A Spectrally Efficient Transmission Scheme for Half-Duplex Decode-and-Forward Relaying

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Abstract—We propose a spectrally efficient transmission scheme for the half-duplex relay channel. In the proposed scheme, the relay combines $N$ detected $r$-dimensional symbols and generates $M$ new $r$-dimensional symbols, where $M < N$, using a linear transformation. The proposed linear transformation preserves the signal energy, and it facilitates decoupled symbol detection at the receiver.

We also present an optimized design for the case of complex scalar modulation ($r = 2$) with $N = 2$ and $M = 1$. This design increases the spectral efficiency by 33% compared to conventional decode- or amplify-and-forward relaying.

I. INTRODUCTION

We study the classical relay channel [1], consisting of a source, a destination, and a relay. The choice of a proper processing algorithm at the relay is the main challenge in the design of a transmission scheme for this relay channel. The relay functionality should efficiently use the available resources, and at the same time help the source to reliably communicate with the destination. In the literature two major protocols are extensively investigated, where the relay either amplifies and retransmits (“amplify-and-forward”) [2]–[5] or decodes and retransmits (“decode-and-forward”) [5]–[8] the received signal. We restrict our study to the decode-and-forward protocol.

Due to practical limitations in the radio hardware, transmission and reception at the relay cannot occur simultaneously. Therefore, the reception and transmission must take place during two non-overlapping timeslots [5], [9]. However, this reduces the transmission rate by half. This motivates us to design a new transmission scheme that maintains the same diversity gain but provides a higher transmission rate, under the half-duplex constraint. In this paper, we propose a structured linear processing at the relay, based on unitary transformations and projection operations. In our proposed scheme the relay uses this linear transformation to combine $N$ detected $r$-dimensional symbols and generate $M$ new $r$-dimensional symbols. We present a particular design for the case of $N = 2$, $M = 1$ and a signal space of dimension $r = 2$. With this design, the source and the relay consume three timeslots to transmit two complex symbols at the destination. By contrast, in the conventional collaborative scheme [5], four timeslots are used to transmit two symbols. Our scheme thereby increases the spectral efficiency by 33%. While providing high data rate transmission, the proposed scheme preserves full diversity gain and enjoys decoupled symbol-by-symbol detection at the destination.

II. THE PROPOSED TRANSMISSION SCHEME

In this section we present our novel spectrally efficient relaying scheme in its most general form. Assume that there are $N$ incoming streams of data into the relay. Each stream contains independent symbols of dimension $r$. (For $r = 2$ this could be complex-valued symbols, for example.) We are looking for a mapping $f(\cdot)$ that transforms the $N$ incoming symbols into $M$ outgoing symbols where $M < N$, i.e.,

$$f : (X^r)^N \mapsto (X^r)^M$$

$$[x_1^{(r)}, \ldots, x_M^{(r)}] = f([x_1, \ldots, x_N])$$

where $x_i$ and $x_i^{(r)}$ denote the $i$th incoming and outgoing $r$-dimensional symbols at the relay, respectively. All quantities are real-valued, and I/Q transmission is simply handled by doubling the dimension $r$. The mapping $f(\cdot)$ can be nonlinear in general. However, we will confine our study to linear mappings. Thereby, the problem reduces to finding a $rM \times rN$ matrix, say $W$, such that

$$[x_1^{(r)}, \ldots, x_M^{(r)}] = W [x_1, \ldots, x_N]$$

$x_i \in X^r$ and $x_i^{(r)} \in X^r$. (1)

We will impose the following structure on the linear mapping $W$ in order to facilitate optimization, analysis, and simplicity of the receiver. The proposed mapping comprises two components: an orthogonal rotation followed by a projection. More precisely, let $\{U_k\}_{k=1}^N$ be a set of $r \times r$ orthogonal matrices that satisfy

$$U_k^T U_k = I, \quad k = 1, \ldots, N$$

(In the special case of $r = 2$, $U_k$ is given either by

$$\begin{bmatrix} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos \theta_k & \sin \theta_k \\ -\sin \theta_k & \cos \theta_k \end{bmatrix}, \quad (2)$$

