A Generalized Multiplicative Directional Distance Function for Efficiency Measurement in DEA

Mahmood Mehdiloozad, Biresh K. Sahoo, Israfil Roshdi

Faculty of Mathematical Sciences and Computer, Kharazmi (Tarbiat Moallem) University, Tehran, Iran
Xavier Institute of Management, Bhubaneswar 751 013, India
Department of Mathematics, Semnan Branch, Islamic Azad University, Semnan, Iran

Abstract

For measuring technical efficiency relative to a log-linear technology, a generalized multiplicative directional distance function (GMDDF) is developed using the framework of multiplicative directional distance function (MDDF). Furthermore, a computational procedure is suggested for its estimation. The GMDDF serves as a comprehensive measure of efficiency in revealing Pareto-efficient targets as it accounts for all possible input and output slacks. This measure satisfies several desirable properties of an ideal efficiency measure such as strong monotonicity, unit invariance, translation invariance, and positive affine transformation invariance. This measure can be easily implemented in any standard DEA software and provides the decision makers with the option of specifying preferable direction vectors for incorporating their decision-making preferences. Finally, to demonstrate the ready applicability of our proposed measure, an illustrative empirical analysis is conducted based on real-life data set of 20 hardware computer companies in India.

Keywords: Data envelopment analysis; Multiplicative directional distance function; Generalized multiplicative directional distance function; Log-convexity; Piece-wise log-linear technology

* Corresponding author.
E-mail: m.mehdiloozad@gmail.com (M. Mehdiloozad)
1. Introduction

Data envelopment analysis (DEA), introduced by Charnes, Cooper, & Rhodes (1978, 1979) and extended by Banker, Charnes, & Cooper (1984), is a linear programming (LP) based non-parametric methodology for measuring the relative efficiency of a set of homogenous decision making units (DMUs) with multiple inputs and outputs. DEA is used to construct a reference technology relative to which the efficiency of individual DMUs can be estimated. The frontier of the reference technology provides a conservative inner (empirical) approximation of a true but unobserved production function.

Based on the axiomatic approach and the types of convexity postulate, two different characterizations of a technology structure – piece-wise linear and piece-wise log-linear – are suggested for estimating relative efficiency in the DEA literature. The piece-wise linear technology is constructed based on normal convexity postulate (Banker, et al., 1984), which requires the marginal products of factor inputs to be non-increasing. The piece-wise log-linear technology is, however, constructed based on geometric convexity (log-convexity) postulate (Banker & Maindiratta, 1986). The motivation for introducing the log-linear technology is that, unlike the linear one, it allows for increasing, constant and decreasing marginal productivities along the production frontier. More precisely, this technology structure is flexible enough to simultaneously capture the three production features – convexity, linearity, and concavity – of a production function.

The piece-wise log-linear technology pertains to a class of DEA models, known as the multiplicative models, which were developed by Charnes, Cooper, Seiford, & Stutz (1982) based on the log-linear envelopment principle introduced by Banker, Charnes, Cooper, & Schinnar (1981) (For more details about the multiplicative models, see e.g., Chang & Guh (1994), Charnes, Cooper, Seiford, & Stutz (1983), Seiford & Zhu (1998), and Sueyoshi & Chang (1989), among others).

The directional distance function (DDF) of Chambers, Chung, & Färe (1996, 1998) is a useful generalization of the Shephard’s (1970) distance functions. Using the concept of DDF, Peyrache & Coelli (2009) recently introduced the concept of multiplicative directional distance function (MDDF), which completely characterizes the technology, and serves as a measure of technical efficiency (TE). The MDDF also encompasses the hyperbolic, modified hyperbolic, and Shephard’s input and output distance functions as special cases. However, if slacks are considered as additional sources of inefficiency, then the MDDF is unable to yield a comprehensive measure of TE in the sense of Pareto-Koopmans (Koopmans, 1951); and hence, as a result, it is not strongly monotonic.
In this paper, first, we demonstrate the capability of the MDDF for evaluating the TE of DMUs operating on the non-concave and non-convex frontier regions of technology by redefining the MDDF relative to the log-linear technology. We also propose a DEA formulation for the MDDF, and a computational method for its estimation. Considering slacks as the additional sources of inefficiency, we then extend the MDDF to a generalized multiplicative directional distance function (GMDDF), which provides a complete characterization of technology. This GMDDF yields a comprehensive measure of TE as it satisfies several desirable properties of an ideal efficiency measure such as strong monotonicity, unit invariance, translation invariance, and positive affine transformation invariance.

The GMDDF-based TE measure yields a straightforward interpretation – it is the product of the geometric means of the input and output efficiencies. As regards the practical advantages of our proposed TE measure, first, due to the flexibility in computer programming, the computations of TE scores and the projections of inefficient observations towards different facets of the efficient frontier can be easily provided in any standard DEA software. Second, it allows for the incorporation of decision makers’ preferences into efficiency assessment, which will generate a more realistic TE measure by assigning weights for individual inputs and outputs that are more consistent with the underlying objectives of decision makers.

While the GMDDF-based TE measure is non-radial in nature, the MDDF-based measure is radial. On comparison between the two measures, one can argue that the former is to be preferred to the latter, on the ground that the MDDF does not encapsulate all sources of inefficiency as it suffers from the problem of slacks, and the GMDDF has the flexibility to enable itself to better reflect the trade-offs between the inputs and/or outputs in its efficiency measures. However, purely on the axiomatic ground of efficiency measurement, the MDDF-based measure can be favored, given its economic interpretation for the class of log-convex monotonic technologies. Furthermore, from a purely practical point of view, the proportional interpretation of the radial MDDF measure facilitates the use of its benchmarking results in managerial contexts. For more details on the debate as to the choice of radial vis-a-vis non-radial measures, see Mahlberg & Sahoo (2011), Sahoo & Acharya (2010, 2012), Sahoo, Luptacik, & Mahlberg (2011), Sahoo & Tone (2009a, 2009b), among others.

The remainder of the paper unfolds as follows. Section 2 deals in detail with the description of technology in an analytical framework, followed by an introduction of the MDDF-based TE measure. Section 3, first, develops the GMDDF-based measure of TE that accounts for all types of slacks; second, it describes a computational method for the practical execution of this measure, and
finally, it presents a discussion of its properties. Section 4 deals with an illustrative empirical application on real-life data set of hardware computer companies in India for the period 2001-2010. Section 5 presents the conclusion with some observations and remarks.

2. Technology and multiplicative directional distance function

2.1 Technology

Let us consider a technology involving \( n \) observed DMUs; each uses \( m \) inputs to produce \( s \) outputs. Let \( x_j = (x_{j1}, \ldots, x_{jm})^T \in \mathbb{R}_{20m}^m \) and \( y_j = (y_{j1}, \ldots, y_{js})^T \in \mathbb{R}_{20s}^s \) be, respectively, the input and output vectors of DMU \( j \), \( j \in J = \{1, \ldots, n\} \). The superscript \( T \) stands for a vector transpose. We further consider \( o \) as the index of DMU under evaluation.

A production technology transforming an input vector \( x \in \mathbb{R}_{20m}^m \) into an output vector \( y \in \mathbb{R}_{20s}^s \) can be characterized by the technology set \( T \subset \mathbb{R}_{20m}^m \times \mathbb{R}_{20s}^s \), defined as

\[
T = \{(x, y) \in \mathbb{R}_{20m}^m \times \mathbb{R}_{20s}^s | x \text{ can produce } y \}.
\] (1)

We assume that \( T \) satisfies the following three postulates:

A.1 (Closure). \( T \) is a closed set.

