \\
\frac{dy(j)}{dt} &= \xi(x(j) - \beta y(j) + \gamma) + D_2 \nabla^2 y(j),
\end{align*}
\]

(165)

where \( j = 1, 2, \ldots, N \), \( f(x) = (x^3/3) - x \), \( \beta = 1.28 \), \( \gamma = 0.12 \), \( \xi = 0.1 \), \( D_1 = 0.05 \), \( D_2 = 0 \) and

\[
\begin{align*}
\nabla^2 x(j) &\triangleq x(j + 1) + x(j - 1) - 2x(j), \\
\nabla^2 y(j) &\triangleq y(j + 1) + y(j - 1) - 2y(j).
\end{align*}
\]

(166)

The discrete FitzHugh–Nagumo reaction–diffusion equation (165) has \( N \) identical cells. Each uncoupled cell has a globally asymptotically-stable equilibrium point, and the cell is locally active. Our numerical simulations show that spatio-temporal oscillations in this equation depend on both the boundary conditions and the size of the arrays. That is, there exists a number \( N_{\text{min}}(\geq 2) \), such that if the size of the array \( N \geq N_{\text{min}} \), then Eq. (165) can exhibit spatio-temporal oscillations. However, if \( N < N_{\text{min}} \), then Eq. (165) cannot
exhibit spatio-temporal oscillations. From computer simulations, the minimum integer $N_{\text{min}}$ is as follows:

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$N_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed boundary (fixed to zero)</td>
<td>2</td>
</tr>
<tr>
<td>zero-flux boundary</td>
<td>-</td>
</tr>
<tr>
<td>periodic boundary</td>
<td>5</td>
</tr>
</tbody>
</table>

Under zero-flux boundary conditions, spatio-temporal oscillations are not observed. Observe that this experimental result is consistent with the hypotheses. Our computer simulations of spatio-temporal oscillations are illustrated in Figs. 17 and 18. The following palette is used to show the state of the cells:

<table>
<thead>
<tr>
<th>State</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1 &lt; x(j, k)$</td>
<td>red</td>
</tr>
<tr>
<td>$-1 \leq x(j, k) \leq 1$</td>
<td>grayscale</td>
</tr>
<tr>
<td>$x(j, k) &lt; -1$</td>
<td>green</td>
</tr>
</tbody>
</table>

Consider next the following two-dimensional array of FitzHugh–Nagumo reaction–diffusion equations

$$
\begin{align*}
\frac{dx(j, k)}{dt} &= -y(j, k) - f(x(j, k)) + D_1 \nabla^2 x(j, k), \\
\frac{dy(j, k)}{dt} &= \xi \left( x(j, k) - \beta y(j, k) + \gamma \right) + D_2 \nabla^2 y(j, k),
\end{align*}
$$

(167)

where $j = 1, 2, \ldots, N$, $k = 1, 2, \ldots, N$, $D_1 = 0.05$, $D_2 = 0$ and

$$
\nabla^2 x(j, k) \triangleq x(j + 1, k) + x(j - 1, k) + x(j, k + 1) + x(j, k - 1) - 4x(j, k), \\
\nabla^2 y(j, k) \triangleq y(j + 1, k) + y(j - 1, k) + y(j, k + 1) + y(j, k - 1) - 4y(j, k).
$$

(168)

From computer simulations, the minimum integer $N_{\text{min}}$ is as follow:

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$N_{\text{min}} \times N_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed boundary (fixed to zero)</td>
<td>2 \times 2</td>
</tr>
<tr>
<td>zero-flux boundary</td>
<td>4 \times 4</td>
</tr>
<tr>
<td>periodic boundary</td>
<td>5 \times 5</td>
</tr>
</tbody>
</table>

Fig. 17. Oscillation of a 2-array under fixed boundary conditions. The sequence of oscillation: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\cdots$. 
Fig. 18. Oscillation of a 5-array under periodic boundary conditions. A red stripe propagates from the left to the right. The sequence of oscillation: (1) ⇒ (2) ⇒ (3) ⇒ (1) ⇒ (2) ⋅⋅⋅.

Fig. 19. Oscillation of a 2 × 2 array under fixed boundary conditions. The sequence of oscillation: (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1) ⇒ (2) ⋅⋅⋅.

Fig. 20. Oscillation of a 4 × 4 array under zero-flux boundary conditions. A spiral wave in red turns around counterclockwise. The sequence of oscillation: (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1) ⇒ (2) ⋅⋅⋅.
Fig. 21. Oscillation of a $5 \times 5$ array under periodic boundary conditions. A red stripe propagates from the left to the right on the torus $T^2$ ($5 \times 5$ array with the periodic boundary). The sequence of oscillation: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1) \Rightarrow (2) \ldots$.

Fig. 22. Oscillation of a $6 \times 6$ array under periodic boundary conditions. A red $3 \times 3$ square turns around clockwise on the torus $T^2$ ($6 \times 6$ array with the periodic boundary). The sequence of oscillation: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (2) \ldots$. 
Observe that this experimental result is also consistent with the hypotheses. Our computer simulations of spatio-temporal oscillation are illustrated in Figs. 19–22. Note that the two-dimensional spatial-array under periodic boundary conditions is equivalent to a two-dimensional torus $T^2$.

From computer simulations, we conclude that spatio-temporal oscillations can occur in arrays of FitzHugh–Nagumo reaction–diffusion equations on the edge of chaos, provided the array size is sufficiently large.

9. Conclusion

In this paper, we have shown that simple dissipations can bring about oscillations in asymptotically-stable nonlinear dynamical systems. If these nonlinear dynamical systems are coupled, then the coupled system can exhibit spatio-temporal patterns. We have also shown that if two nonidentical dynamical systems are coupled by diffusion, then periodic oscillations can occur. Furthermore, we found that spatio-temporal oscillations can occur in spatially discrete reaction–diffusion equations on the edge of chaos, provided the array size is sufficiently large.

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References