where $\theta_k$ is a design parameter that can be optimized.) Then

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we form $W$ according to

$$W = \begin{bmatrix} \Pi e_1 U_1 & \Pi e_2 U_1 & \cdots & \Pi e_M U_1 \\ \Pi e_1 U_2 & \Pi e_2 U_2 & \cdots & \Pi e_M U_2 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi e_1 U_M & \Pi e_2 U_M & \cdots & \Pi e_M U_M \end{bmatrix}$$

where $e_i \triangleq \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ is the $i$th column of $I_r$ (the $r \times r$ identity matrix) and $\Pi e_1 \triangleq e(e^T e)^{-1}e^T$. We can write the $k$th output symbol at the relay ($x_k^{(r)}$) explicitly:

$$x_k^{(r)} = \sum_{i=1}^{N} \Pi e_i U_k x_i, \quad k = 1, \cdots, M$$

The interpretation of (4) is that each incoming symbol of dimension $r$ is first rotated by an orthogonal transformation (i.e., $U_k$) and then projected onto $N$ orthogonal basis vectors. Finally, the relay transmits the sum of all such projected $r$-dimensional constellation points. The proposed structure is illustrated in Figure 1.

The proposed mapping at the relay has the following attractive properties, which also serve as a motivation for the scheme.

Energy conservation: Suppose that the incoming $r$-dimensional symbols at the relay have zero mean and uncorrelated elements with power $\rho^2$,

$$E[x_i] = 0 \quad \text{and} \quad E[x_i x_i^T] = \rho^2 I_r$$

where $E[\cdot]$ denotes expectation. The average output power is

$$E \| x_k^{(r)} \|^2 = E \left[ \sum_{i=1}^{N} \Pi e_i U_k x_i \|^2 \right]$$

$$= \rho^2 \sum_{i=1}^{N} \text{Tr} \left( \Pi e_i U_k U_k^T \Pi e_i \right)$$

$$= \rho^2 \sum_{i=1}^{N} \text{Tr} \left( \Pi e_i \right) = N \rho^2 = \frac{N}{r} E \| x_k \|^2$$

where we used the following equality

$$\left\| \sum_{i=1}^{N} \Pi e_i U_k x_i \right\|^2 = \sum_{i=1}^{N} \| \Pi e_i U_k x_i \|^2$$

and the facts that $U_k U_k^T = I_r$; $\Pi e_i \cdot \Pi e_i = \Pi e_i$; $\Pi e_i \cdot \Pi e_j = 0$ if $i \neq j$; and $\text{Tr}(\Pi e_i) = 1$. If $N = r$, we have $E \| x_k^{(r)} \|^2 = E \| x_k \|^2$.

More generally, if $N = r$, and if additionally the incoming symbols are uncorrelated: $E[x_i x_j^T] = 0$ for $i \neq j$, then the elements of $x_k^{(r)}$ are uncorrelated and have the same power:

$$E \left[ x_k^{(r)} x_k^{(r)}^T \right] = E \left[ \sum_{i=1}^{N} (\Pi e_i U_k x_i) \sum_{j=1}^{N} (\Pi e_j U_k x_j)^T \right]$$

$$= \sum_{i=1}^{N} \Pi e_i U_k E[x_i x_i^T] U_k^T \Pi e_i$$

$$+ \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \Pi e_i U_k E[x_i x_j^T] U_k^T \Pi e_j$$

$$= \rho^2 \sum_{i=1}^{N} \Pi e_i U_k U_k^T \Pi e_i = \rho^2 \sum_{i=1}^{N} \Pi e_i = \rho^2 I_r$$

(7)

Decoupled detection: The received signals at the destination are given by

$$y_i = \begin{cases} h_{sd} x_i + z_i & i = 1, \cdots, N \\ h_{rd} x_i^{(r)} + z_i & i = N + 1, \cdots, N + M \end{cases}$$

for the direct link and the relayed transmission, respectively. In (8), $h_{sd}$ and $h_{rd}$ are the channel gains of the source-destination and the relay-destination links and $\{z_i\}$ are additive white Gaussian vectors with zero mean and covariance matrix $\frac{N}{r} I_r$.