A.2 (Boundedness). The set \( A(x) = \{(u, y) \in T | u \leq x\} \) is bounded for each \( x \in \mathbb{R}_{20m}^m \).

A.3 (Free disposability). \( T \) satisfies strong (free) disposability for all the inputs and outputs, i.e., if \( (x, y) \in T \) and \( (x, -y) \leq (x', -y') \), then \( (x', y') \in T \).

For details of these assumptions and the resulting properties, see Banker, et al. (1984), Färe, Grosskopf, & Lovell (1985), and Shephard (1974).

In addition, we consider the convexity postulate for \( T \) in two alternative forms:

C.1 (Normal convexity). If \( (x_1, y_1) \in T \) and \( (x_2, y_2) \in T \), then \( (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in T \) for any \( \lambda \in [0,1] \).

C.2 (Geometric convexity or log-convexity). If \( (x_1, y_1) \in T \) and \( (x_2, y_2) \in T \), then \( \left(x_1^{(\lambda)}, y_1^{(\lambda)}, x_2^{(\lambda)}, y_2^{(\lambda)}\right) \in T \) for any \( \lambda \in [0,1] \).

To further clarify the definition of C.2, we define the log form of \( T \) as

\[
\log T := \{(\log x, \log y) \mid (x, y) \in T\},
\] (2)
where \( \log x = (\log x_1, \ldots, \log x_m) \) and \( \log y = (\log y_1, \ldots, \log y_s) \). The strict monotonicity property of the logarithmic transformation establishes a one-to-one correspondence between \( \log T \) and \( T \). This acknowledges that the log-convexity of \( T \) is equivalent to the normal convexity of \( \log T \). Formally, \( T \) is log-convex if and only if
\[
(\lambda \log x_i + (1-\lambda) \log x_j, \lambda \log y_i + (1-\lambda) \log y_j) \in \log T
\]
holds for any \( (\log x_i, \log y_i), (\log x_j, \log y_j) \in \log T \) and any \( \lambda \in [0,1] \).

Following Banker, et al. (1984) and Banker & Maindiratta (1986), we now formulate two different non-parametric representations of \( T \) as follows:

(i) Assuming the inputs and outputs to be non-negative, we define the piece-wise linear technology set \( T_1 \) that is constructed from the observed DMUs under Postulates A.1–A.3 and C.1.

(ii) Assuming the inputs and outputs to be strictly positive, we define the piece-wise log-linear technology \( T_2 \) that is constructed from the observed DMUs under Postulates A.1–A.3 and C.2.

Since assuming log-convexity for \( T \) is tantamount to assuming convexity for \( \log T \), \( \log T \) will be piece-wise linear provided \( T \) is log-convex, and as a result, \( T_2 \) is called piece-wise log-linear.

The explicit DEA-based representations of \( T_1 \) and \( T_2 \) in variable returns to scale (VRS) environment are, respectively, given as
\[
T_1 = \left\{ (x, y) \in \mathbb{R}_{\geq 0}^{m+s} \mid x \geq \sum_{j \in J} \lambda_j x_{ij}, y \leq \sum_{j \in J} \lambda_j y_{ij}, i = 1, \ldots, m, r = 1, \ldots, s, \lambda \in \Lambda \right\},
\]
\[
T_2 = \left\{ (x, y) \in \mathbb{R}_{> 0}^{m+s} \mid x \geq \prod_{j \in J} x_{ij}^{\lambda_j}, y \leq \prod_{j \in J} y_{ij}^{\lambda_j}, i = 1, \ldots, m, r = 1, \ldots, s, \lambda \in \Lambda \right\},
\]
where
\[
\Lambda = \left\{ \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0, j \in J \right\}.
\]

While \( T_1 \) requires the marginal products of factor inputs to be non-increasing; \( T_2 \) is free from this restriction, and allows for increasing, constant and decreasing marginal products (see Example).

We now turn to elaborate on the technological structures of \( T_1, T_2 \) and \( \log T_2 \) with the help of an example. Let us consider a single-input-single-output technology comprising six DMUs whose data are all exhibited in Table 1. The empirical technological structures of \( T_2, \log T_2 \) and \( T_1 \) are, respectively, depicted in Fig.1 (a), Fig.1 (b) and Fig.1 (c).
Table 1. Input and output data for Example 1

<table>
<thead>
<tr>
<th></th>
<th>DMU(_1)</th>
<th>DMU(_2)</th>
<th>DMU(_3)</th>
<th>DMU(_4)</th>
<th>DMU(_5)</th>
<th>DMU(_6)</th>
<th>DMU(_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input (x)</td>
<td>exp(0.75)</td>
<td>exp(1.00)</td>
<td>exp(1.50)</td>
<td>exp(2.00)</td>
<td>exp(0.90)</td>
<td>exp(2.25)</td>
<td>exp(1.70)</td>
</tr>
<tr>
<td>Output (y)</td>
<td>exp(0.50)</td>
<td>exp(1.00)</td>
<td>exp(1.50)</td>
<td>exp(1.75)</td>
<td>exp(0.80)</td>
<td>exp(1.60)</td>
<td>exp(1.60)</td>
</tr>
</tbody>
</table>

As can be seen in Fig. 1 (a), the frontier of \( T_2 \) is not everywhere concave and comprises three distinct parts:

(i) The first part is a convex parabola \( y = e^{-1}x^2 \) connecting DMU\(_1\) to DMU\(_2\) (represented with red color) where the marginal product is increasing.

(ii) The second part is a straight line \( y = x \) joining DMU\(_2\) and DMU\(_3\) (represented with green color) where the marginal product is constant.

(iii) The third part is a concave curve \( y = e^{0.75}\sqrt{x} \) connecting DMU\(_3\) to DMU\(_4\) (represented with blue color) where the marginal product is decreasing.

(a) Piece-wise log-linear technology \( (T_2) \)  
(b) Transformed piece-wise linear set \( (logT_2) \)  
(c) Piece-wise linear technology \( (T_1) \)

Fig. 1 Geometrical construction of the piece-wise linear and log-linear technologies
$T_2$ allows for all possible types of production structures – convexity, linearity, and concavity – thus capturing all the three types of variation – increasing, constant, and decreasing – in the marginal product of its factor input, which are the natural consequences of the log-convexity property of $T_2$.

Fig. 1 (b) exhibits $\log T_2$. As mentioned earlier, there exists a one-to-one correspondence between $T_2$ and $\log T_2$, including the frontier points. The frontier segments of both $\log T_2$ and $T_2$ are marked with the same colors. For example, the red line segment $\log y = 2 \log x - 1$ joining $\log \text{DMU}_1$ and $\log \text{DMU}_2$ in the set $\log T_2$ in Fig. 1 (b) corresponds to the red parabola $y = e^{-x^2}$ connecting $\text{DMU}_1$ to $\text{DMU}_2$ through $\text{DMU}_5$ in $T_2$ in Fig. 1 (a). One can observe from Fig. 1 (b) that the frontier of $\log T_2$ is a concave function, which follows from the fact that it admits normal convexity postulate.

### 2.2 Multiplicative directional distance function

Extending the concept of DDF, Peyrache & Coelli (2009) developed the concept of MDDF to measure technical efficiency. This MDDF encompasses the hyperbolic, the modified hyperbolic, and the Shephard’s input and output distance functions as special cases. Since the MDDF is based on the convex technology $T_1$, it yields a non-linear programming (NLP) problem that is hard to solve with standard linear programming technique. Therefore, it is imperative to reformulate the MDDF based on the log-convex technology $T_2$ so that it can be easily converted into an LP program. Moreover, this reformulation is useful in revealing the correct TE scores of DMUs operating on the non-concave and non-convex frontier regions of technology.

