Minimax controls for uncertain parabolic systems

NADIR ARADA\textsuperscript{1} MAÏTINE BERGOUNIoux\textsuperscript{2} JEAN-PIERRE RAYMOND\textsuperscript{3}

Abstract. We consider systems governed by a nonlinear parabolic equation, with a distributed control and a disturbance in the initial condition. We prove the existence of solutions to a corresponding minimax problem, and we obtain necessary optimality conditions in the form of Pontryagin’s principles.

Keywords: uncertain systems, minimax, semilinear parabolic equations, existence of optimal solutions, necessary optimality conditions, Pontryagin’s principle.

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1. Introduction. In this paper we consider an uncertain system described by the parabolic equation

\begin{equation}
\frac{\partial y}{\partial t} + Ay + \Phi(x,t,y) = u \text{ in } Q, \quad y = \psi \text{ on } \Sigma, \quad y(x,0) = y_0(x) + g(x) \text{ in } \Omega,
\end{equation}

where $Q = \Omega \times [0,T]$, $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\Sigma = \Gamma \times [0,T]$, $\Gamma$ is the boundary of $\Omega$, and $A$ is a second order elliptic operator, $\psi \in L^\infty(\Sigma)$. The function $u$ is a control variable, the initial condition is not completely known. We only suppose that $g$ belongs to $G_{ad}$, where $G_{ad}$ is a closed convex subset of $L^\infty(\Omega)$ (not necessarily reduced to a unique element). For this reason, system (1.1) is called “uncertain system”. Many physical systems may be described by equations involving disturbances, noises or uncertainties. Here we suppose that the disturbance only appears through the initial condition but the results of this paper may be extended to other classical situations. We can for example consider systems with a disturbance or a control in the boundary condition (see [2]). Let us denote by $y(u,g)$ the solution of equation (1.1) corresponding to $(u,g)$, and for a given $g \in G_{ad}$, consider the problem

\begin{equation}
(P_g) \quad \inf \{ I(y(u,g),u,g) \mid u \in U_{ad} \},
\end{equation}

where $U_{ad}$ is a given control set, and $I$ a cost functional that we explicit hereafter. We denote by $\text{Argmin}(P_g)$ the set of solutions to $(P_g)$, and set $J(u,g) = I(y(u,g),u,g)$. The control problem that we consider is the following:

\begin{equation}
(P) \quad \left\{ \begin{array}{l}
\text{Find } \bar{u} \in U_{ad} \text{ such that } \bar{u} \in \text{Argmin} \ (P_g) \text{ for some } \bar{g} \in G_{ad}, \text{ and } \\
J(u_g,g) \leq J(\bar{u},\bar{g}) \text{ for all } g \in G_{ad} \text{ and all } u_g \in \text{Argmin} \ (P_g).
\end{array} \right.
\end{equation}

This problem may be expressed in an equivalent form as:

\begin{equation}
\left\{ \begin{array}{l}
\text{Find } \bar{g} \in G_{ad} \text{ and } u_{\bar{g}} \in \text{Argmin} \ (P_g) \text{ such that } \\
J(u_g,g) \leq J(u_{\bar{g}},\bar{g}) \text{ for all } g \in G_{ad} \text{ and all } u_g \in \text{Argmin} \ (P_g),
\end{array} \right.
\end{equation}

or

\begin{equation}
\max_{g \in G_{ad}} \min_{u \in U_{ad}} J(u,g).
\end{equation}

\textsuperscript{1}UMR-CNRS 5640, UFR MIG, Université Paul Sabatier, 31062 Toulouse Cedex 4, France.
E-mail: arada@mip.ups-tlse.fr.
\textsuperscript{2}UMR-CNRS 6628, Université d’Orléans, U.F.R. Sciences, B.P. 6759, F-45067 Orléans Cedex 2, France.
E-mail: Maitine.Bergounioux@labomath.univ-orleans.fr.
\textsuperscript{3}UMR-CNRS 5640, UFR MIG, Université Paul Sabatier, 31062 Toulouse Cedex 4, France.
E-mail: raymond@mip.ups-tlse.fr.
The cost functional that we consider is defined as follows:

\[ I(y, u, g) = \int_Q (F(x, t, y) + H(x, t, u)) \, dx \, dt + \int_\Omega (\ell(x, y(T)) + L(x, g)) \, dx, \]

where \( H \) is convex with respect to \( u \), and \( L \) is concave with respect to \( g \). We prove the existence of optimal solutions to \((P)\) (Theorem 4.3), and we establish optimality conditions in the form of Pontryagin’s principles (Theorems 2.1 and 2.2). The proof of the Pontryagin’s principle for an optimal solution \( \bar{g} \) is derived via Taylor’s expansions of the state variable and the cost functional for diffuse perturbations of \( \bar{g} \) (see Theorem 5.2). This is the main part of the paper. When the state equation is linear, and when \( F \) and \( \ell \) are convex, the proof is much simpler (see Theorem 2.3 and its proof in Section 6.3).

We may relate this kind of problem to the concept of robustness, since considering a min-max problem is equivalent to find the best control which takes into account the “worst” disturbance in the initial value. For linear equations, such problems have been studied by J. L. Lions [11]. The notion of “least regret controls” in [11] corresponds to our definition of robust controls.

Problem \((P)\) is also connected to \( H_\infty \)-control problems. Indeed for a quadratic functional of the form

\[ I(y, u, g) = \int_Q |y - y_d|^2 \, dx \, dt + \int_Q u^2 \, dx \, dt - \gamma \int_\Omega g^2 \, dx \quad (\gamma > 0), \]

for linear equations (\( \Phi \equiv 0 \)), and for the control sets \( U_{ad} = L^2(Q) \), \( G_{ad} = L^2(\Omega) \), robust controls that we consider are suboptimal solutions to some \( H_\infty \)-control problems (see [5], p. 218). However the terminology of “robust control” is not adapted here since we always deal with open-loop systems.

A series of papers has been widely devoted to uncertain systems [14], [1]. To analyse the different original contributions, and to compare our results with the previous ones, we must distinguish the nature of the system under consideration (i.e. the state equation), and the definition of optimal solutions. Firstly examine the second point. Ahmed and Xiang [1], and Mordukhovitch and Zhang [12] establish optimality conditions for saddle points, that is optimal strategies \((\bar{u}, \bar{g})\) satisfying

\[ \sup_{\bar{g} \in G_{ad}} \inf_{u \in U_{ad}} J(u, g) = \inf_{\bar{g} \in G_{ad}} \sup_{u \in U_{ad}} J(u, g) = J(\bar{u}, \bar{g}). \]

The inequality

\[ \sup_{\bar{g} \in G_{ad}} \inf_{u \in U_{ad}} J(u, g) \leq \inf_{\bar{g} \in G_{ad}} \sup_{u \in U_{ad}} J(u, g) \]

is always satisfied, and it is well known that for linear equations and convex cost functionals, the equality holds [6]. We do not know similar results for nonlinear equations as the ones considered here. Thus our optimal solutions are different from the ones of Ahmed and Xiang. Indeed, for a saddle point strategy, that is for \((\bar{u}, \bar{g}) \in U_{ad} \times G_{ad} \) satisfying

\[ J(\bar{u}, g) \leq J(\bar{u}, \bar{g}) \leq J(u, \bar{g}) \quad \text{for all } u \in U_{ad}, \text{ all } g \in G_{ad}, \]

optimality conditions for \( \bar{u} \) and \( \bar{g} \) can be obtained by considering separately two optimization problems [1]. Such an approach is not applicable for the problem that we consider. Mordukhovitch and Zhang [12] study minimax problems in the presence of pointwise state constraints. Optimality conditions are established for saddle point solutions satisfying the state constraints. Taking advantage of the linear structure of the state equation, the saddle point problem is splitted into two optimization problems. Still in this case, optimality conditions for \( \bar{u} \) and \( \bar{g} \) can be obtained separately.

In our knowledge, the optimality conditions established in Theorem 2.2, for a minimax problem of the form

\[ \sup_{\bar{g} \in G_{ad}} \inf_{u \in U_{ad}} J(u, g), \]
is a completely new result. Papageorgiou [14], Ahmed and Xiang [1] consider uncertain systems where the noise is modelled by parametrized measures. These authors consider a control problem where the uncertain term appears as a distributed measure. Our purpose is to consider an initial perturbation because in many situations, for example in data assimilation, the initial condition is not well known. In this case, the disturbance can be considered as an impulsive disturbance, and the analysis is more complicated (see the Taylor’s expansions stated in Theorem 5.2, and the role played by the parameter \( \tau \)).

The systems considered in [1] are more general than the one considered here, but the results in [1] cannot be applied to problems with uncertain initial conditions. Since the case of convection-diffusion equations considered in Section 5 in [1], is interesting from the point of view of applications, we explain in Section 7 how to extend our results to this kind of equations.

Finally, mention that existence of solutions for minimax control problems governed by variational inequalities are proven in [14].

