Rate-Maximizing Mappings for Memoryless Relaying

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Abstract—We study the problem of optimal design of relay mappings for the Gaussian relay channel in order to maximize the reliable transmission rate. We consider both Gaussian and modulation constrained signaling at the source. To optimize the relay mapping, we use an iterative integral equation as a necessary condition for optimality.

The optimized relay mappings demonstrate significant rate improvement over conventional linear relaying (amplify-and-forward). The optimized mappings allow an efficient utilization of the side information received via the source–destination link at the destination. Hence, the proposed mappings can be considered as an analog, memoryless approach to implementing compress-and-forward relaying with Wyner–Ziv compression in the relay.

I. INTRODUCTION

The relay channel consists of one sender (source), one receiver (destination) and a number of intermediate nodes (relays) whose purpose is to help the communication between the source and the destination. The capacity of the general relay channel is an open problem. Cover and El Gamal investigated the capacity of the single node relay channel in [1], where they derived upper and lower bounds. The achievable rates over the relay channel depend on the functionality of relay nodes, therefore considerable effort is put into finding optimal relaying strategies. The two most well-known strategies are compress-and-forward (CF) and decode-and-forward (DF), and it was shown in [2] that DF achieves the capacity when the relay is close to the source and CF achieves the capacity when the relay is close to the destination. It should be noted that both DF and CF are based on the assumption that the relay has infinite memory. In contrast, we in this paper assume that the relay is memoryless and its output consequently depends only on the current received signal. One main motivation for memoryless relaying is applications in wireless sensor networks where a massive number of inexpensive low-delay sensors with limited signal processing capability may need to be implemented.

In this paper, we propose design of memoryless (single-input single-output) relaying strategies to maximize the reliable information transmission rate from the source with an arbitrary input source alphabet. We assume that the relay and the source transmit signals to the destination in disjoint frequency bands or disjoint time slots. Related work has been done in [3]– [7]. In [3], amplify-and-forward (AF) strategy was shown to be capacity optimal when there are large of relays in network. For one relay node case, it was shown in [4] that sawtooth relaying outperforms AF when the source uses a Gaussian codebook. In [5], design of relay mappings for higher-order modulation schemes using demodulate-and-forward relay was investigated, and it was shown that a scheme so-called constellation re-arrangement at the relay can provide significant gains when the relay is placed close to the source. The paper [6] considers two-hop relay channel without direct link from the source to the destination, and shows that estimate-and-forward (EF) is optimal for BPSK signaling. Related work for optimized mappings in the two-way relay channel scenario appears in [7].

II. SYSTEM MODEL AND DESIGN CRITERION

We consider the three-node memoryless relay channel with source node $S$, memoryless relay node $R$ and destination node $D$ as shown in Fig. 1. We assume that the $S − D$ and $R − D$ links are orthogonal, and the relay is memoryless (single-input single-output). The signals received at $D$ are given by

$$Y = X + Z_1$$

(1)

$$Y_1 = g(W) + Z_3$$

(2)

where $W = X + Z_2$ is the received signal at $R$. The symbol $X$ is produced by $S$ with a given probability distribution $f_X(x)$ and is the input to the relay channel with the average power $P_s$. The symbols $(Y, Y_1)$ denote the outputs of the relay channel. The relay mapping is denoted by $g(W)$ which only depends on present received symbol $W$ at the relay. We assume the average power constraint $P_r$ at $R$. The variables $Z_i \sim \mathcal{N}(0, N_i)$ for $i \in \{1, 2, 3\}$, are independent AWGN components. Our goal is to find optimal relay mappings in order to maximize the reliable transmission rate for a given probability distribution $f_X(x)$ over the source alphabet.
A. Achievable Transmission Rate

The maximum rate with which information can be transmitted reliably from $S$ with a fixed probability distribution $f_X(x)$, to $D$, in the memoryless relaying scenario under discussion is

$$R_{\text{max}} = \max_{g(W): E[g^2(W)] \leq P_r} \frac{1}{2} I(X; Y, Y_1) \text{ [bpcu]}$$  \hspace{1cm} (3)

where the maximization is carried out over all memoryless relay mappings $g(.)$ subject to the average power constraint at $R$. The rate in (3) is measured in bits per channel use [bpcu].