Following Peyrache & Coelli (2009), we present the definition of MDDF.

**Definition 2.2.1.** Let $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ be an input-output vector and $g = (-g^-, g^+) \in (-\mathbb{R}^n_+) \times \mathbb{R}^m_+$ be a user-specified direction vector. Then, the function

$$
\bar{M} (x, y; -g^-, g^+) = \sup \left\{ \beta \left| \left( \beta^{x^*}, \beta^{y^*} \right) \in T_2 \right\}
$$

is called the *multiplicative directional distance function*.

Seeking for both expansion in the outputs and contraction in the inputs, the MDDF, with a given $g$, projects $(x, y)$ onto the frontier of $T_2$ at $\left( \beta^{x^*}, \beta^{y^*} \right)$ where $\beta^* = \bar{M} (\cdot)$. The properties of
\( \tilde{M} (\cdot) \) are similar to those of the MDDF developed by Peyrache & Coelli (2009), which is based on the convex technology. See Peyrache & Coelli (2009) for more details on these properties.

From the strict monotonicity property of \( \log \) function, it follows that

\[
\log \tilde{M} (x, y; -g^-, g^+) = \log \left( \text{Sup} \left\{ \beta \left( \beta^{-x} x, \beta^{y} y \right) \in T_2 \right\} \right) \\
= \text{Sup} \left\{ \log \beta \left( \beta^{-x} x, \beta^{y} y \right) \in T_2 \right\} \\
= \text{Sup} \left\{ \log \beta \left( \log \left( \beta^{-x} x \right), \log \left( \beta^{y} y \right) \right) \in \log T_2 \right\} \\
= \text{Sup} \left\{ \log \beta \left( \log x - g^- \log \beta, \log y + g^+ \log \beta \right) \in \log T_2 \right\} \\
= \text{Sup} \left\{ \log \beta \left( \log x, \log y \right) + g \log \beta \in \log T_2 \right\}.
\]

Or, equivalently,

\[
\tilde{M} (x, y; -g^-, g^+) = \exp \left( \text{Sup} \left\{ \log \beta \left( \log x, \log y \right) + g \log \beta \in \log T_2 \right\} \right).
\]

The equation (9) indicates that the pre-assigned direction vector \( g \) in \( \tilde{M} (\cdot) \) specifies the direction along which the vector \( (\log x, \log y) \) is projected onto the frontier of \( \log T_2 \) at \( (\log x - g^- \log \beta^-, \log y + g^+ \log \beta^+) \), where \( \beta^+ = \tilde{M} (\cdot) \) (see Fig.2).

![Fig.2 The multiplicative directional distance function](image)

For any pre-determined direction vector \( g \), \( \tilde{M} (\cdot) \) can be computed for DMUo by solving the following NLP problem:
In order to solve the NLP (10), using the natural logarithmic transformation, we transform it into the following equivalent LP problem:

\[
\log \tilde{M} \left(x_o, y_o; -g^-, g^+\right) = \max \beta \\
\text{s.t.} \quad \prod_{j \in J} x_{ij}^\lambda \leq \beta^{-\delta_i} x_{wo}, \quad i = 1, \ldots, m, \\
\prod_{j \in J} y_{ij}^\lambda \geq \beta^\delta_j y_{wo}, \quad r = 1, \ldots, s, \\
\lambda \in \Lambda.
\]  

From the strict monotonicity properties of \(\log\) and \(\exp\) functions, it follows that the constraints of (10) and (11) are equivalent, i.e., \(\left(\beta', \lambda', \forall j \in J\right)\) is a feasible solution to the NLP (10) if and only if \(\left(\log \beta', \lambda', \forall j \in J\right)\) is a feasible solution to the LP (11).

Let \(\left(\log \beta', \lambda', j \in J\right)\) is an optimal solution to (11). Then, since the constraints of (10) and (11) are equivalent, \(\left(\beta', \lambda', j \in J\right)\) is an optimal solution to (10); otherwise, the optimality of \(\left(\log \beta', \lambda', j \in J\right)\) would be violated. Thus, an optimal solution for (11) can be easily transformed into an optimal solution for (10).

We now turn to show how \(\tilde{M} (\cdot)\) can be adopted as a measure of TE. Because \(\left(\beta = 1, \lambda_c = 0, j \neq 0\right)\) is a feasible solution to the NLP (10), the optimal objective value becomes \(\tilde{M} (\cdot) \geq 1\). Full technical efficiency is attained when \(\tilde{M} (\cdot) = 1\), otherwise \(\tilde{M} (\cdot) > 1\). Since the value of \(\tilde{M} (\cdot)\) in its current form cannot be considered a measure of TE, we use its multiplicative inverse as the measure of TE, i.e.,

\[
E_o (g) := \frac{1}{\tilde{M} \left(x_o, y_o; -g^-, g^+\right)}.
\]  

One can now observe that \(0 < E_o (g) \leq 1\), and hence, \(E_o (g)\) satisfies the property of efficiency requirement (Sueyoshi & Sekitani, 2009).
**Definition 2.2.2.** DMU\(_o\) is MDDF-efficient (weak log-efficient) if and only if \( E_o(g) = 1 \).

It is worth noting that the output-oriented multiplicative model developed by Banker & Maindiratta (1986) can be obtained from the MDDF, by using the following direction vector:

\[
g_i^- = 0, \quad g_i^+ = 1, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \tag{13}
\]

Note that the MDDF fails to account for the additional sources of inefficiency caused by the non-zero exponential slacks: \( \exp(s_i^-), \quad i = 1, \ldots, m \), and \( \exp(s_r^+), \quad r = 1, \ldots, s \). So, from the closedness property of \( T_2 \), it follows that DMU\(_o\) is MDDF-efficient if and only if it belongs to the weakly efficient frontier of \( T_2 \) defined by

\[
\partial^n(T_2) = \{(x, y) \in T_2 | (x', -y') < (x, -y) \Rightarrow (x', y') \notin T_2\}. \tag{14}
\]

As a result, \( E_o(g) \) does not reflect full efficiency in the sense of Pareto-Koopmans (Koopmans, 1951). Thus, the MDDF overestimates the TE of those DMUs suffering from slacks. Therefore, in order for a TE measure to satisfy the Pareto-Koopmans definition of efficiency, the efficiency projection should be made onto the strongly efficient frontier of \( T_2 \) defined by

\[
\partial^s(T_2) = \{(x, y) \in T_2 | (x', -y') \leq (x, -y) \Rightarrow (x', y') \notin T_2\}. \tag{15}
\]

In order to develop such a TE measure, we now introduce in the immediately following section a generalized MDDF-based measure of TE.

**3. Generalized MDDF-based measure of efficiency**

**3.1 Generalized multiplicative directional distance function**

Consider the following definition.