At the destination, the maximum-likelihood (ML) detector is optimal (for equiprobable input symbols). The ML detector maximizes (with respect to $x_1, \cdots, x_N$) the conditional pdf

$$\frac{1}{N!} \prod_{i=1}^{N} f(z_i | x_1, \cdots, x_N)$$

where

$$f(z_i | x_1, \cdots, x_N) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( - \frac{1}{2\sigma^2} (z_i - h_{sd} x_i)^2 \right)$$

for the direct link and

$$f(z_i | x_1, \cdots, x_N) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( - \frac{1}{2\sigma^2} (z_i - h_{rd} x_i^{(r)})^2 \right)$$

for the relayed link.

1All quantities in (8), including the channels $h_{sd}$, $h_{rd}$, $h_{sr}$, are real-valued. In particular, if the quantities in (8) represent complex baseband signals (for IQ) transmission, then this requires that all receivers are able to perfectly track the carrier phase. (Then $h_{sd}$, $h_{rd}$, $h_{sr}$ represent the channel magnitudes.) The noise power is $N_0/2$ per real dimension.
of the received signals:

\[
p(y_1, \cdots, y_{N+M} | x_1, \cdots, x_N) = \prod_{i=1}^{N} \frac{1}{\pi N_0^{\frac{1}{2}}} \exp\left(-\frac{\|y_i - h_{sd}x_i\|^2}{N_0}\right) \times \prod_{i=N+1}^{N+M} \frac{1}{\pi N_0^{\frac{1}{2}}} \exp\left(-\frac{\|y_i - h_{rd}x_{i-r}\|^2}{N_0}\right)
\]

By taking the logarithm and neglecting constant terms, the ML detector simplifies to

\[
\arg \min_{x_1, \cdots, x_N} \left\{ \sum_{i=1}^{N} \|y_i - h_{sd}x_i\|^2 + \sum_{k=1}^{M} \|y_{N+k} - h_{rd} \sum_{i=1}^{N} \Pi_i U_k x_i\|^2 \right\}
\]

where the first term corresponds to the direct link and the second term is the contribution from the relay. The second term can be further simplified as follows:

\[
\sum_{k=1}^{M} \|y_{N+k} - h_{rd} \sum_{i=1}^{N} \Pi_i U_k x_i\|^2 = \max_{k=1} \left\{ \|y_{N+k}\|^2 \right\}
\]

\[
+ \sum_{i=1}^{N} \|h_{rd} \sum_{i=1}^{N} \Pi_i U_k x_i\|^2 - 2h_{rd} \sum_{i=1}^{N} y_{N+k}^{T} \Pi_i U_k x_i \right\}
\]

\[
= \sum_{k=1}^{M} \|y_{N+k}\|^2
\]

Equation (12) shows that the decision metric is a sum of terms, where each term depends on exactly one symbol \(x_i\). This means that the individual (\(r\)-dimensional) symbols \(x_1, \cdots, x_N\) can be detected independently, on a symbol-by-symbol basis, using

\[
\arg \min_{x_i} \left\{ \|y_i - h_{sd}x_i\|^2 + \sum_{k=1}^{M} \|y_{N+k} - h_{rd} \sum_{i=1}^{N} \Pi_i U_k x_i\|^2 \right\}
\]

In other words, the detection of \(x_k\) is decoupled from the detection of \(x_i\) for \(k \neq i\).

**Spectral efficiency:** The spectral efficiency of the proposed scheme is \(\beta = \frac{q}{N+M} \frac{q}{2} \) bits per dimension, where \(2^q\) is the number of points in the constellation. By contrast, for conventional decode-and-forward relaying \((M = N, \ W = I_{rN})\) the spectral efficiency is \(\beta = \frac{q}{N} \) bits per dimension.

The proposed scheme thus increases the spectral efficiency by a factor \(\frac{N}{N+M}\).

The spectral efficiency increase of the scheme effectively comes from an increase in the constellation size for the signal transmitted by the relay. This results from several symbols being combined via a linear operation consisting of rotations and projections. The concept of using a rotation matrix has some connection to the ideas in [10].