We tailor the general problem to a special case with noiseless communications via $S-D$ and $R-D$ links in relay channel.

A relay mapping $g^*(.)$ which maximizes transmission rate in (3) is given by

$$g^*(W) = \arg \max_{g(W): E[g^2(W)] \leq P_r} I(X; Y, Y_1)$$  \hspace{1cm} (4)

We refer to $g^*(.)$ as a rate maximizing relay mapping.

III. MAPPING OPTIMIZATION FOR AN ARBITRARY INPUT DISTRIBUTION

In order to find rate maximizing relay mapping for a given source probability distribution $f_X(x)$, we construct the Lagrangian objective function

$$J = I(X; Y, Y_1) + \lambda \{E[g^2(W)] - P_r\}$$  \hspace{1cm} (5)

where $\lambda \in \mathbb{R}^+$. The necessary condition for optimality is given by

$$\frac{\partial J}{\partial g(r)} = 0, \quad \forall r \in \mathbb{R}$$  \hspace{1cm} (6)

After some mathematical manipulations, (6) simplifies to the following condition

$$g^*(r) = \frac{1}{\lambda f_W(r)} \int \int \int (y_1 - g^*(r)) f_{Y_1|X}(y_1|x) f_{Y_1|W}(y_1|r) f_W|X(r|x) f_X(x) \log \frac{f_{Y_1|Y}(y_1|y)}{f_{Y_1}(y_1)} dydy_1 dx$$  \hspace{1cm} (7)

where $\lambda$ is chosen to ensure $\int g^2(r) f_W(w) dw = P_r$. Thus, a relay mapping which is optimal in the sense of rate maximization, for a given source distribution $f_X(x)$, has to satisfy equation (7). It is important to note that the integrations in (7) cannot be computed because $g^*(r)$ is unknown $\forall r \in \mathbb{R}$. Therefore, we resort to numerical methods to tackle this issue, as will be explained in the sequel.

Relay Channel with Noiseless $S-R$ Link: In the following, we tailor the general problem to a special case with noiseless $S-R$ link. This special case provides asymptotic performance and helps to gain better insight into the problem. The expression in (3) therefore simplifies to

$$I(X; Y, Y_1) = h(Y_1) - h(Y_1|X)$$  \hspace{1cm} (8)

$$= h(Y_1) - h(Z, Z_2)$$  \hspace{1cm} (9)

Thus, maximizing mutual information $I(X; YY_1)$ is equivalent to maximizing joint entropy $h(Y_1)$ for noise-free $S-R$ link case. We again use Lagrange multiplier method to derive necessary condition for the simplified optimization problem. The necessary condition for the optimality can be shown to be

$$g^*(r) = \frac{1}{\lambda f_X(r)} \int \int (\log f_{Y_1}(y_1) + 1) \frac{\partial f_{Y_1}(y_1)}{\partial g(r)} dydy_1$$  \hspace{1cm} (10)

where $\lambda$ is chosen to meet $\int g^2(x) f_X(x) dx = P_r$.

IV. CONTINUOUS ALPHABET SOURCE

In this section, we restrict our discussion to the continuous alphabet source with Gaussian distribution. This scenario is motivated by the well-known fact that in the absence of $S$, information is transmitted at maximum rate (channel capacity) to $D$ using a Gaussian randomly generated codebook at the source$^1$. We assume a unit variance Gaussian source ($X \sim \mathcal{N}(0,1)$), and employ the fixed point iteration method [8] to solve (7) for any given values of $S-D$ SNR, $S-R$ SNR and $R-D$ SNR. We denote the SNRs of $S-D$, $S-R$ and $R-D$ links by $\gamma_1$, $\gamma_2$ and $\gamma_3$, respectively, where $\gamma_1 = P_s/N_1, \gamma_2 = P_s/N_2, \gamma_3 = P_r/N_3$. In the fixed point iteration method, we initialize the relay mapping with linear mapping (amplify-and-forward), and perform iterations until relay function converges to rate maximizing relay function with a desired degree of accuracy.