**Definition 3.1.1.** Let \((x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^s\) be an input-output vector and \( g = (-g^-, g^+) \in (-\mathbb{R}_+^m) \times \mathbb{R}_+^s\) be a user-specified direction vector. Then, the function

\[
\bar{M}_G(x, y; -g^-, g^+) = \text{Sup} \left\{ \left( \prod_{i=1}^m \alpha_i \right) \left( \prod_{r=1}^s \beta_r \right)^{-1} \left| \left( \alpha_1 x_1, \ldots, \alpha_m x_m, \beta_1 y_1, \ldots, \beta_s y_s \right) \in T_2 \right. \right\}, \tag{16}
\]

is called the generalized multiplicative directional distance function (GMDDF).
The GMDDF possesses the following properties:

**P.1 (Representation).** \( \tilde{M}_G(\cdot) \) completely characterizes \( T_2 \), i.e., \( \tilde{M}_G(x, y; \cdot; g^-, g^+) \geq 1 \) if and only if \( (x, y) \in T_2 \).

**P.2 (Homogeneity).** \( \log \tilde{M}_G(x, y; \cdot; g^-, g^+) \) is homogeneous of degree minus one in the direction vector \( (g^-, g^+) \), i.e.,
\[
\frac{1}{\log \tilde{M}_G(x, y; \cdot; g^-, g^+)} \quad \text{is homogeneous of degree minus one in} \quad (g^-, g^+),
\]
\[
\tilde{M}_G(x, y; \cdot; g^-, g^+) = \varphi^{-1} \log \tilde{M}_G(x, y; \cdot; g^-, g^+), \quad \varphi > 0.
\]

**P.3 (Almost Homogeneity or Log-Translation).** The GMDDF is almost homogeneous (Cuesta, Lovell, & Zofio, 2007; Cuesta & Zofio, 2005) of degree \( (g^-, g^+, -2) \), i.e.,
\[
\tilde{M}_G\left(\mu^{g^-, g^+} x, \mu^{g^-, g^+} y; g^-, g^+\right) = \mu^{-2} \tilde{M}_G(x, y; g^-, g^+), \quad \mu > 0.
\]
This property states that if one transforms \((x, y)\) into \((\mu^{g^-, g^+} x, \mu^{g^-, g^+} y)\), then the value of \( \tilde{M}_G(\cdot) \) will be divided by the scalar \( \mu^{-2} \).

**P.4 (Multiplicative Concavity).** \( \tilde{M}_G(\cdot) \) is a multiplicative concave function on \( T_2 \), i.e., \( \log \tilde{M}_G(\cdot) \) is a concave function over \( \log T_2 \).

The proofs of Properties P.1–P.4 can be found in Appendix A.

For any given \( g \), \( \tilde{M}_G(\cdot) \) can be computed for DMU_0 by setting up the following NLP problem:
\[
\tilde{M}_G(x_o, y_o; g^-, g^+) = \max \left( \prod_{i=1}^{m} \alpha_i \right)^{\frac{1}{m}} \times \left( \prod_{j=1}^{s} \beta_j \right)^{\frac{1}{s}}
\]
\[
s.t. \quad \prod_{j \in J} x_{ij}^{\lambda_j} \leq \alpha_i \cdot x_{io}, \quad i = 1, \ldots, m,
\]
\[
\prod_{j \in J} y_{ij}^{\lambda_j} \geq \beta_i \cdot y_{io}, \quad r = 1, \ldots, s,
\]
\[
\alpha_i \geq 1, \quad \beta_j \geq 1, \quad \forall i, \quad \forall r,
\]
\[
\lambda \in \Lambda.
\]

Using the natural logarithmic transformation, we transform the NLP (18) into the following equivalent LP problem.
\[
\log \tilde{M}_G \left( x_o, y_o; -g^-, g^+ \right) = \max \frac{1}{m} \sum_{i=1}^{m} \log \alpha_i + \frac{1}{s} \sum_{r=1}^{s} \log \beta_r,
\]
\[
\text{s.t. } \sum_{j \in J} \lambda_j \log x_{ij} \leq \log x_{io} - g_i \log \alpha_i, \quad i = 1, \ldots, m,
\]
\[
\sum_{j \in J} \lambda_j \log y_{jr} \geq \log y_{ro} + g_j \log \beta_j, \quad r = 1, \ldots, s,
\]
\[
\log \alpha_i \geq 0, \quad \log \beta_j \geq 0, \quad \forall i, \forall r, \lambda \in \Lambda.
\]

Let \((\log \alpha_i^*, \log \beta_j^*, \lambda_j^*, \forall i, r, j)\) is an optimal solution to (19). Then, since the constraints of (18) and (19) are equivalent, \((\alpha_i^*, \beta_j^*, \lambda_j^*, \forall i, r, j)\) is an optimal solution to (18).

Obviously, \((\alpha_i = 1, \beta_j = 1, \forall i, r, \lambda = 0, j \neq o)\) is a feasible solution to (18). This follows that \(\tilde{M}_G (\cdot) \geq 1\). Similar to the MDDF-based TE measure, we define the multiplicative inverse of \(\tilde{M}_G (\cdot)\) as the GMDDF-based TE measure, i.e.,
\[
E_o^G (g) := \frac{1}{\tilde{M}_G \left( x_o, y_o; -g^-, g^+ \right)}.
\]

Since \(\tilde{M}_G (\cdot) \geq 1\), this measure satisfies the property of efficiency requirement, i.e.,
\[
0 < E_o^G (g) \leq 1.
\]

**Definition 3.1.2.** DMU\(_o\) is GMDDF-efficient (strong log-efficient) if and only if \(E_o^G (g) = 1\).

The GMDDF-based efficiency has a straightforward interpretation. If one expresses (20) as
\[
E_o^G (g) = \left( \prod_{i=1}^{m} \frac{1}{\alpha_i^*} \right)^\frac{1}{m} \times \left( \prod_{r=1}^{s} \frac{1}{\beta_r^*} \right)^\frac{1}{s},
\]
then the GMDDF-based efficiency can be interpreted as the product of the geometric means of the input and output efficiencies, respectively: \(E_i^G (g) = \left( \prod_{i=1}^{m} \frac{1}{\alpha_i^*} \right)^\frac{1}{m}\) and \(E_o^G (g) = \left( \prod_{r=1}^{s} \frac{1}{\beta_r^*} \right)^\frac{1}{s}\).

Based on the optimal solutions of the NLP (18), we have the following proposition.

**Proposition 3.1.1.** \(E_o^G (g) = 1\) if and only if \(\alpha_i^* = 1, \quad i = 1, \ldots, m,\) and \(\beta_j^* = 1, \quad r = 1, \ldots, s\).
Note that all of the input and output inequality constraints in (18) (and (19)) are satisfied at the optimality, and hence, can be replaced by equalities without affecting the optimal value.

Now, we have the following theorem.

**Theorem 3.1.1.** DMU<sub>o</sub> is Pareto-efficient if and only if it is GMDDF-efficient.

**Proof.** Let \((\alpha_i^*, \beta_i^*, \lambda_j^*, \forall i, r, j)\) be an optimal solution to the NLP (18) for DMU<sub>o</sub>. According to Proposition 3.1.1, the statement is tantamount to proving that DMU<sub>o</sub> is Pareto-efficient in \(T_2\) if and only if \(\alpha_i^* = 1, i = 1, ..., m\), and \(\beta_r^* = 1, r = 1, ..., s\).