The paper is organized as follows. After setting the problem we present the main assumptions and results in Section 2. Section 3 is devoted to the study of solutions for the state and adjoint equations. We prove existence results for the control problems in Section 4. In Section 5, we establish some Taylor expansions that are used to perform the proofs of the main results in Section 6. Extensions to convection-diffusion equations are stated in Section 7.

2. Assumptions and main results.

2.1. Assumptions. Throughout the paper, \( \Omega \) is a bounded open and connected subset in \( \mathbb{R}^N \) (\( N \geq 2 \)) of class \( C^{2+\gamma} \), for some \( 0 < \gamma \leq 1 \), \( q \) is a positive constant satisfying \( q > \frac{N}{2} + 1 \). The operator \( A \) is of the form \( Ay(x) = -\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_jy(x)) \) , where \( D_i \) denotes the partial derivative with respect to \( x_i \). The coefficients \( a_{ij} \) of \( A \) belong to \( C^{1+\gamma}(\overline{\Omega}) \) and satisfy the condition:

\[
a_{ij}(x) = a_{ji}(x) \quad \text{for all } i,j \in \{1,\ldots,N\}, \quad m_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j,
\]

for every \( \xi \in \mathbb{R}^N \) and every \( x \in \overline{\Omega} \), with \( m_0 > 0 \).

Let us precise the notation.

- The conormal derivative of \( y \) with respect to \( A \) is denoted by \( \frac{\partial y}{\partial n_A} \), that is

\[
\frac{\partial y}{\partial n_A}(s,t) = \sum_{i,j} a_{ij}(s)D_jy(s,t)n_i(s),
\]

where \( n = (n_1,\ldots,n_N) \) is the unit normal to \( \Gamma \) outward \( \Omega \).

- \( \overline{\Omega} = \overline{\Omega} \times \{0\}, \overline{\Omega}_T = \overline{\Omega} \times \{T\} \). For any \( \tau > 0 \), we set \( Q_{\tau} = \Omega \times ]\tau,\infty[, \Omega_{\tau} = \{ x \in \Omega \mid d(x,\Gamma) > \tau \}, Q^r = \Omega_\tau \times ]\tau,\infty[, T^r = \{ (s,t) \mid \tau < s < T \} \) (\( d \) is the Euclidean distance).

- For every \( 1 \leq d \leq \infty \), the usual norms in the spaces \( L^d(\Omega) \), \( L^d(Q) \), \( L^d(\Sigma) \) will be denoted by \( \| \cdot \|_{d,\Omega}, \| \cdot \|_{d,Q}, \| \cdot \|_{d,\Sigma} \).

- If \( \mathcal{O} \) is a locally compact subset of \( \mathbb{R}^{N+1} \), \( \mathcal{C}_b(\mathcal{O}) \) denotes the space of bounded continuous functions on \( \mathcal{O} \), and \( \mathcal{C}_c(\mathcal{O}) \) the space of all continuous functions from \( \mathcal{O} \) into \( \mathbb{R} \) vanishing at infinity. The dual space of \( \mathcal{C}_c(\mathcal{O}) \) is denoted by \( \mathcal{M}_b(\mathcal{O}) \) (it is the space of bounded Radon measures on \( \mathcal{O} \)).

Now, let us set the assumptions.
A1 - The control set is defined as

$$U_{ad} = \{ u \in L^2(Q) \mid u(x,t) \in K_U(x,t) \text{ for almost all } (x,t) \in Q \},$$

where $K_U(\cdot)$ is a measurable multivalued mapping with nonempty, convex and closed values in $\mathcal{P}(IR)$. The set of constraints on $g$ is defined by

$$G_{ad} = \{ g \in L^\infty(\Omega) \mid g(x) \in K_G(x) \subset G \text{ for almost all } x \in \Omega \},$$

where $K_G(\cdot)$ is a measurable multivalued mapping with nonempty, convex and closed values in $\mathcal{P}(IR)$, and $G$ is a compact subset of $IR$. We suppose that $U_{ad}$ and $G_{ad}$ are nonempty.

**Remark 2.1.** Observe that $U_{ad}$ is a closed, convex subset of $L^2(Q)$. Similarly, $G_{ad}$ is convex and bounded in $L^\infty(\Omega)$, and closed in $L^s(\Omega)$, for all $1 \leq s \leq +\infty$.

A2 - $\Phi$ is a Carathéodory function from $Q \times IR$ into $IR$. For almost every $(x,t) \in Q$, $\Phi(x,t,\cdot)$ is of class $C^1$. Moreover, the following estimates hold

$$|\Phi(x,t,0)| \leq \Phi_1(x,t), \quad 0 \leq \Phi'_y(x,t,y) \leq \Phi_1(x,t) \eta(|y|),$$

where $\Phi_1 \in L^2(Q)$, and $\eta$ is a nondecreasing function from $IR^+$ to $IR^+$.

A3 - $F$ and $H$ are Carathéodory functions from $Q \times IR$ to $IR$. For almost all $(x,t) \in Q$, $F(x,t,\cdot)$ is of class $C^1$ and $H(x,t,\cdot)$ is convex. Moreover, the following estimates hold

$$-C_1|y|^a \leq F(x,t,y) \leq F_1(x,t) \eta(|y|), \quad |F'_y(x,t,y)| \leq F_2(x,t) \eta(|y|),$$

$$C_1|u|^q \leq H(x,t,u) \leq H_1(x,t) + C_2 |u|^q,$$

where $C_1 > 0, C_2 > 0, 1 \leq \sigma < q$, $F_1 \in L^1(Q)$, $H_1 \in L^1(Q)$, $F_2 \in L^m(Q)$ with $m > 1$, and $\eta$ is defined as in A2.

A4 - $\ell$ and $L$ are Carathéodory functions from $\Omega \times IR$ into $IR$. For almost all $x \in \Omega$, $\ell(x,\cdot)$ is $C^1$, $L(x,\cdot)$ is concave and the following estimates hold

$$-C_1|y|^\sigma \leq \ell(x,y) \leq \ell_1(x) \eta(|y|), \quad |\ell'_y(x,y)| \leq \ell_2(x) \eta(|y|), \quad |L(x,g)| \leq L_1(x) \eta(|g|),$$

where $\ell_1 \in L^1(\Omega)$, $L_1 \in L^1(\Omega)$, $\ell_2 \in L^m(\Omega)$; $m > 1$ and $\sigma$ are the same exponents as in A3, and $\eta$ is as in A2.

### 2.2. Statement of the main results.

Let us define the Hamiltonian functions:

$$\mathcal{H}_Q(x,t,u,p) = H(x,t,u) - pu \text{ for all } (x,t,u,p) \in Q \times IR^2,$$

$$\mathcal{H}_\Omega(x,y,w,p) = L(x,w) - pw \text{ for all } (x,w,p) \in \Omega \times IR^2.$$

The following result provides necessary optimality conditions (as a Pontryagin’s principle) for solutions to $(P_g)$, where $g \in G_{ad}$ is fixed.

**Theorem 2.1.** Suppose that A1-A4 are fulfilled. For any $g \in G_{ad}$, let $u_g$ be a solution of $(P_g)$. Then there exists $p_g \in L^1(0,T;W^{1,1}_o(\Omega))$, such that

$$\frac{\partial p_g}{\partial t} + Ap_g + \Phi'_y(x,t,y(u_g,g))p_g + F'_y(x,t,y(u_g,g)) = 0 \quad \text{ in } Q,$$

$$p_g(x,T) + \ell'_y(x,y(u_g,g)(T)) = 0 \quad \text{ in } \Omega,$$

$$\mathcal{H}_Q(x,t,u,p) = \mathcal{H}_\Omega(x,y,w,p) = 0,$$

$$p_g(x,t) = 0 \quad \text{ on } \partial Q.$$
and
\[ \mathcal{H}_Q(x, t, u_g(x, t), p_g(x, t)) = \min_{u \in K_U(x, t)} \mathcal{H}_Q(x, t, u, p(x, t)) \quad \text{for a.e. } (x, t) \in Q. \]

Next, we are concerned with necessary optimality conditions for solutions to problem \((P)\).

**Theorem 2.2.** Assume A1-A4. Then \((P)\) admits at least a solution \(\bar{g}\). In addition, there exists a solution \(\bar{u}\) to \((P_\bar{g})\), and \(\bar{p} \in L^1(0, T; W^{1,1}_0(\Omega))\), such that
\[
\begin{aligned}
-\frac{\partial \bar{p}}{\partial t} + A\bar{p} + \Phi'((x, t, \bar{u}, \bar{g})) \bar{p} + F'(x, t, y(\bar{u}, \bar{g})) &= 0 \quad \text{in } Q, \\
\bar{p}(x, T) + \ell'(x, y(\bar{u}, \bar{g}))(T) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

and
\[ \mathcal{H}_Q(x, t, \bar{u}(x, t), \bar{p}(x, t)) = \min_{u \in K_U(x, t)} \mathcal{H}_Q(x, t, u, \bar{p}(x, t)) \quad \text{for a.e. } (x, t) \in Q, \]
\[ \mathcal{H}_\Omega(x, \bar{g}(x), \bar{p}(x, 0)) = \max_{g \in K_G(x)} \mathcal{H}_\Omega(x, g, \bar{p}(x, 0)) \quad \text{for a.e. } x \in \Omega. \]

In the case of linear equations, and when the cost functional is convex with respect to the state variable, a more accurate statement is given below.