A. Shape of Optimized Mappings

Fig. 2 shows two typical examples of rate maximizing relay mappings obtained for two set of SNRs ($\gamma_1$, $\gamma_2$ and $\gamma_3$). Interestingly, optimized relay mappings are non-invertible and have oscillating (periodic-like) shape. These mappings at relay make efficient use of available transmit power and thus maximizes information transmission rate under relay power constraint. Such power efficient, non-invertible, periodic-like relay function has become possible due to availability of side information via direct $S-D$ link. Message from rate maximizing relay can be decoded at destination with the help of information coming directly from source. This relay function is anti-symmetric about origin and drops to zero for high received values where the probability of occurrence of these input values is almost zero.

B. Performance of Optimized Mappings

In order to evaluate the performance of our optimized relay functions, we compute achievable rates numerically and plot them as a function of $S-D$ SNR and $S-R$ SNR in Figs. 3 and 4 respectively. For comparison, we also plot the maximum transmission rate achieved by the conventional amplify-and-forward (AF) scheme, which is given by

$$R_{AF} = \frac{1}{4} \log_2 \left( 1 + \frac{\gamma_1 + \frac{\gamma_2 \gamma_3}{1 + \gamma_2 + \gamma_3}}{\gamma_1 + \gamma_2 + \gamma_3} \right) \text{ [bpcu]}$$  \hspace{1cm} (11)

$^1$The results in this paper are, of course, to be interpreted as “achievable rates by a random coding argument.”
An upper bound to the achievable rate (capacity) with Gaussian source distribution is given by

$$R_{u_1} = \frac{1}{4} \min \left\{ \log_2(1 + \gamma_1) + \log_2(1 + \gamma_3), \right.$$  
$$\log_2(1 + \gamma_1 + \gamma_2) \left. \right\} \text{ [bpcu]}$$  \hspace{1cm} (12)

which is also plotted in Figs. 3 and 4.

In Fig. 3, we fix $\gamma_2$ and $\gamma_3$ to 25 dB and 5 dB respectively, and plot achievable rate as a function of $\gamma_1$. It is clear that our optimized relay mappings outperform AF relaying. The performance gap between the rate maximizing relay and AF is very small when $S - D$ link is either very weak or very strong as compared to other links. With weak $S - D$ link, the information available via the direct link is not reliable enough to help estimate signal coming from relay. Thus, the relay cannot employ power efficient mappings by utilizing side information at destination. With strong $S - D$ link compared to $S - R$ link, information received via direct link is already very reliable and relay is not helpful in such situation. We can also see that the number of cycles in the periodic-like rate maximizing mapping increases as the reliability of $S - D$ link improves. This makes it possible for the relay to employ a more power efficient mapping.

In Fig. 4, we fix $\gamma_1$ and $\gamma_3$ to 5 dB each, and plot achievable rates as a function of $\gamma_2$. The performance gap between the rate maximizing relay and AF increases monotonically as the $S - R$ link gets stronger. When the $S - R$ link is weak, the rate maximizing relay is very similar to AF and gives almost the same transmission rate. As $S - R$ link becomes stronger, the effect of the relay becomes significant by employing power efficient mappings depending on quality of $S - D$ direct link.

C. Noise-free $S - R$ Link

It is now interesting to consider a special case with noise-free $S - R$ link. The corresponding maximizing relay mapping can be found numerically by employing fixed point iteration method to solve (10). Maximum transmission rate achieved with optimized relay mapping is also plotted in Fig. 3. The upper bound given in (12) for noisy $S - R$ link, now becomes

$$R_{u_2} = \frac{1}{4} \left\{ \log_2(1 + \gamma_1) + \log_2(1 + \gamma_3) \right\} \text{ [bpcu]}$$  \hspace{1cm} (13)

which is also plotted in Fig. 3. We can see that rate maximizing relay performs very close to upper bound for all SNRs.