We, first, prove that if DMU<sub>o</sub> is Pareto-efficient in \(T_2\), then \(\alpha_i^* = 1, i = 1, ..., m\), and \(\beta_r^* = 1, r = 1, ..., s\). By contradiction, we assume that there exists \(i'\) or \(r'\) such that \(\alpha_{i'}^* > 1\) or \(\beta_{r'}^* > 1\). We then have \(\hat{x}_{i'} := (\alpha_i^*)^{-1} x_{i'} \leq x_{i'}, i = 1, ..., m\), and \(\hat{y}_{r} := (\beta_r^*)^{-1} y_{r} \geq y_{r}, r = 1, ..., s\), such that for \(i'\) or \(r'\), the previous inequality is strict. As mentioned earlier, all of the input and output constraints in the NLP (18) are active at the optimality. This follows that \((\hat{x}, \hat{y}) \in T_2\). Thus, \((\hat{x}, \hat{y})\) dominates DMU<sub>o</sub>, thus leading to the contradiction that DMU<sub>o</sub> is Pareto-efficient.

We now prove that if \(\alpha_i^* = 1, i = 1, ..., m\), and \(\beta_r^* = 1, r = 1, ..., s\), then DMU<sub>o</sub> is Pareto-efficient in \(T_2\). By contradiction, let us assume that there exists \((\hat{x}, \hat{y}) \in T_2\) with \((\hat{x}, \hat{y}) \neq (x_o, y_o)\) such that \(\hat{x}_i \leq x_{i'}, i = 1, ..., m\), and \(\hat{y}_r \geq y_{r}, r = 1, ..., s\). Thus, we have \(\hat{\delta}_i := \frac{x_{i'}}{\hat{x}_i} \geq 1, i = 1, ..., m\), and \(\hat{\phi}_r := \frac{\hat{y}_r}{y_{r}} \geq 1, r = 1, ..., s\), such that there exists some \(i'\) or \(r'\) with \(\hat{\delta}_i > 1\) (\(\hat{\phi}_r > 1\)). Since \((\hat{x}, \hat{y}) \in T_2\), there exists \(\hat{\lambda} \in \Lambda\), such that \(\prod_{i\in J} x_{i'}^\lambda \leq \hat{x}_i, i = 1, ..., m\), and \(\prod_{r \in J} y_{r}^\lambda \geq \hat{y}_r, r = 1, ..., s\). Now, we define \(\hat{\alpha}_i := \exp \left(\frac{1}{g_i} \log \hat{\delta}_i\right), i = 1, ..., m\), and \(\hat{\beta}_r := \exp \left(\frac{1}{g_r} \log \hat{\phi}_r\right), r = 1, ..., s\). Then, it is easy to check that \((\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \forall i, r, j)\) is a feasible solution to the NLP (10) for DMU<sub>o</sub>. Moreover, the value of
objective function corresponding to this solution is greater than that of \((\alpha_i^+, \beta_i^+, \lambda_j^+, \forall i, r, j)\), which is clearly a contradiction.

Theorem 3.1.1 reveals that the GMDDF declares a DMU fully efficient if and only if it operates on the strongly efficient subset of \(T_2, \partial^S (T_2)\).

Unlike \(E_o (g), \ E_o^G (g)\) is “complete” in the sense of Cooper, Park, & Pastor (1999) as it takes into account all possible non-zero exponential slacks as the additional sources of inefficiency. Therefore, the following relationship holds:

\[
E_o^G (g) \leq E_o (g). \tag{23}
\]

The inequality (23) reveals the higher discriminatory power of the GMDDF over the MDDF.

It is worth noting that the NLP (18) turns to the invariant multiplicative model of Charnes, et al. (1983) when both the input and output direction vectors take on unity values, i.e.,

\[
g_i^- = 1, \ g_i^+ = 1, \ i = 1, \ldots, m, \ r = 1, \ldots, s. \tag{24}
\]

3.2 Properties of the GMDDF

We now turn to investigate several useful properties of the GMDDF-based TE measure. While some of these are precisely the minimum requirement conditions for an efficiency measure (Cooper, et al., 1999; Sueyoshi & Sekitani, 2009; Tone, 2001), others are related to the special structure of GMDDF-based measure, and are important from both theoretical and practical viewpoints. To facilitate the discussion, we first provide a precise definition of each property and then verify whether the GMDDF-based measure of efficiency satisfies it.

We provide a formal definition of the reference set dependence (Cooper, Seiford, & Tone, 2007) of an efficiency measure:

**Definition 3.2.1.** An efficiency measure is called reference set dependent if its optimal value is not dependent on the whole data set but on a subset of DMUs, known as the peers or reference set to the DMU under evaluation.

Based on this definition, we find \(E_o^G (g)\) to be reference set dependent.
We extend the definition of weak and strong monotonicity axioms by Färe & Lovell (1978) to the multiplicative DEA measures as follows:

**Definition 3.2.2.** A multiplicative DEA measure is said to be *weakly monotonic* if by changing, \( x_{io} \) to \( \delta x_{io} \) for any \( i \) with \( \delta > 1 \), or \( y_{ro} \) to \( \gamma y_{ro} \) for any \( r \) with \( \gamma < 1 \), holding all other inputs and outputs constant, the efficiency (inefficiency) score of DMU\(_o\) is not increased (increased).

**Definition 3.2.3.** A multiplicative DEA measure is said to be *strongly monotonic* if by changing, \( x_{io} \) to \( \delta x_{io} \) for any \( i \) with \( \delta > 1 \), or \( y_{ro} \) to \( \gamma y_{ro} \) for any \( r \) with \( \gamma < 1 \), holding all other inputs and outputs constant, the efficiency (inefficiency) score of DMU\(_o\) is decreased (increased).

We now have the following theorem.

**Theorem 3.2.1.** For any given \( g \), \( E^G_o (g) \) is strongly monotonic.

**Proof.** See Appendix A.

Now, we show that the GMDDF-based TE score is not influenced by units in which inputs and outputs are measured, i.e., the GMDDF is *units (scale) invariant* (Cooper, et al., 1999; Lovell & Pastor, 1995; Seiford & Zhu, 1998; Sueyoshi & Sekitani, 2009). According to Seiford & Zhu (1998), the unit invariance property of a multiplicative efficiency measure in the original input-output space is equivalent to the translation invariance property of this measure in the log-transformed space, i.e., log-translation invariant, and vice versa.

**Theorem 3.2.2.** \( E^G_o (g) \) is unit invariant as long as the specified direction vector is not affected by the change in the units of measurement of the data.

**Proof.** See Appendix A.

As an immediate consequence, the GMDDF-based measure is unit invariant for any fixed direction vector (e.g., \( g \) in (24)). Furthermore, it is unit invariant in the case of the following non-fixed direction vectors that satisfy Theorem 3.2.2:
\begin{align}
  g_i^- &= \log x_{i\omega} - \min_{j \in J} \{ \log x_{ij} \}, \quad g_r^+ = \max_{j \in J} \{ \log y_{ij} \} - \log y_{r\omega}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s; \quad (25) \\
  g_i^- &= \max_{j \in J} \{ \log x_{ij} \} - \min_{j \in J} \{ \log x_{i\omega} \}, \quad g_r^+ = \max_{j \in J} \{ \log y_{ij} \} - \min_{j \in J} \{ \log y_{r\omega} \}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \quad (26)
\end{align}

Similar to the translation invariance property of the traditional DEA measures (see, e.g., Ali & Seiford, 1990; Cooper, et al., 1999; Lovell & Pastor, 1995; and Sueyoshi & Sekitani, 2009, among others), we now introduce its analogous equivalent in the case of multiplicative DEA measures.

**Definition 3.2.4.** A multiplicative efficiency measure is said to be *translation invariant* if changing the original data of, the $i$th input, i.e., $x_{ij}$, $i = 1, \ldots, m$, to $x_{ij}^c$, with $c_i > 0$ and the $r$th output, i.e., $y_{rj}$, $r = 1, \ldots, s$, to $y_{rj}^d$, with $d_r > 0$, for $j \in J$, will not affect the solution set or alter the value of objective function.