**Theorem 2.3.** Suppose that A1-A4 are fulfilled. Suppose in addition that \(\Phi\) is of the form \(\Phi(\cdot, y) = a(\cdot)y + b(\cdot)\) (with \(a \in L^\infty(Q)\), \(a \geq 0\), \(b \in L^3(Q)\)), and that \(F(x, t, \cdot)\) and \(\ell(x, \cdot)\) are convex.

Let \(\bar{g}\) be a solution of \((P)\) and let \(u_g\) be in \(\text{Arg} \inf(P_{\bar{g}})\). Then, there exists \(\bar{p} \in L^1(0, T; W^{1,1}_0(\Omega))\) satisfying the equation
\[
\begin{aligned}
-\frac{\partial \bar{p}}{\partial t} + A\bar{p} + \Phi'(x, t, y(u_g, \bar{g})) \bar{p} + F'(x, t, y(u_g, \bar{g})) &= 0 \quad \text{in } Q, \\
\bar{p}(x, T) + \ell'(x, y(u_g, \bar{g}))(T) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

and such that
\[ \mathcal{H}_Q(x, t, u_g(x, t), \bar{p}(x, t)) = \min_{u \in K_U(x, t)} \mathcal{H}_Q(x, t, u, \bar{p}(x, t)) \quad \text{for a.e. } (x, t) \in Q, \]
\[ \mathcal{H}_\Omega(x, \bar{g}(x), \bar{p}(x, 0)) = \max_{g \in K_G(x)} \mathcal{H}_\Omega(x, g, \bar{p}(x, 0)) \quad \text{for a.e. } x \in \Omega. \]


3.1. Existence and regularity for the solution of state equation. Let \(a\) be a nonnegative function in \(L^3(Q)\), let \(\phi\) be in \(L^3(Q)\), \(f\) in \(L^\infty(\Sigma)\), and \(w\) in \(L^\infty(\Omega)\). Consider the following equation:

\[
\frac{\partial y}{\partial t} + Ay + ay = \phi \quad \text{in } Q, \quad y = f \quad \text{on } \Sigma, \quad y(0) = w \quad \text{in } \Omega.
\]

**Definition 3.1.** A function \(y \in L^1(Q)\) is a weak solution of equation (3.1) if, and only if, \(a y \in L^1(Q)\) and
\[
\int_Q y (-\frac{\partial z}{\partial t} + Az + az) \, dx \, dt = \int_Q \phi z \, dx \, dt + \int_{\Omega} w(z(0)) \, dx - \int_{\Sigma} f \frac{\partial z}{\partial n_A} \, ds \, dt
\]
for all $z \in C^2(\overline{Q})$ such that $z(T) = 0$ and $z_{|\Sigma} = 0$.

**Proposition 3.1** ([2], Proposition 3.6). Equation (3.1) admits a unique weak solution $y \in L^1(Q)$. This solution belongs to $C_0(Q \cup \Omega_T)$ and satisfies 
\[ \|y\|_{\infty,Q} \leq C(\|\phi\|_{\infty,Q} + \|f\|_{\infty,\Sigma} + \|w\|_{\infty,\Omega}), \]

where $C \equiv C(T, \Omega, N, q)$ does not depend on $a$.

**Proposition 3.2** ([2], Proposition 3.7). Let $a$ be a nonnegative function in $L^q(Q)$ such that $\|a\|_{q,Q} \leq M$. For every $\tau > 0$, the weak solution $y$ of (3.1) is H"older continuous on $\overline{Q^T}$ and satisfies 
\[ \|y\|_{C^{\nu/2}(\overline{Q^T})} \leq C(\tau)(\|\phi\|_{q,Q} + \|f\|_{\infty,\Sigma} + \|w\|_{\infty,\Omega}) \quad \text{for some } 0 < \nu < 1, \]

where $C(\tau) \equiv C(T, \Omega, N, M, q, \tau)$.

Now, we recall some results for the (nonlinear) state equation.

**Definition 3.2.** A function $y \in L^1(Q)$ is a weak solution of equation (1.1) if, and only if, 
\[ \Phi(\cdot, y(\cdot)) \in L^1(Q) \]

and satisfies:
\[ \int_Q y \left(- \frac{\partial z}{\partial t} + Az\right) dx \, dt = \int_Q \left(\Phi(x, t, y) - u\right) z \, dx \, dt = -\int_\Sigma \psi \frac{\partial z}{\partial n_A} \, ds \, dt + \int_\Omega (y_o + g) z(0) \, dx \]

for all $z \in C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z_{|\Sigma} = 0$.

**Theorem 3.1** ([2], Theorem 3.9). Let $u$ be in $L^q(Q)$, $g$ in $L^\infty(\Omega)$, and $y_o$ in $L^\infty(\Omega)$. Equation (1.1) admits a unique weak solution $y(u, g)$. This solution belongs to $C_0(Q \cup \Omega_T)$ and satisfies 
\[ \|y(u, g)\|_{\infty,Q} \leq C(\|u\|_{q,Q} + \|\psi\|_{\infty,\Sigma} + \|g\|_{\infty,\Omega} + \|y_o\|_{\infty,\Omega} + 1), \]

where $C = C(T, \Omega, N, q)$.

**Theorem 3.2** ([2], Theorem 3.10). For every $M > 0$ and every $\tau > 0$, there exists a positive constant $C(\tau) \equiv C(T, \Omega, N, q, \tau, M)$ and $\nu > 0$ such that, for every $(u, g) \in L^q(Q) \times L^\infty(\Omega)$ satisfying $\|u\|_{q,Q} + \|g\|_{\infty,\Omega} \leq M$, the weak solution $y(u, g)$ of (1.1) corresponding to $(u, g)$ is H"older continuous on $\overline{Q^T}$ and satisfies:
\[ \|y(u, g)\|_{C^{\nu/2}(\overline{Q^T})} \leq C(\tau). \]

### 3.2. Adjoint equation.

Consider the following terminal boundary value problem:

\[ -\frac{\partial p}{\partial t} + Ap + a p = \phi \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(T) = w \quad \text{in } \Omega_T, \]

where $a$ is a nonnegative function in $L^q(Q)$, $\phi \in L^1(Q)$, and $w \in L^1(\Omega)$.

**Definition 3.3.** A function $p \in L^1(0, T; W^{1,1}_{loc}(\Omega))$ is a weak solution of (3.3) if, and only if, 
\[ a p \in L^1(Q) \]

and
\[ \int_Q \left(p \frac{\partial z}{\partial t} + \sum_{i,j=1}^N a_{ij} D_j p D_i z + a z p\right) \, dx \, dt = \int_Q \phi z \, dx \, dt + \int_\Omega wz(T) \, dx \]

for all $z \in C^1(\overline{Q}) \cap C_0(Q \cup \Omega_T)$. 

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THEOREM 3.3 ([2], Theorem 4.2). Let $\phi \in L^1(Q)$, $w \in L^1(\Omega)$, and let $a$ be a nonnegative function in $L^q(\Omega)$ satisfying $\|a\|_{q,\Omega} \leq M$. Then, equation (3.3) admits a unique weak solution $p$ in $L^1(0,T;W_{\sigma}^{1,1}(\Omega))$. This solution belongs to $L^q(0,T;W_{\sigma}^{1,1}(\Omega))$, for all $(\delta,d)$ satisfying $\delta > 1$, $d > 1$, $N\frac{1}{2d} + \frac{1}{3} > \frac{N + 1}{2}$. Moreover, there exists a function in $L^1(\Omega)$, denoted by $\varphi_{p,\sigma}$, and a function in $L^1(\Omega)$, denoted by $p(\varphi_{p,\sigma})$, such that:

$$
\int_Q \left( \frac{\partial z}{\partial t} + Az + az \right) dx dt - \int_{\Omega} z \frac{\partial p}{\partial n_A} ds dt + \int_Q z(0) p(0) dx = \int_Q \phi z dx dt + \int_{\Omega} wz(T) dx
$$

for all $z \in \mathcal{C}(\Omega)$.

4. Existence results for min and max problems. Before dealing with existence results for the different problems, we have to establish some lower semicontinuity properties for the cost functional. As the state variable is implicitly involved in the cost functional the following result is crucial for the sequel.