The matrices \(U_k\) may be optimized. One approach to this is to assume that the relay decodes without error, and then consider the resulting error probability at the destination. If we assume perfect detection at the relay (e.g., the relay and source are very close, or there is a mechanism that can detect erroneous decisions by the relay and prevent these from being forwarded), the pairwise error probability (PEP) between two vectors \(x_i\) and \(x_i'\) for fixed channels follows as

\[
P_2(x_i \rightarrow x_i' | h_{sd}, h_{rd}) = Q\left(\sqrt{\frac{h_{rd}^2 \|x_i - x_i'\|^2 + \sum_{k=1}^{M} h_{rd}^2 \|U_k(x_i - x_i')\|^2}{2N_0}}\right)
\]

In i.i.d. Rayleigh fading with unit average gain, the PEP averaged over the channels can be bounded by the Chernoff bound

\[
P_2(x_i \rightarrow x_i') \leq \frac{1}{1 + \frac{\|x_i - x_i'\|^2}{4N_0}} \cdot \frac{1}{1 + \sum_{k=1}^{M} \frac{\|U_k(x_i - x_i')\|^2}{4N_0}}
\]

\[
\leq \left(\frac{\|x_i - x_i'\|^2}{\sum_{k=1}^{M} \|U_k(x_i - x_i')\|^2}\right) \cdot (4N_0)^2
\]

**III. OPTIMIZED DESIGN FOR N = 2, M = 1, AND r = 2**

In the following, we specialize the design to the case of \(N = 2, M = 1\) and complex signal constellations, i.e., \(r = 2\).

Having detected the received signals at the relay, the detected symbols are rotated by the matrix \(U\) and then projected onto the columns of the identity matrix. A combination of both projected signal points is then transmitted. Figure 2 shows a block diagram of the resulting linear relay processing. Effectively, the relay rotates the phase of the two received complex symbols, and then transmits the I-component of the
first rotated symbol plus the Q-component of the second rotated symbol.

This provides for decoupled detection at the destination. This follows from the general discussion in Section II. We can also see this explicitly as follows. Introduce the following notational convention

\[ v_i = \begin{bmatrix} \bar{v}_i \\ e_i \end{bmatrix} \]

for any vector \( v \). We have

\[ \Pi_{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_{e_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

With \( U \) as in (2), we have

\[ \Pi_{e_1} U x_1 = \begin{bmatrix} \bar{x}_1 \cos \theta - \bar{x}_1 \sin \theta \\ 0 \end{bmatrix} \]

\[ \Pi_{e_2} U x_2 = \begin{bmatrix} 0 \\ \bar{x}_2 \sin \theta + \bar{x}_2 \cos \theta \end{bmatrix} \]

Hence the optimal receiver (12) is

\[ \min_{\bar{x}_1, \bar{x}_2} \left| \bar{y}_1 - h_{sd} \bar{x}_1 \right|^2 \left| \bar{y}_3 - h_{rd} (\bar{x}_1 \cos \theta - \bar{x}_1 \sin \theta) \right|^2 \]

and

\[ \min_{\bar{x}_2, \bar{x}_2} \left| \bar{y}_2 - h_{sd} \bar{x}_2 \right|^2 \left| \bar{y}_3 - h_{rd} (\bar{x}_2 \sin \theta + \bar{x}_2 \cos \theta) \right|^2 \]

Clearly the detection of \( x_1, x_2 \) is decoupled, but the detection of the I/Q components is coupled due to the rotation introduced by \( U \).

The receiver can also be expressed in complex number notation. Let \( Z = \bar{z} + j \bar{z} \) for any tuple \( \bar{z}, \bar{z} \). Then the receiver is

\[ \min_{X_1} \left| Y_1 - h_{sd} X_1 \right|^2 + \left| \text{Re}(Y_3 - e^{j \theta} h_{rd} X_1) \right|^2 \]

\[ \min_{X_2} \left| Y_2 - h_{sd} X_2 \right|^2 + \left| \text{Im}(Y_3 - e^{j \theta} h_{rd} X_2) \right|^2 \]

Basically this receiver combines \( Y_1 \) with the real part of \( Y_3 \) to take decisions on \( X_1 \) and it combines \( Y_2 \) with the imaginary part of \( Y_3 \) to take decisions on \( X_2 \). This is illustrated schematically in Figure 3.

\[ \text{Fig. 3. Schematic of the receiver for the scheme in Section III. ("c" stands for combining.)} \]