As regards the GMDDF-based TE measure, we have the following theorem.

**Theorem 3.2.3.** Let the direction vector $g$ be such that $g_i^-, \quad i = 1, \ldots, m$, and $g_r^+ = \max_{j \in J} \{ \log y_{ij} \}$, $i = 1, \ldots, m$, and $r = 1, \ldots, s$, have, respectively, the same units of measurement of the $i$th logarithmic input and the $r$th logarithmic output. Then, $E_o^G (g)$ is translation invariant.

**Proof:** See Appendix A.

As an immediate consequence of Theorem 3.2.3, the translation invariance property of a multiplicative measure with respect to the original inputs and outputs implies the unit invariance property of its corresponding logarithmic based measure with respect to the logarithmic inputs and outputs, i.e., log-unit invariant, and vice versa. Thus, $E_o^G (g)$ is not translation invariant for any fixed direction vector. This follows that the output-oriented multiplicative measure of Banker & Maindiratta (1986) and the invariant multiplicative measure of Charnes, et al. (1983) are not translation invariant.

In addition to (25) and (26), the following direction vectors satisfy Theorem 3.2.3:

\begin{align}
  g_i^- &= \log x_{i\omega}, \quad g_r^+ = \log y_{r\omega}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s; \quad (27) \\
  g_i^- &= \max_{j \in J} \{ \log x_{ij} \}, \quad g_r^+ = \max_{j \in J} \{ \log y_{ij} \}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \quad (28)
\end{align}
Although the direction vector $g$ was assumed to be strictly positive in Definitions 2.2.1 and 3.1.1, some data may take on less than unity values in any empirical application, in which case one should be careful of using the associated direction vectors. For example, consider the direction - (27) or (28).

Recently, Färe & Grosskopf (2013) have discussed a general data transformation while measuring the DDF-based efficiency in the VRS framework, which they call it positive, affine data transformation. We offer a similar definition in the case of multiplicative DEA measures.

**Definition 3.2.5.** A multiplicative DEA measure is said to be positive, affine invariant if changing the original data of the $i$th input, i.e., $x_{ij}$, $i = 1, \ldots, m$, to $a_i x_{ij}$ with $a_i, c_i > 0$, and the $r$th output, i.e., $y_{rj}$, $r = 1, \ldots, s$, to $b_r y_{rj}$ with $b_r, d_r > 0$, for $j \in J$ will not affect the solution set or alter the value of objective function.

Note that $E^G_o(g)$ is positive, affine invariant as long as the specified direction vector simultaneously satisfies the conditions of Theorems 3.2.2 and 3.2.3. For example, consider the directions - (25) and (26).

### 4. An illustrative empirical application

For an illustrative empirical illustration, we analyze sample data of 20 hardware computer companies operating in India over the period 2001-2010, which was originally used and analyzed in an earlier study by Sahoo, Kerstens & Tone (2012). The data consists of one output, i.e., gross sales, and three inputs - manufacturing cost, overhead cost and maintenance cost.

The computer hardware market comprises Indian branded players, MNC players, and assembly players. To introduce efficiency and competition, the government undertook a series of policy measures that were aimed at developing the Indian information technology (IT) industry as a major global player, and to bring benefits of IT to every walk of life. The informational contents of our efficiency estimates are not only useful to policy makers in evaluating the outcomes of their reforms, but also to regulators who need to understand and monitor the consequences of their regulations. This analysis enables us to establish a connection between these reforms and efficiency performance.

We now present our discussion concerning only average TE performance of the computer hardware companies using the MDDF- and GMDDF-based measures over the years across models.
However, the detailed estimates on these are available in Appendix B (Table B.1 (a), Table B.1 (b), Table B.2 (a), Table B.2 (b)). For empirical implementation of these efficiency measures, we have used the direction vectors (25) and (26). To carry out all the computations in this example, we have developed a computer program given in Appendix C, using the GAMS software.

The 10-year average TE scores of companies across two models and across two direction vectors are all exhibited in Table 2. In the case of direction vector (25), while MDDF-based TE scores varies from 0.965 to 1 with a standard deviation of 1.07 percent, the GMDDF-based ones range from 0.774 to 1 with a standard deviation of 7.48 percent. This finding reveals the higher discriminatory power of the latter measure over the former one. Besides three companies that are found efficient in both the measures, the GMDDF-based measure exhibit substantially lower TE scores as compared to the MDDF-based one. This finding explains the evidence of slacks present in the latter measure.

In the case of direction vector (26), the efficiency variation as measured by the standard deviation is reduced, from 1.07 percent to 0.58 percent in the case of MDDF-based measure (as TE varies from 0.981 to 1); and from 7.48 percent to 4.14 percent in the case of GMDDF-based measure (as TE ranges from 0.857 to 1). The finding of higher average TE in both of the measures holds as well at the level of individual companies. This is purely due to the choice of direction vector. One can therefore infer that no matter which direction vector is chosen, the MDDF-based measure always overestimates the TE scores of companies suffering from slacks; and the choice of direction vector plays an important role in explaining the efficiency differential in each efficiency measure, i.e., the second direction vector always yields a higher TE score than the first one.

Now, we turn to present year-wise companies’ average TE across models to examine how hardware computer industry has performed over the years (see Table 3 and Fig.3). Since the MDDF-based TE estimates have upward bias in the presence of slacks, we concentrate only on the GMDDF-based estimates with respect to the direction vectors – (25) and (26).

From both Table 3 and Fig.3, it can be noted that the hardware computer industry has not progressed in becoming continuously efficient over time, if a monotonically increasing time trend is taken to indicate progress. Since the industry average performance trend is one of decline and rise, there is a significant heterogeneity visible in the TE patterns not only for the industry as a whole but also for the individual companies within each of these 10 years.
Table 2. Average TE of hardware computer companies over years across models

<table>
<thead>
<tr>
<th>Companies</th>
<th>Direction vector (25)</th>
<th>Direction vector (26)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MDDF</td>
<td>GMDDF</td>
</tr>
<tr>
<td>A C I Infocom Ltd.</td>
<td>0.985</td>
<td>0.847</td>
</tr>
<tr>
<td>Abee Info-Consumables Ltd.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Accel Transmatic Ltd.</td>
<td>0.986</td>
<td>0.961</td>
</tr>
<tr>
<td>C C S Infotech Ltd.</td>
<td>0.985</td>
<td>0.812</td>
</tr>
<tr>
<td>C M S Computers Ltd.</td>
<td>0.983</td>
<td>0.9</td>
</tr>
<tr>
<td>Compuage Infocom Ltd.</td>
<td>0.997</td>
<td>0.931</td>
</tr>
<tr>
<td>Computer Point Ltd.</td>
<td>0.982</td>
<td>0.8</td>
</tr>
<tr>
<td>Dynamos Systems &amp; Solutions Ltd.</td>
<td>0.985</td>
<td>0.845</td>
</tr>
<tr>
<td>Gemini Communication Ltd.</td>
<td>0.977</td>
<td>0.832</td>
</tr>
<tr>
<td>Lipi Data Systems Ltd.</td>
<td>0.971</td>
<td>0.799</td>
</tr>
<tr>
<td>Moser Baer India Ltd.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>P C S Technology Ltd.</td>
<td>0.99</td>
<td>0.931</td>
</tr>
<tr>
<td>Saarc Net Ltd.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Savex Computers Ltd.</td>
<td>0.999</td>
<td>0.987</td>
</tr>
<tr>
<td>Smartlink Network Systems Ltd.</td>
<td>0.981</td>
<td>0.853</td>
</tr>
<tr>
<td>T V S Electronics Ltd.</td>
<td>0.986</td>
<td>0.867</td>
</tr>
<tr>
<td>V X L Instruments Ltd.</td>
<td>0.965</td>
<td>0.774</td>
</tr>
<tr>
<td>Vintron Informatics Ltd.</td>
<td>0.972</td>
<td>0.872</td>
</tr>
<tr>
<td>X O Infotech Ltd.</td>
<td>0.968</td>
<td>0.806</td>
</tr>
<tr>
<td>Zenith Computers Ltd.</td>
<td>0.986</td>
<td>0.876</td>
</tr>
</tbody>
</table>