THEOREM 4.1. For every $\tau \in [0,T]$, the mapping $(u,g) \mapsto y(u,g)$ is sequentially continuous from $U_{ad} \times G_{ad}$, endowed with its weak-$L^q(\Omega) \times$ weak-star-$L^\infty(\Omega)$ topology, into $C(\overline{Q}^r)$.

Proof - Let $(u_n,g_n)_n$ be a sequence converging to $(\bar{u},\bar{g})$ for the weak-$L^q(\Omega) \times$ weak-star-$L^\infty(\Omega)$ topology. Let $y_n$ be the solution of (1.1), corresponding to $(u_n,g_n)$. Due to (3.2), the sequence $(y_n)_n$ is bounded in $L^\infty(\Omega)$. Then there exists a subsequence, still indexed by $n$, and $\tilde{y} \in L^\infty(\Omega)$ such that $(y_n)_n$ converges to $\tilde{y}$ for the weak-star topology of $L^\infty(\Omega)$. Moreover, from Theorem 3.2, the sequence $(y_n)_n$ is bounded in $C^{2,\nu/2}(\overline{Q}^r)$, for some $\nu > 0$, and for all $\tau \in [0,T]$. Since the embedding from $C^{2,\nu/2}(\overline{Q}^r)$ into $C(\overline{Q}^r)$ is compact, $(y_n)_n$ converges to $\tilde{y}$ uniformly on $\overline{Q}^r$, for all $\tau > 0$. On the other hand, observe that $y_n$ satisfies

$$
\int_Q y_n \left( \frac{\partial z}{\partial t} + Az \right) dx dt + \int_Q \left( \Phi(x,t,y_n) - u_n \right) z dx dt = - \int_{\Omega} \psi \frac{\partial z}{\partial n_A} ds dt + \int_Q \left( y_n + g_n \right) z(0) dx
$$

for all $z \in C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z|_{\Gamma} = 0$. With assumptions $A2$ on $\Phi$ and Lebesgue's theorem, we can pass to the limit when $n$ tends to infinity, and we obtain

$$
\int_Q \tilde{y} \left( \frac{\partial z}{\partial t} + Az \right) dx dt + \int_Q \left( \Phi(x,t,\tilde{y}) - \bar{u} \right) z dx dt = - \int_{\Omega} \psi \frac{\partial z}{\partial n_A} ds dt + \int_Q \left( y_{n,\tau} + \tilde{g} \right) z(0) dx
$$

for all $z \in C^2(\overline{Q})$ satisfying $z(T) = 0$ and $z|_{\Gamma} = 0$. Therefore $\tilde{y}$ is the solution of (1.1) corresponding to $(\bar{u},\bar{g})$.

COROLLARY 4.1. For all $g \in G_{ad}$, the mapping $u \mapsto J(u,g)$ is sequentially lower semicontinuous on $U_{ad}$, endowed with the weak-topology of $L^q(\Omega)$.

Proof - Let $g$ be in $G_{ad}$, and let $(u_n)_n$ be a sequence converging to some $\bar{u}$ for the weak topology of $L^q(\Omega)$. Let $y_n = y(u_n,g)$ and $\tilde{y} = y(\bar{u},g)$ be the associated solutions of (1.1). We observe that

$$
J(\bar{u},g) - J(u_n,g) = \int_Q F_n(x,t)(\bar{y} - y_n) dx dt + \int_{\Omega} \ell_n(\bar{y} - y_n)(T) dx + \int_Q H(x,t,\bar{u}) - H(x,t,u_n) \right) dx dt
$$

where

$$
F_n(x,t) = \int_0^1 F'(x,t,(1 - \theta) y_n + \theta \tilde{y}) d\theta \quad \text{and} \ell_n(x) = \int_0^1 \ell'(x,(1 - \theta) y_n(T) + \theta \bar{y}(T)) d\theta.
$$
Due to Theorem 4.1, the sequence \((y_n)_n\) converges to \(\tilde{y}\) uniformly on \(\overline{Q}\), for all \(\tau > 0\). With assumptions on \(F\) and \(\ell\), and Lebesgue’s theorem, we obtain

\[
(4.1) \lim_{n \to +\infty} \int_Q F_n(x, t) (\tilde{y} - y_n) \, dx \, dt = \lim_{n \to +\infty} \int_\Omega \ell_n(x) (\tilde{y} - y_n)(T) \, dx = 0.
\]

On the other hand, from [9] Theorem 2.1, Chapter 8, we deduce that

\[
(4.2) \int_Q H(x, t, \tilde{u}) \, dx \, dt \leq \lim \inf_{n \to +\infty} \int_Q H(x, t, u_n) \, dx \, dt.
\]

The sequential lower semicontinuity of \(J(\cdot, g)\) follows from (4.1) and (4.2).

Now, we are able to give an existence result of solutions to problem \(\mathcal{P}_g\).

**Theorem 4.2.** Let \(g\) be in \(G_{ad}\). If \(A1\text{-}A4\) are fulfilled, then problem \(\mathcal{P}_g\) admits at least one solution.

**Proof** - Let \(g\) be in \(G_{ad}\) and let \(u\) be in \(U_{ad}\). From \(A3\text{-}A4\) and Theorem 3.1, it follows that

\[
(4.3) J(u, g) \geq C_1 ||u||_{q,Q}^q - C(||u||_{q,q}^q + ||\Psi||_{1,\Omega}^q + ||y_0||_{\infty,\Omega} + ||g||_{\infty,\Omega} + 1) - ||L_1||_{1,\Omega} ||\eta(g + y_0)||_{\infty,\Omega}.
\]

With Young’s inequality we can prove that the infimum of \(\mathcal{P}_g\) belongs to \(IR\). Let \((u_n)_n\) be a minimizing sequence for \(\mathcal{P}_g\). Due to (4.3), the sequence \((u_n)_n\) is bounded in \(L^q(Q)\). Then, there exist a subsequence, still indexed by \(n\), and \(u \in U_{ad}\), such that \((u_n)_n\) converges to \(u\) for the weak topology of \(L^q(Q)\). Due to Corollary 4.1, we have

\[
J(u, g) \leq \lim \inf_{n} J(u_n, g) \leq \inf \mathcal{P}_g,
\]

and the proof is complete.

To study the existence of solutions to problem \((\mathcal{P})\), we have to set some continuity results.

**Proposition 4.1.** Let \((g_n)_n \subset G_{ad}\) be a sequence converging to some \(g \in G_{ad}\) for the weak-star topology of \(L^\infty(\Omega)\). Let \(u_{g_n}\) be an element in \(\text{Argmin} \ (\mathcal{P}_{g_n})\). Then, there exists \(u_g \in \text{Argmin} \ (\mathcal{P}_g)\) such that the following conditions hold:

- 
  \((u_{g_n})_n\) converges (up to a subsequence) to \(u_g\) for the weak topology of \(L^q(Q)\),

- 
  \((y(u_{g_n}, g_n))_n\) converges (up to a subsequence) to \(y(u, g)\) in \(C(\overline{Q})\), for all \(\tau > 0\).

**Proof** - Due to (4.3), since \((g_n)_n\) is bounded in \(L^\infty(\Omega)\), and since \(J(u_{g_n}, g_n) \leq J(u_{\sigma}, g_n) \leq M\) (for some \(u_\sigma\) fixed in \(U_{ad}\), the sequence \((u_{g_n})_n\) is bounded in \(U_{ad}\). Then there exists a subsequence, still indexed by \(n\), and \(\tilde{u}\), such that \((u_{g_n})_n\) converges to \(\tilde{u}\) for the weak topology of \(L^q(Q)\). Since \(U_{ad}\) is closed and convex in \(L^q(Q)\), it is also weakly closed and \(\tilde{u} \in U_{ad}\). Due to Theorem 4.1, the sequence \((y(u_{g_n}, g_n))_n\) converges to \(y(\tilde{u}, g)\) in \(C(\overline{Q})\), for all \(\tau > 0\). Let us prove that \(\tilde{u}\) belongs to \(\text{Argmin} \ (\mathcal{P}_g)\). By definition, \(u_{g_n}\) satisfies

\[
(4.4) J(u_{g_n}, g_n) \leq J(u, g_n) \quad \text{for all} \ u \in U_{ad}.
\]

With arguments similar to those used in Corollary 4.1, we can prove that

\[
(4.5) \lim_{n \to +\infty} \int_Q F(x, t, y(u_{g_n}, g_n)) \, dx \, dt = \int_Q F(x, t, y(\tilde{u}, g)) \, dx \, dt,
\]

\[
(4.6) \lim_{n \to +\infty} \int_Q F(x, t, y(u, g_n)) \, dx \, dt = \int_Q F(x, t, y(u, g)) \, dx \, dt,
\]

\[
\int_Q \ell_n(x) (\tilde{y} - y_n)(T) \, dx = 0.
\]
On the other hand, with Proposition 4.1, we know that 
\begin{equation}
(4.10)
\end{equation}

First we observe that problem (4.10) is equivalent to
\[
\max \{ J(u_g, g) \mid u_g \in \text{Argmin} (P_g); \ g \in G_{ad} \}.
\]