A. Optimum Rotation Angle for Uncoded Transmission

We next obtain the optimum rotation angle \( \theta \) by minimizing the symbol error probability using the union bound (for uncoded transmission). In this calculation, for simplicity we assume that the source-relay link is error-free. The average symbol error probability \( (P_{s}) \), assuming equiprobable source symbols, can be quite accurately approximated at high SNR by the union bound which is given as

\[ P_s \leq \frac{1}{2^q} \sum_{m=1}^{2^q} \sum_{n \neq m} P_2(x_m^{(m)} \rightarrow x_n^{(n)}) \]

(15)

where \( 2^q \) is the number of constellation points and \( P_2(x_m^{(m)} \rightarrow x_n) \) denotes the pairwise error probability when the transmitted symbol \( x_m^{(m)} \) is detected as \( x_n \).

The optimal \( \theta \) can be found by inserting (14) into (15) and minimizing with respect to \( \theta \). For QPSK modulation \( X_i = \pm/\sqrt{2} \pm/\sqrt{2}j \) we obtain

\[ \theta \approx 29.6^\circ \]

B. Outage Probability for Coded Transmission

Next we give an expression for the outage probability (for coded transmission) of the proposed scheme. For completeness we also review the outage probabilities for non-cooperative transmission (direct link only) and conventional decode-and-forward relaying. A radio link with channel gain (magnitude) \( h \), SNR \( \rho \), and spectral efficiency \( \beta \) [per bit per channel use] is in outage when the instantaneous achievable spectral efficiency is less than \( \beta \). Throughout, we denote this outage event by \( O(h^2 \rho, \beta) \) \( \Leftrightarrow \log_2(1 + h^2 \rho) < \beta \).

*Non-collaborative transmission*: The direct link fails if \( O(h_{sd}^2 \rho, \beta) \). The outage probability in Rayleigh fading is given by

\[ P_{\text{out}} = \Pr \left\{ h_{sd}^2 < \frac{2^\beta - 1}{\rho} \right\} = \frac{2^\beta - 1}{\rho} + O \left( \frac{1}{\rho^2} \right). \]

(16)

*Conventional decode-and-forward relaying*: In this baseline scheme, if the relay successfully decodes the received message from the source, it forwards the decoded bits using repetition coding. Otherwise the relay remains silent. When the relay cooperates the destination receives two copies of the message. If the destination performs maximum-ratio combining of these two received signals, the outage event is

\[ O(h_{sd}^2 \rho, 2\beta) \bigcap \bigcup \left[ O(h_{rd}^2 \rho, 2\beta) \right] \]

and one can show that the outage probability is given by [11]

\[ P_{\text{out}} = 1.5 \left( 1 - 2^{2\beta} \right) \frac{1}{\rho^2} + O \left( \frac{1}{\rho^3} \right). \]

(17)

Equation (17) indicates that this scheme provides diversity of order two.

*Proposed collaborative scheme*: The mutual information between \( Y_1, Y_2, Y_3 \) and \( X_1, X_2 \) is

\[ I(Y_1, Y_2, Y_3; X_1, X_2) = \log_2 \left( 1 + \rho (h_{sd}^2 + h_{rd}^2) \right) \]
The probability of the scheme does not depend on $\theta$. The outage performance increase over classical relaying is about 1 and 6 dB, respectively.

The ideas we have presented motivate further research on the design of high-rate transmission protocols based on partial relay cooperation and linear relay processing. Future work may include the performance study of optimal and suboptimal receivers, and the design of good linear transformations at the relay for an arbitrary number of incoming and outgoing data streams, and for arbitrary signal space dimensions $r$. Scalability of the scheme to larger networks is also an open issue.

IV. CONCLUSIONS

We presented a novel and general spectrally efficient transmission scheme for the half-duplex relay channel, based on the decode-and-forward protocol. Our proposed scheme supports high data rate applications by performing partial cooperation, i.e., the relay transmits only a part of the regenerated data. More precisely, the relay computes a linear transformation of $N$ detected $r$-dimensional symbols, and creates $M$ $r$-dimensional symbols (where $M < N$) which are then transmitted to the destination. The scheme is able to achieve second order diversity (with coded transmission), and it enjoys decoupled symbol-by-symbol detection via linear processing at the destination. We also introduced a specific design for the special case of two-dimensional symbols ($r = 2$, $N = 2$ and $M = 1$).

REFERENCES