Table 3. Year-wise average TEs of hardware computer companies across models

<table>
<thead>
<tr>
<th>Direction vector</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direction vector (25)</td>
<td>MDDF</td>
<td>0.984</td>
<td>0.953</td>
<td>0.981</td>
<td>0.986</td>
<td>0.994</td>
<td>0.989</td>
<td>0.987</td>
<td>0.991</td>
<td>0.991</td>
</tr>
<tr>
<td></td>
<td>GMDDF</td>
<td>0.878</td>
<td>0.816</td>
<td>0.875</td>
<td>0.874</td>
<td>0.913</td>
<td>0.879</td>
<td>0.863</td>
<td>0.899</td>
<td>0.921</td>
</tr>
<tr>
<td>Direction vector (26)</td>
<td>MDDF</td>
<td>0.992</td>
<td>0.973</td>
<td>0.99</td>
<td>0.993</td>
<td>0.997</td>
<td>0.995</td>
<td>0.994</td>
<td>0.995</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>GMDDF</td>
<td>0.932</td>
<td>0.887</td>
<td>0.929</td>
<td>0.941</td>
<td>0.962</td>
<td>0.942</td>
<td>0.942</td>
<td>0.957</td>
<td>0.957</td>
</tr>
</tbody>
</table>

A year-specific analysis shows that the performance pattern shows resurgence just after one year of declining performance from 2001 to 2002. Fortunately, this resurgence has continued up to 2005. The industry performance shows again a declining performance until 2007 after which the
performance shows an increasing trend till the end of our study period. This continuous increase in performance in the later years seems to indicate that the policy measures recently undertaken by the government to introduce efficiency and competition, are perhaps exhibiting effective results on the ground.

Fig. 3. Companies’ average TE over years

[Note: GMDDF (1) is based on (25), and GMDDF (2) is based on (26)]

5. Concluding remarks

The analysis of efficiency relies extensively on the types of convexity postulate used in the construction of non-parametric technology. Two different characterizations of the non-parametric technology – piece-wise linear and piece-wise log-linear – are generally used to estimate technical efficiency. While the linear technology requires normal convexity, the log-linear one uses geometric convexity. The latter technology is flexible enough to accommodate three production structures – convexity, linearity, and concavity, which are somewhat more consistent with the empirical description of several real-life production processes.

Based on the log-linear technology, two multiplicative DEA models are suggested to describe several real-life production correspondences. The first one is the MDDF of Peyrache & Coelli (2009). However, if slacks are considered important in revealing proper efficiency behavior, then this model produces an upward bias in its efficiency estimates as it does not consider the input and output
slacks as the additional sources of inefficiency. The second model is our proposed GMDDF, which provides a comprehensive measure of efficiency. The GMDDF-based measure of efficiency satisfies several desirable properties of an ideal efficiency measure such as strong monotonicity, unit invariance, translation invariance, and positive affine transformation invariance.

Both the MDDF- and GMDDF-based efficiency measures can be empirically applied as they can be easily implemented in any standard DEA software due to flexible computer programming, and they provide the decision makers with the option of specifying the preferable direction vectors to incorporate their decision-making preferences while evaluating efficiency. Furthermore, they, usually, include the conventional multiplicative DEA measures as special cases.

As regards the debate on the preferred choice between the MDDF- and GMDDF-based efficiency measures, we take no position. While the GMDDF-based efficiency measure is non-radial in nature, the MDDF-based measure is radial. The issue of which measure to use in any empirical application depends on whether the slacks are viewed as additional sources of inefficiency, in which case the GMDDF can be advantageous. However, the MDDF takes the lead if the proportional interpretation for the class of log-convex monotonic technologies that facilitates the utilization of its benchmarking results in managerial context is important.

The current study points to avenue for future research in three directions. Investigation of local returns to scale is crucial in some regions: the non-concave region of production function (DMU₁-DMU₅-DMU₂) and the non-convex region (DMU₃-DMU₄) in Fig.1 (a). This is because in these regions the piece-wise log-linear technology can help revealing all possible types of returns to scale as against only increasing/decreasing returns to scale by the linear technology in Fig.1 (c). Therefore, based on the work of Fukuyama (2003), the extension of the prior research results on returns to scale in the multiplicative models (Banker & Maindiratta, 1986; Banker, Cooper, Thrall, Seiford, & Zhu, 2004; Zarepisheh, Khorram, & Jahanshahloo, 2010) can be made to develop an MDDF-based technique to determine local returns to scale. Second, in the spirit of Chambers, et al. (1998), an investigation of the duality of the MDDF should be made to define a multiplicative profit efficiency measure, and then establishes its linkage with the MDDF. Third, in the spirit of Emrouznejad & Cabanda (2010), an attempt can be made to develop a directional multiplicative corporate performance model.

Notes

1 This felicitous reformulation was suggested by an anonymous referee.
Acknowledgements

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References


**Appendix A**

**Proof of P.1.** Clearly, if \((x, y)\in T_2\), then \(\bar{M}_G (x, y; -g^-, g^+) \geq 1\). Conversely, assume that \(\bar{M}_G (x, y; -g^-, g^+) \geq 1\). By Definition 3.1.1, we have \((\alpha_1^{g^+} x_1, ..., \alpha_m^{g^+} x_m, \beta_1^{g^+} y_1, ..., \beta_s^{g^+} y_s) \in T_2\), and \((x_1, ..., x_m, -y_1, ..., -y_s) \geq (\alpha_1^{g^+} x_1, ..., \alpha_m^{g^+} x_m, -\beta_1^{g^+} y_1, ..., -\beta_s^{g^+} y_s)\). Then, the result immediately follows from the free disposability assumption on \(T_2\) in A.3.  

\[\square\]
Proof of P.2. Similar to the relationship (8), we have

\[
\log \tilde{M}_G(x, y; -g^-, g^+) = \sup \left\{ \frac{1}{m} \log \alpha + \frac{1}{s} \log \beta \left| \begin{array}{c}
\log x - g^- \otimes \log \alpha, \\
\log y + g^+ \otimes \log \beta 
\end{array} \right| \in \mathbb{T}_2, \log \alpha \geq 0_m, \log \beta \geq 0_s \right\},
\]

where \( \log \alpha = (\log \alpha_1, \ldots, \log \alpha_m) \), \( \log \beta = (\log \beta_1, \ldots, \log \beta_s) \); \( 1_m(0_m) \) and \( 1_s(0_s) \) are the unity (zero) vectors of appropriate orders; \( \cdot \) and \( \otimes \) respectively stand for the inner and component-wise multiplication of vectors.