Let \((g_n)_n\) be a maximizing sequence for \((P)\). Since \(G_{ad}\) is bounded in \(L^\infty(\Omega)\), there exist a subsequence, still indexed by \(n\), and \(\tilde{g}\), such that \((g_n)_n\) converges to \(\tilde{g}\) for the weak-star topology of \(L^\infty(\Omega)\). In addition, \(\tilde{g}\) is also the weak limit of \(g_n\) in \(L^s(\Omega)\) (for all \(s \geq 1\)). Since \(G_{ad}\) is convex and closed in \(L^s(\Omega)\), it is also weakly closed and \(\tilde{g} \in G_{ad}\). From assumption \(A4\) on \(L\), we deduce that
\begin{equation}
(4.11)
\end{equation}

As in the proof of Proposition 4.1 we have
\begin{equation}
(4.12)
\end{equation}

Consequently, with (4.11), (4.10), (4.12), and (4.13), it follows that
\[
\sup_{g \in G_{ad}} \{ J(u_g, g) \mid u_g \in \text{Argmin}(P_g) \} = \lim_{n \to +\infty} J(u_{g_n}, g_n) = \limsup_{n \to +\infty} J(u_{g_n}, g_n) \leq J(u_{\tilde{g}}, \tilde{g}).
\]

Since \(\tilde{g}\) belongs to \(G_{ad}\) and \(u_{\tilde{g}}\) belongs to \(\text{Arg inf}(P_{\tilde{g}})\), \(\tilde{g}\) is a solution to \((P)\).
5. Taylor expansions. In order to give optimality conditions we need some Taylor expansions. In the sequel, \( \mathcal{L}^{N+1} \) denotes the \((N+1)\)-dimensional Lebesgue measure.

**Theorem 5.1.** (Taylor expansion of \( y \) with respect to \( u \)). Let \( \rho \) be in \([0,1] \). For every \( u_1, u_2 \in U_{ad} \), and every \( g \in G_{ad} \), there exist measurable subsets \( Q_\rho \subset Q \) such that:

\[
\mathcal{L}^{N+1}(Q_\rho) = \rho \mathcal{L}^{N+1}(Q),
\]

\[
\int_{Q_\rho} (H(x,t,u_2) - H(x,t,u_1)) \, dx \, dt = \rho \int_Q (H(x,t,u_2) - H(x,t,u_1)) \, dx \, dt,
\]

\[
y_\rho = y_1 + \rho z + r_\rho, \quad \text{with} \quad \lim_{\rho \to 0} \frac{1}{\rho} ||r_\rho||_{L^1(Q_\rho)} = 0,
\]

\[
J(u_\rho, g) = J(u_1, g) + \rho \left( J'(u_1, g_1) z + \int_Q (H(x,t,u_2) - H(x,t,u_1)) \, dx \, dt \right) + o(\rho),
\]

where

\[
u_\rho(x,t) = \begin{cases} u_1(x,t) & \text{in } Q \setminus Q_\rho, \\ u_2(x,t) & \text{in } Q_\rho; \end{cases}
\]

\( y_\rho \) and \( y_1 \) are the solutions of (1.1) corresponding to \( (u_\rho, g) \) and \( (u_1, g) \), \( z \) is the solution of:

\[
\frac{\partial z}{\partial t} + Az + \Phi_y'(\cdot, y_1) z = u_2 - u_1 \text{ in } Q, \quad z = 0 \text{ on } \Sigma, \quad z(0) = 0 \text{ in } \Omega.
\]

**Proof - See [16], Theorem 4.1.** □

**Theorem 5.2.** (Taylor expansion of \( y \) with respect to \( g \)). Let \( \rho \) be in \([0,1] \), and set \( \tau_\rho = \rho^{m' + \frac{q}{1-q}} \), where \( q > \bar{q} > \frac{N}{2} + 1 \) (\( m > 1 \) is the exponent in A4, \( m' \) is the conjugate exponent to \( m \)). For every \( g_1, g_2 \in G_{ad} \), there exist measurable subsets \( \Omega_\rho \subset \Omega \), and there exists \( u_{g_2} \in \text{Arginf} (P_{g_2}) \), such that the following holds

\[
\mathcal{L}^{N}(\Omega_\rho) = \rho \mathcal{L}^{N}(\Omega),
\]

\[
\int_{\Omega_\rho} (L(x,g_2) - L(x,g_1)) \, dx = \rho \int_{\Omega} (L(x,g_2) - L(x,g_1)) \, dx,
\]

\[
y(u_{g_\rho}, g_\rho) = y(u_{g_\rho}, g_1) + \rho z_1 + r_\rho, \quad \text{with} \quad \lim_{\rho \to 0} \frac{1}{\rho} ||r_\rho||_{L^1(\Omega_\rho, \tau)} = 0,
\]

\[
J(u_{g_\rho}, g_\rho) = J(u_{g_\rho}, g_1) + \rho \left( J'_y(u_{g_\rho}, g_1) z_1 + \int_{\Omega} (L(x,g_2) - L(x,g_1)) \, dx \right) + o(\rho),
\]

where

\[
g_\rho(x) = \begin{cases} g_1(x) & \text{on } \Omega \setminus \Omega_\rho, \\ g_2(x) & \text{on } \Omega_\rho \end{cases}, \quad u_{g_\rho} \in \text{Arginf} (P_{g_\rho}).
\]
y(u_{g_x}, g_{1}) and y(u_{g_x}, g_1) are the solutions of (1.1) corresponding to (u_{ρ}, g_{ρ}) and (u_{ρ}, g_{1}), z_1 is the solution of

\[ \frac{∂z}{∂t} + Az + Φ'_y(x, t, y(u_{g_1}, g_1))z = 0 \text{ in } Q, \quad z = 0 \text{ on } Σ, \quad z(0) = g_2 - g_1 \text{ in } Ω. \]

The proof is based on the following lemmas.

**Lemma 5.1** ([16], Lemma 4.1). Let \( g_1, g_2 \) be in \( L^∞(Ω) \). For every \( ρ ∈ ]0, 1[ \), there exist a sequence of measurable subsets \((Ω^n_ρ)\) in \( Ω \), such that:

\[ L^N(Ω^n_ρ) = ρL^N(Ω), \]

\[ \int_{Ω^n_ρ} (L(x, g_2) - L(x, g_1)) dx = ρ \int_{Ω} (L(x, g_2) - L(x, g_1)) dx, \]

\[ \left( \frac{χ_{Ω^n_ρ}}{ρ} \right)_n \text{ converges to } 1 \text{ for the weak-star topology of } L^∞(Ω), \]

where \( χ_{Ω^n_ρ} \) denotes the characteristic function of \( Ω^n_ρ \).

**Lemma 5.2.** Let \( ρ ∈ ]0, 1[ \), and \((Ω^n_ρ)\) the sequence of measurable subsets defined in Lemma 5.1. Set

\[ g^n_ρ(x) = \begin{cases} g_1(x) & \text{ in } Ω \setminus Ω^n_ρ, \\ g_2(x) & \text{ in } Ω^n_ρ. \end{cases} \]

Then the sequence \((g^n_ρ)_n\) converges to \( ρg_2 + (1 - ρ)g_1 \) for the weak-star topology of \( L^∞(Ω) \).

**Proof.** Let be \( ϕ ∈ L^1(Ω) \). We have

\[ \left| \int_{Ω} g^n_ρ ϕ dx - \int_{Ω} (ρg_2 + (1 - ρ)g_1) ϕ dx \right| = \left| \int_{Ω} (g^n_ρ - ρg_2 - (1 - ρ)g_1) ϕ dx \right| = \left| \int_{Ω} \frac{χ_{Ω^n_ρ}}{ρ} - 1)g_2 - g_1 \right| \varphi dx \to 0 \text{ as } n → ∞. \]

**Proof of Theorem 5.2.** Let \( u_{g_x}^n \) be in Argmin \((P^n_{g_x})\). With Lemma 5.2 and Proposition 4.1, we can prove that \((u_{g_x}^n)_n\) (or at least a subsequence) weakly converges to some \( u_ρ ∈ L^q(Ω), u_ρ ∈ \text{Argmin} \{(P_{(ρg_2 + (1 - ρ)g_1)})\), and

\[ \lim_{n → ∞} \|y(u_{g_x}^n, g_ρ^n) - y(u_ρ, ρg_2 + (1 - ρ)g_1)\|_{C(Ω')} = 0 \text{ for all } τ > 0. \]

With similar arguments we prove that \((u_ρ)_ρ\) weakly converges to some \( u_{g_1} ∈ \text{Argmin} \,(P_{g_1})\), and

\[ \lim_{ρ → 0} \|y(u_{g_1}, g_1) - y(u_ρ, ρg_2 + (1 - ρ)g_1)\|_{C(Ω')} = 0 \text{ for all } τ > 0. \]