We get some motivation for rate maximizing function by considering joint entropy $h(Y, Y_1)$, since we have shown in section III that maximizing joint entropy is equivalent to maximizing mutual information for noise-free $S - R$ case. The joint entropy $h(Y, Y_1)$ is only a function of the pdf $f_{Y, Y_1}(y, y_1)$. We know that jointly Gaussian pdf maximizes joint entropy over all distributions with same covariance matrix, and joint entropy of jointly Gaussian variables is maximum when they are uncorrelated which yields upper bound given in (13). This provides motivation to look at joint pdf $f_{Y, Y_1}(y, y_1)$ obtained with rate maximizing relay and AF relay, and then to compare their closeness to uncorrelated Gaussian pdf. We can numerically compute joint pdf $f_{Y, Y_1}(y, y_1)$ for any relay
S − D

Each Fig. contains two optimized mappings, for both weak SNR (dB). By looking at these rate maximizing relay mappings, we find that rate maximizing relay makes hard decisions and forwards detected symbols to destination without any constellation re-arrangement since side information available via S − D link is not reliable enough.

For relatively weak S − R link and strong S − D link, relay provides soft information to destination due to unreliable S − D link, and do not utilize side information due to unreliable S − D link.

B. Performance of Optimal Mappings

In order to evaluate the performance of our proposed rate maximizing relay, we plot achievable 16-P AM source rates with rate maximizing relay and AF relay as a function of S − D SNR and S − R SNR in Fig. 8 and Fig. 9 respectively.

In Fig. 8, we fix γ2 and γ3 to 25 dB and 5 dB respectively,
and plot achievable rate as a function of $\gamma_1$. We see that rate maximizing relay significantly outperforms AF relaying. The performance gap reduces for higher $S-D$ SNRs since the relay is less useful, e.g., at very high $S-D$ SNR, 2 [bpcu], which is maximum achievable rate with 16-PAM source, can be achieved by only using $S-D$ link. This figure also shows optimized mappings at three $S-D$ SNR values, and again, we see that as the $S-D$ link becomes stronger, the relay makes a more efficient use of power by relying on side information. Achievable rates with rate maximizing relay for noise-free $S-R$ link (perfect link) are also plotted in the same figure, providing asymptotic gain over AF relaying. Achievable rates with Gaussian source and rate maximizing relay for both noisy and perfect $S-R$ link are also reproduced in this figure for comparison. It is clear that a Gaussian codebook provides higher rates than when restricting to a PAM alphabet. Since the rate of the 16-PAM source resolution-limited the gap between achievable rate of 16-PAM and Gaussian source significantly increases at high $S-D$ SNRs.

Now, we fix $\gamma_1$ and $\gamma_3$ to 5 dB each, and plot achievable rates with rate maximizing and AF relaying as a function of $\gamma_2$ in Fig. 9. As $S-R$ SNR increases, the relay becomes more significant, and performance gap between rate maximizing relay and conventional AF relaying increases. This figure also shows optimized mappings at three $S-R$ SNR values, and again, we see that as the $S-R$ link becomes stronger, rate maximizing relay fills channel space more efficiently by utilizing more reliable side information. Transmission rates achieved with Gaussian source and rate maximizing and AF relay are also plotted in the same figure for comparison, where we can see that Gaussian source achieves higher rate than 16-PAM source.

VI. Conclusions

We investigated the orthogonal half-duplex three-node AWGN relay channel, with a memoryless relay. We studied the design of relay mappings that maximize the achievable reliable transmission rate subject to a relay power constraint, for a given source alphabet and the corresponding distribution. We presented numerous examples of rate maximizing mappings for Gaussian (continuous) source and uniform PAM (discrete) source. Numerical results demonstrate that the rate maximizing relay makes efficient use of power, depending on the availability of side information available via the source-destination link, and also finds an optimal trade-off between soft and hard decisions depending on the quality of all links. Memoryless relaying with optimal mappings achieves superior rates compared to the conventional amplify-and-forward relaying. Interestingly, with the noise-free source-relay link, memoryless relay with optimal mapping performs very close to the upper bound. Since the transmission rate depends on source probability distribution; therefore, an interesting extension of our work would be the maximization of rate over all source probability distributions in addition to the relay mapping.

REFERENCES