**Step 1.** We first establish (5.2). The function \( ζ_ρ^n = \frac{1}{ρ} \left( y(u_{g_x}^n, g_ρ^n) - y(u_{g_x}^n, g_1) \right) - z_1 \) belongs to \( C_b(Ω \cup Ω_T) \), and it is the solution of

\[ \frac{∂ζ}{∂t} + Aζ + β^n_ρ ζ = h^n_ρ \text{ in } Q, \quad ζ = 0 \text{ on } Σ, \quad ζ(0) = f^n_ρ \text{ in } Ω. \]
where
\[ \beta_{\rho}^{n} = \int_{0}^{1} \Phi'_{y} \left( \cdot, \theta \cdot y(u_{g_{\rho}^{n}}, g_{\rho}^{n}) + (1 - \theta) y(u_{g_{\rho}^{n}}, g_{1}) \right) d\theta, \]

\[ h_{\rho}^{n} = \left( \Phi'_{y}(\cdot, y(u_{g_{1}}), g_{1}) \right) \beta_{\rho}^{n} z_{1}, \]

\[ f_{\rho}^{n} = (1 - \frac{1}{\rho} \chi_{\Omega_{\rho}})(g_{1} - g_{2}). \]

We look for \( n \in \mathbb{N}^{+} \) as a function of \( \rho \), say \( n(\rho) \), such that:

\( \lim_{\rho \searrow 0} \| v_{n(\rho)} \|_{C(\overline{Q}_{r_{\rho}})} = 0. \)  

Set \( \zeta_{\rho}^{n} = \zeta_{\rho}^{n,1} + \zeta_{\rho}^{n,2} \), where \( \zeta_{\rho}^{n,1} \in C(\overline{Q}) \) is the solution of

\[ \frac{\partial \zeta}{\partial t} + A \zeta + \beta_{\rho} \zeta = h_{\rho}^{n} \text{ in } Q, \quad \zeta = 0 \text{ on } \Sigma, \quad \zeta(0) = 0 \text{ in } \Omega, \]

and \( \zeta_{\rho}^{n,2} \in C_{b}(Q \cup \Omega_{T}) \) is the solution of

\[ \frac{\partial \zeta}{\partial t} + A \zeta + \beta_{\rho} \zeta = 0 \text{ in } Q, \quad \zeta = 0 \text{ on } \Sigma, \quad \zeta(0) = f_{\rho}^{n} \text{ in } \Omega. \]

Let \( \eta_{\rho}^{n} \in C_{b}(Q \cup \Omega_{T}) \) be the solution of

\[ \frac{\partial \eta}{\partial t} + A \eta + \beta \eta = 0 \text{ in } Q, \quad \eta = 0 \text{ on } \Sigma, \quad \eta(0) = f_{\rho}^{n} \text{ in } \Omega, \]

where \( \beta(\cdot) = \Phi'_{y}(\cdot, y(u_{g_{1}}), g_{1}) \). The operator \( T \) which associates \( \zeta \), the solution of

\[ \frac{\partial \zeta}{\partial t} + A \zeta + \beta \zeta = 0 \text{ in } Q, \quad \zeta = 0 \text{ on } \Sigma, \quad \zeta(0) = w \text{ in } \Omega, \]

with \( w \in L^{\infty}(\Omega) \), is continuous from \( L^{\infty}(\Omega) \) into \( C^{\nu,\nu/2}(\overline{Q}_{r_{\rho}}) \) for some \( 0 < \nu < 1 \) (see Proposition 3.2). Since the imbedding from \( C^{\nu,\nu/2}(\overline{Q}_{r_{\rho}}) \) into \( C(\overline{Q}_{r_{\rho}}) \) is compact, \( T \) can be considered as a compact operator from \( L^{\infty}(\Omega) \) into \( C(\overline{Q}_{r_{\rho}}) \). Since the sequence \( (f_{\rho}^{n})_{n} \) converges to 0 for the weak-star topology of \( L^{\infty}(\Omega) \), we obtain

\[ \lim_{n \to \infty} \| \eta_{\rho}^{n} \|_{C(\overline{Q}_{r_{\rho}})} = 0. \]

From (5.4) and (5.7), we deduce the existence of some integer \( n(\rho) \) such that

\[ \| y \left( u_{g_{\rho}^{n(\rho)}}, g_{\rho}^{n(\rho)} \right) - y (u_{\rho}, \rho g_{2} + (1 - \rho)g_{1}) \|_{C(\overline{Q}_{r_{\rho}})} + \| \eta_{\rho}^{n(\rho)} \|_{C(\overline{Q}_{r_{\rho}})} \leq \rho. \]

On the other hand, the function \( \zeta_{\rho}^{n(\rho),2} - \eta_{\rho}^{n(\rho)} \) belongs to \( C(\overline{Q}) \) and satisfies:

\[ \frac{\partial \zeta}{\partial t} + A \zeta + \beta_{\rho}(\cdot) \zeta = (\beta - \beta_{\rho}(\cdot)) \eta_{\rho}^{n(\rho)} \text{ in } Q, \quad \zeta = 0 \text{ on } \Sigma, \quad \zeta(0) = 0 \text{ in } \Omega. \]

There exists \( C \equiv C(T, \Omega, q, N) > 0 \), independent of \( \rho \), such that

\[ \| \zeta_{\rho}^{n(\rho),1} \|_{C(\overline{Q})} \leq C \| h_{\rho}^{n(\rho)} \|_{Q, \overline{Q}}. \]
\[ \| \zeta_p^{n(\rho),2} - \eta_p^{n(\rho)} \|_{C(Q)} \leq C \| (\beta - \beta_p^{n(\rho)}) \eta_p^{n(\rho)} \|_{q,Q} \leq C \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \| \eta_p^{n(\rho)} \|_{r,Q}, \]

where \( \frac{1}{r} = \frac{1}{q} - \frac{1}{q} (\bar{q}) \) obeys \( q > \bar{q} > \frac{N}{2} + 1 \), see assumptions. It follows that

\[ \| \zeta_p^{n(\rho),2} - \eta_p^{n(\rho)} \|_{C(Q)} \leq C \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \left( \| \eta_p^{n(\rho)} \|_{C(Q_{\tau,T})} + (L^{N+1}(Q \setminus Q_{\tau,T})) \right) \| \eta_p^{n(\rho)} \|_{\infty,Q}, \]

\[ \leq C \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \left( \| \eta_p^{n(\rho)} \|_{C(Q_{\tau,T})} + \frac{1}{r} \| f_p^{n(\rho)} \|_{\infty,Q} \right) \]

\[ \leq C \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \left( \| \eta_p^{n(\rho)} \|_{C(Q_{\tau,T})} + \frac{1}{r} \| g_1 - g_2 \|_{\infty,Q} \right) \]

\[ \leq C \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \left( \| \beta - \beta_p^{n(\rho)} \|_{q,Q} \rho + \rho^{\frac{N-2}{2}} \right) \]

Therefore,

\[ \| \zeta_p^{n(\rho)} \|_{C(Q_{\tau,T})} \leq \| \zeta_p^{n(\rho),1} \|_{C(Q)} + \| \zeta_p^{n(\rho),2} - \eta_p^{n(\rho,\tau)} \|_{C(Q)} + \| \eta_p^{n(\rho)} \|_{C(Q_{\tau,T})} \]

\[ \leq C \left( \| \eta_p^{n(\rho)} \|_{q,Q} + \| \beta - \beta_p^{n(\rho)} \|_{q,Q} (\rho + \rho^{\frac{N-2}{2}}) \right). \]

With assumption A2 and with Lebesgue’s theorem of dominated convergence, we can prove that \( \left( h_p^{n(\rho)} \right) \) converges to 0 in \( L^q(Q) \). Thus (5.6) follows from the last inequality. Let us set \( \Omega_p = \Omega_p^{n(\rho)}, g_p = g_p^{n(\rho)} \). We have \( y(u_{g_p}, g_p) = y \left( u_{g_p^{n(\rho)}}, g_p^{n(\rho)} \right), y(u_{g_p}, g_1) = y \left( u_{g_p^{n(\rho)}}, g_1 \right) \), \( \frac{r}{\rho} = \zeta_p^{n(\rho)} \), and (5.2) is proved.

**Step 2.** Now we establish (5.3). First observe that, with (5.8), we have

\[ \| y(u_{g_p}, g_p) - y(u_{g_1}, g_1) \|_{C(Q^\tau)} \leq \| y(u_{g_p}, g_p) - y(u_{g_p}, g_2 + (1 - \rho)g_1) \|_{C(Q^\tau)} + \| y(u_{g_1}, g_1) - y(u_{g_p}, g_2 + (1 - \rho)g_1) \|_{C(Q^\tau)} \]

\[ \leq \rho + \| y(u_{g_1}, g_1) - y(u_{g_p}, g_2 + (1 - \rho)g_1) \|_{C(Q^\tau)} \]

for all \( \tau > 0 \).

Therefore, from (5.5) it follows that:

\[ \lim_{\rho \to 0} \| y(u_{g_p}, g_p) - y(u_{g_1}, g_1) \|_{C(Q^\tau)} = 0 \text{ for all } \tau > 0. \]

On the other hand

\[ \left| \frac{J(u_{g_p}, g_p) - J(u_{g_1}, g_1)}{\rho} - \Delta J \right| \]

\[ \leq \left| \int_Q \left( \frac{F(x,t,y(u_{g_p}, g_p)) - F(x,t,y(u_{g_1}, g_1))}{\rho} - F'(x,t,y(u_{g_1}, g_1))y_1(x,t) \right) dx \right| \]

\[ \leq \left| \int_Q \left( \frac{F(x,t,y(u_{g_p}, g_p)) - F(x,t,y(u_{g_1}, g_1))}{\rho} - F'(x,t,y(u_{g_1}, g_1))y_1(x,t) \right) dx \right|. \]
Thus \( \lim_{\rho \to 0} I_\rho = I_\rho^1 + I_\rho^2 \).

(Due to (5.1), the integrand \( L \) does not appear in the above estimate.) From the equation satisfied by \( \frac{r_\rho}{\rho} = \zeta_{\rho}^{(\rho)} \), we deduce that

\[
\| \frac{r_\rho}{\rho} \|_{\infty, Q} \leq C \left( \| h_{\rho}^{n(\rho)} \|_{\ell, Q} + \| f_\rho^{n(\rho)} \|_{\infty, Q} \right) \leq C \left( \| h_{\rho}^{n(\rho)} \|_{\ell, Q} + 1 \right).
\]

With A3, we obtain

\[
I_\rho^1 \leq \| F_\rho \|_{\infty, Q} + \| F_\rho - F_\rho' (\cdot, y(u_{g_1}, g_1)) \|_1 \leq C \left( \| F_\rho \|_{\infty, Q} + \| F_\rho - F_\rho' (\cdot, y(u_{g_1}, g_1)) \|_1 \right)
\]

where

\[
F_\rho = \int_0^1 F_\rho' (\cdot, (1 - \theta) y(u_{g_1}, g_1) + \theta y(u_{g_2}, g_2)) d\theta,
\]

and \( C \) is a positive constant independent of \( \rho \). Similarly to the calculus of \( I_\rho^1 \), and due to A4, we prove that

\[
I_\rho^2 \leq C \left( \rho + (\tau_\rho)^{\frac{1}{\theta}} \| h_{\rho}^{n(\rho)} \|_{\ell, Q} + \rho^{\frac{\alpha q}{\alpha - q}} + \| \ell_\rho (\cdot) - \ell_\rho' (\cdot, y(u_{g_1}, g_1)(T)) \|_{1, \Omega} \right),
\]

where

\[
\ell_\rho (\cdot) = \int_0^1 \ell_\rho' (\cdot, (1 - \theta) y(u_{g_1}, g_1)(T) + \theta y(u_{g_2}, g_2)(T)) d\theta,
\]

and where \( C \) is a positive constant independent of \( \rho \). Due to (5.9), by assumptions on \( F \) and \( \ell \), and Lebesgue's theorem of dominated convergence, we have

\[
\lim_{\rho \to 0} \| F_\rho - F_\rho' (\cdot, y(u_{g_1}, g_1)) \|_1, \Omega = 0 \quad \text{and} \quad \lim_{\rho \to 0} \| L_\rho (T) - L_\rho' (\cdot, y(u_{g_1}, g_1)(T)) \|_{1, \Omega} = 0.
\]

Thus \( \lim_{\rho \to 0} I_\rho^1 = \lim_{\rho \to 0} I_\rho^2 = 0 \), and the proof is complete.

6. Proof of the optimality conditions.

6.1. Proof of Theorem 2.1: Optimal Conditions for \( (P_\rho) \). The proof is similar to the one given in [2], Theorem 2.1. We reiterate it for the convenience of the reader. Let \( \rho \) be in [0, 1]. Let \( g \) be in \( G_{ad}, u_g \) in Argmin \( (P_\rho) \), and \( u \) in \( U_{ad} \). Due to Theorem 5.1, there exists a measurable subset \( Q_\rho \) such that \( L_{N+1} (Q_\rho) = \rho L_{N+1} (Q) \), and

\[
y(u_\rho (g), g) = y(u_g, g) + \rho z_g + r_\rho, \quad \text{with} \quad \lim_{\rho \to 0} \| r_\rho \|_{C(\overline{Q})} = 0.
\]
\[ J(u_\rho(g), g) = J(u_g, g) + \rho \Delta J + o(\rho), \]

where \( u_\rho(g) \) is defined by
\[ u_\rho(g)(x, t) = \begin{cases} u_g(x, t) & \text{in } Q \setminus Q_\rho, \\ u(x, t) & \text{in } Q_\rho. \end{cases} \]

\( y_\rho(g, g) \) is the solution to (1.1) corresponding to \( (u_\rho(g), g) \), \( z_g \) is the weak solution of
\[
\frac{\partial z}{\partial t} + A z + \Phi'_y(x, t, y(u_g, g)) z = u - u_g \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(0) = 0 \quad \text{in } \Omega,
\]

and
\[
\Delta J = \int_Q \left( F'_y(x, t, y(u_g, g)) z_g + (H(x, t, u) - H(x, t, u_g)) \right) dx dt + \int_{Q_\rho} \ell'_y(x, y(u_g, g)) z_g(T) dx.
\]

Since \( (u_\rho, g) \) is admissible for \( (P_g) \), it follows that \( J(u_g, g) \leq J(u_\rho(g), g) \) and
\[
-\Delta J \leq \lim_{\rho \to 0} \frac{J(u_g, g) - J(u_\rho(g), g)}{\rho} \leq 0.
\]

Let \( p_g \) be the weak solution of (2.1). Using the Green formula of Theorem 3.3, we obtain
\[
-\int_Q F'_y(x, t, y(u_g, g)) z_g dx dt - \int_{\Omega} \ell'_y(x, y(u_g, g)) z_g(T) dx
\]
\[
= \int_Q p_g \left( \frac{\partial z_g}{\partial t} + A z_g + \Phi'_y(x, t, y(u_g, g)) z_g \right) dx dt = \int_Q p_g (u - u_g) dx dt.
\]

Taking the definition of \( \Delta J \) into account, we have
\[
\Delta J = \int_Q (H(x, t, u) - H(x, t, u_g)) dx dt - \int_Q p_g(x, t)(u(x, t) - u_g(x, t)) dx dt.
\]

From (6.1) and (6.2), we finally obtain:
\[
\int_Q H(x, t, u_g, p_g) dx dt \leq \int_Q H(x, t, u, p_g) dx dt \quad \text{for all } u \in U_{ad}.
\]

The pointwise Pontryagin’s principle (2.2), is next obtained by the method developed in [16], Section 5.2.

**6.2. Proof of Theorem 2.2 : Optimality Conditions for (P).** Let \( \rho \) be in \( [0, 1] \), \( \tau_\rho \) be as in Theorem 5.2, and let \( g \in G_{ad} \). We recall that \( \bar{g} \) is the optimal solution to \( (P_g) \) that we want to characterize. Due to Theorem 5.2, there exist \( \bar{u} \in \text{Argmin} (P_g) \), and measurable subsets \( \bar{\Omega}_\rho \) such that
\[
\mathcal{L}^N(\bar{\Omega}_\rho) = \rho \mathcal{L}^N(\bar{\Omega}),
\]
\[
y(u_g, g_\rho) = y(u_g, \bar{g}) + \rho \tilde{z} + \tilde{r}_\rho, \quad \text{with } \lim_{\rho \to 0} \frac{1}{\rho} \| \tilde{r}_\rho \|_{C(Q_{\tau_\rho T})} = 0,
\]

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Using the Green formula of Theorem 3.3, we have
\[ J(u_{g^*}, g_{\rho}) = J(u_{g^*}, \bar{g}) + \rho \left( J'_g(\bar{u}, \bar{g}) \bar{z} + \int_{\Omega} (L(x, g) - L(x, \bar{g})) dx \right) + o(\rho), \]
where
\[ g_{\rho}(x) = \begin{cases} \bar{g}(x) & \text{in } \Omega \setminus \Omega_{\rho}, \\ g(x) & \text{in } \Omega_{\rho}, \end{cases} \]
\[ u_{g^*} \in \text{Argmin } (P_{g^*}), \]
\[ \bar{z} \text{ is the solution of the equation} \]
\[ \frac{\partial z}{\partial t} + A z + \Phi'(\cdot, y(\bar{u}, \bar{g})) z = 0 \text{ in } Q, z = 0 \text{ on } \Sigma, z(0) = g - \bar{g} \text{ in } \Omega. \]
Since \( \bar{g} \) is a solution to (P) and \( \bar{u} \in \text{Argmin}(P_{\bar{g}}) \), we see that
\[ 0 \leq \frac{J(\bar{u}, \bar{g}) - J(u_{g^*}, g_{\rho})}{\rho} \leq \frac{J(u_{g^*}, \bar{g}) - J(u_{g^*}, g_{\rho})}{\rho}. \]
With (6.3), we obtain
\[ 0 \leq \lim_{\rho \to 0} \frac{J(u_{g^*}, \bar{g}) - J(u_{g^*}, g_{\rho})}{\rho} \]
\[ = -\int_Q F'_y(x, t, y(\bar{u}, \bar{g})) \bar{z} dx dt - \int_{\Omega} \ell'(y(\bar{u}, \bar{g}), \bar{g})(T) \bar{z}(T) dx - \int_{\Omega} (L(x, g) - L(x, \bar{g})) dx. \]
Using the Green formula of Theorem 3.3, we have
\[ -\int_Q F'_y(x, t, y(\bar{u}, \bar{g})) \bar{z} dx dt - \int_{\Omega} \ell'(y(\bar{u}, \bar{g}), \bar{g})(T) \bar{z}(T) dx = \int_{\Omega} \bar{p}(x, 0) (g - \bar{g})(x) dx. \]
Therefore,
\[ \int_{\Omega} \mathcal{H}_0(x, g, \bar{p}(x, 0)) dx \leq \int_{\Omega} \mathcal{H}_0(x, \bar{g}, \bar{p}(x, 0)) dx \text{ for all } g \in G_{ad}. \]
The pointwise Pontryagin’s principle (2.5) is obtained by the method developed in [16], Section 5.2. Finally, since \( \bar{u} \) belongs to \( \text{Argmin}(P_{\bar{g}}) \), (2.4) follows from Theorem 2.1. \[ \blacksquare \]

6.3. Proof of Theorem 2.3. The optimality condition for \( u_{\bar{g}} \) is a direct consequence of Theorem 2.1. Let us prove optimality conditions for \( \bar{g} \). Let \( g \) be in \( G_{ad} \) and \( u_g \) be an element of \( \text{Arg inf}(P_g) \). For \( \rho \) in \( [0,1[ \), set \( y_{\rho} = (1 - \rho) y(u_{\bar{g}}, g) + \rho y(u_{g^*}, g) \). Since \( \Phi \) is affine with respect to \( y \), \( y_{\rho} \) is the solution of (1.1) corresponding to \( ((1 - \rho) u_{\bar{g}} + \rho u_g, (1 - \rho) \bar{g} + \rho g) \). Due to the convexity of \( F \) and \( \ell \), we have
\[ \int_Q F(x, t, y_{\rho}) dx dt + \int_{\Omega} \ell(x, y_{\rho}(T)) dx \leq (1 - \rho) \left( \int_Q F(x, t, y(u_{\bar{g}}, \bar{g})) dx dt + \int_{\Omega} \ell(x, y(u_{\bar{g}}, \bar{g})(T)) dx \right) \]
\[ + \rho \left( \int_Q F(x, t, y(u_g, g)) dx dt + \int_{\Omega} \ell(x, y(u_g, g)(T)) dx \right). \]
Set
\[ J_\rho = \int_Q F(x, t, y_{\rho}) dx dt + \int_{\Omega} \ell(x, y_{\rho}(T)) dx \]
\[ + (1 - \rho) \left( \int_Q H(x, t, u_{\bar{g}}) dx dt + \int_{\Omega} L(x, \bar{g}) dx \right) + \rho \left( \int_Q H(x, t, u_g) dx dt + \int_{\Omega} L(x, g) dx \right). \]
With the previous inequality, we obtain
\[ J_\rho \leq (1 - \rho)J(u_\bar{g}, \bar{g}) + \rho J(u_g, g). \]
From the optimality of \(\bar{g}\), we have \( J(u_g, g) \leq J(u_\bar{g}, \bar{g}) \), and thus,
\[ J_\rho \leq (1 - \rho)J(u_\bar{g}, \bar{g}) + \rho J(u_g, g) \leq J(u_\bar{g}, \bar{g}). \]
It follows that:
\[
0 \leq \lim_{\rho \to 0} \frac{J(u_\bar{g}, \bar{g}) - J_\rho}{\rho} = \int_Q F'_y(x, t, y(u_\bar{g}, \bar{g})) \left( y(u_\bar{g}, \bar{g}) - y(u_g, g) \right) \, dx \, dt \\
+ \int_\Omega \ell'_y(x, y(u_\bar{g}, \bar{g})(T)), (y(u_\bar{g}, \bar{g}) - y(u_g, g)(T)) \, dx \\
+ \int_Q \left( H(x, t, u_\bar{g}) - H(x, t, u_g) \right) \, dx \, dt + \int_\Omega \left( L(x, \bar{g}) - L(x, g) \right) \, dx.
\]
From (6.4), by using the Green formula of Theorem 3.3 (with \( z = y(u_\bar{g}, \bar{g}) - y(u_g, g) \)), we obtain
\[
\int_Q \bar{p}(u_\bar{g} - u_g) \, dx \, dt + \int_\Omega \bar{p}(0)(\bar{g} - g) \, dx \leq \int_Q \left( H(x, t, u_\bar{g}) - H(x, t, u_g) \right) \, dx \, dt + \int_\Omega \left( L(x, \bar{g}) - L(x, g) \right) \, dx.
\]
Recalling (2.7), we deduce that
\[
\int_\Omega (L(x, g) - \bar{p}(0)g) \, dx - \int_\Omega (L(x, \bar{g}) - \bar{p}(0)\bar{g}) \, dx \leq \int_Q \left( H(x, t, u_\bar{g}) - \bar{p}u_\bar{g} \right) \, dx \, dt - \int_Q \left( H(x, t, u_g) - \bar{p}u_g \right) \, dx \, dt \leq 0.
\]
The proof is complete. \(\blacksquare\)

7. Some extensions. We can also consider models which take into account the first order derivatives of the state variable both in the equation and the cost functional. For example, consider the parabolic equation:
\[
(7.1) \quad \frac{\partial y}{\partial t} + Ay + \nabla V \cdot \nabla y + \Phi(x, t, y) = u \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \quad y(x, 0) = y_0(x) + g(x) \text{ in } \Omega,
\]
where \(A, \Phi, u\) satisfy the assumptions of Section 2, \(\nabla V\) belongs to \(L^\infty(0, T; (L^\infty(\Omega))^N)\), and the cost functional defined by
\[
(7.2) \quad \mathcal{I}(y, u, g) = \int_Q (F(\cdot, y) + G(\cdot, \nabla y) + H(\cdot, u)) \, dx \, dt + \int_\Omega \left( \ell(\cdot, y(T)) + L(\cdot, g) \right) \, dx.
\]
The regularity results used in the proof of Theorem 5.2 are still true for the above equation [10], [8]. The adjoint equation corresponding to a solution \(\bar{y}\) of (7.1) is
\[
(7.3) \quad \begin{cases}
\frac{\partial p}{\partial t} + Ap - \nabla V \cdot \nabla p + \Phi'_y(\cdot, \bar{y})p + F'_y(\cdot, \bar{y}) - \text{div} \, (G'_z(\cdot, \nabla \bar{y})) = 0 & \text{in } Q, \\
p(\cdot, T) + \ell'_y(\cdot, \bar{y}(T)) = 0 & \text{in } \Omega,
\end{cases}
\]
where \(G'_z\) denotes the derivative of \(G\) with respect to \(\nabla y\). If \(G'_z(\cdot, \nabla \bar{y})\) belongs to \(L^r(Q)\) with \(r > 1\), then equation (7.3) admits a unique solution in \(L^1(0, T; W^{1,1}_0(\Omega))\). The condition \(G'_z(\cdot, \nabla \bar{y}) \in L^r(Q)\) can be easily checked if \(G'_z\) satisfy a suitable growth condition. For \(\nabla V \equiv 0\) and \(\Phi'_y \equiv 0\), the existence to
equation (7.3) can be deduced from [17]. For the general case, that is if $\nabla V \neq 0$ and $\Phi'_y \neq 0$, the existence can be shown by using a fixed point technique as in [13]. Therefore Theorem 2.2 may be extended (with obvious modifications) to problem $(P)$ corresponding to the equation (7.1) and the functional (7.2). Notice that we have considered homogeneous boundary conditions, as in the example of convection-diffusion systems studied in [1]. The analysis corresponding to nonhomogeneous boundary conditions of the form $y = \psi \in L^\infty(\Sigma)$ is more delicate since we only know that $\nabla y$ belongs to $L^1_{loc}(Q)$.

For a state equation with a nonlinear term depending on the gradient $\nabla y$, the analysis is more complicated and some additional material is needed (see for example [7], Chapter 4). The results presented in our paper do not recover this case, and the extension to such models requires another lengthy study of the state equation which cannot be included here.

REFERENCES