According to (A1), the result follows by changing variables \( \delta_i := \alpha_i^\varphi, \ i = 1, \ldots, m \), and \( \rho_r := \beta_r^\varphi \), \( r = 1, \ldots, s \), in \( \log \tilde{M}_G(x, y; -\varphi g^-, \varphi g^+) \).

\[\Box\]

Proof of P.3. From (A1), we have

\[
\log \tilde{M}_G(\mu^{-\varphi} x, \mu^\varphi y; -g^-, g^+) = \sup \left\{ \frac{1}{m} \log \alpha + \frac{1}{s} \log \beta \left| \begin{array}{c}
\log x - g^- \otimes \log \mu \mu - g^- \otimes \log \alpha, \\
\log y + g^+ \otimes \log \mu \mu + g^+ \otimes \log \beta 
\end{array} \right| \in \mathbb{T}_2, \log \alpha \geq 0_m, \log \beta \geq 0_s \right\}
\]

\[= \sup \left\{ \frac{1}{m} \log \alpha + \frac{1}{s} \log \beta \left| \begin{array}{c}
\log x - g^- \otimes \log \mu \alpha, \\
\log y + g^+ \otimes \log \mu \beta 
\end{array} \right| \in \mathbb{T}_2, \log \alpha \geq 0_m, \log \beta \geq 0_s \right\}.
\]

Thus, the result follows immediately by changing variables \( \theta_i := \mu \alpha_i, \ i = 1, \ldots, m \), and \( \delta_r := \mu \beta_r \), \( r = 1, \ldots, s \).

\[\Box\]

Proof of P.4. Assume that \((x_1, y_1)\) and \((x_2, y_2)\) are two arbitrary points in \(\mathbb{T}_2\).

From (A1), we have

\[
\left( \log x^1 - g^- \otimes \log \alpha^\varphi, \log y^1 + g_1^+ \otimes \log \beta^\varphi \right) \in \log \mathbb{T}_2,
\]

\[
\left( \log x^2 - g^- \otimes \log \alpha^\varphi, \log y^2 + g_2^+ \otimes \log \beta^\varphi \right) \in \log \mathbb{T}_2,
\]

where \( \left( \log \alpha^\varphi, \log \beta^\varphi \right) \) and \( \left( \log \alpha^\varphi, \log \beta^\varphi \right) \) are respectively the optimal solutions to (A1) for the points \((x^1, y^1)\) and \((x^2, y^2)\). Then, from the convexity of \(\log \mathbb{T}_2\), each convex combination of the points in (A3) lies in \(\log \mathbb{T}_2\). That is,
\[
\begin{align*}
\Big( \theta \log x^1 + (1-\theta) \log x^2 - g^- \otimes \Big( \theta \log \alpha^1 + (1-\theta) \log \alpha^2 \Big), \\
\theta \log y^1 + (1-\theta) \log y^2 + g^+_1 \otimes \Big( \theta \log \beta^1 + (1-\theta) \log \beta^2 \Big) \Big) \in \log T_2, \\
(A4)
\end{align*}
\]
for any \( \theta \) between 0 and 1, i.e., \( 0 \leq \theta \leq 1 \).

Thus, if \( \frac{1}{m} \log \alpha^* + \frac{1}{s} \log \beta^* \) is the optimal objective of (A1) for a point such as (A4), then
\[
\frac{1}{m} \log \alpha^* + \frac{1}{s} \log \beta^* \leq \frac{1}{m} \log \alpha^*_1 + \frac{1}{s} \log \beta^*_1.
\]

Or, equivalently
\[
\theta \left( \frac{1}{m} \log \alpha^*_1 + \frac{1}{s} \log \beta^*_1 \right) + \left( 1-\theta \right) \left( \frac{1}{m} \log \alpha^*_2 + \frac{1}{s} \log \beta^*_2 \right) \leq \frac{1}{m} \log \alpha^* + \frac{1}{s} \log \beta^*.
\]

This follows
\[
{\left[ \prod_{i=1}^{m} \alpha^*_{i_1} \right]}^{-\theta} \times {\left( \prod_{r=1}^{s} \beta^*_{r_1} \right)}^{\theta} \times {\left[ \prod_{i=1}^{m} \alpha^*_{i_2} \right]}^{\theta} \times {\left( \prod_{r=1}^{s} \beta^*_{r_2} \right)}^{\theta} \leq \prod_{i=1}^{m} \alpha^* \times \prod_{r=1}^{s} \beta^* \quad (A7)
\]
which can be alternatively expressed as
\[
\bar{M}_G \left( x_1, y_1 ; g^- , g^+ \right)^{\theta} \times \bar{M}_G \left( x_2, y_2 ; g^- , g^+ \right)^{(1-\theta)} \leq \bar{M}_G \left( x_1^{\theta}, y_1^{\theta} ; g^- , g^+ \right) \quad (A8)
\]
This completes the proof.

**Proof of Theorem 3.2.1.** For a direction vector \( g \), let \( \left( \alpha^*_i, \beta^*_j, \lambda^*_i, \forall i, r, j \right) \) be an optimal solution to the NLP (18) with the optimal objective value \( \left( \prod_{i=1}^{m} \alpha^* \right)^\theta \times \left( \prod_{r=1}^{s} \beta^* \right)^{\theta} \) when DMU \( o \) is evaluated. To establish the strong monotonicity property of \( E^G_o (g) \), we rate two units, which have the same values for all of their inputs and outputs excepting one. Without less of generality, we prove the theorem for the case in which two units differ in one input. Let \( \text{DMU}_p = \left( x_p, y_p \right) \) is a hypothetical point as given by \( \left( x_{i_1}, \ldots, x_{(h-1)_o}, \delta x_{i_o}, x_{(h+1)_o}, \ldots, x_{m_o}, y_{i_o}, \ldots, y_{s_o} \right) \) with \( \delta > 1 \). We have to show that the value of \( \bar{M}_G \left( x, y ; g^- , g^+ \right) \) for DMU \( p \) is greater than that for DMU \( o \). Obviously,
\( \alpha'_h = \frac{\alpha^*_j}{\delta}, \alpha'_i = \alpha^*_i, \forall i \neq h, \beta'_r = \beta^*_r, \lambda'_j = \lambda^*_j, \forall r, j \) is a feasible solution to the NLP (18) for DMU \( p \) with the objective value of \( \frac{1}{\delta} \prod_{i=1}^{m} \alpha^*_i \times \prod_{r=1}^{s} \beta^*_r \). Since \( \delta > 1 \) and the NLP (18) is a maximization problem, the optimal objective value for DMU \( p \) is greater than that for DMU \( o \), thus completing the proof.

**Proof of Theorem 3.2.2.** Let \( (x'_j, y'_j), j \in J \), be a transformed input-output vector of DMU \( j \) given by:

\[
x'_j = c_i x_j, \quad y'_j = d_r y_j; \quad i = 1, \ldots, m, \quad r = 1, \ldots, s,
\]

where \( c_i, i = 1, \ldots, m \), and \( d_r, r = 1, \ldots, s \), are any collection of positive constants. According to the assumption of the theorem, we have

\[
g'_i = g_i^-, \quad g'_r = g_r^+; \quad i = 1, \ldots, m, \quad r = 1, \ldots, s,
\]

where \( g' \) is the new direction corresponding to the transformed data. By substituting (A9) in the constraints of the NLP (18), we have

\[
\prod_{j \in J} c_i^{\lambda_i} \times \prod_{j \in J} y_j^{\lambda_j} \leq \alpha^*_i c_i x_i, \quad i = 1, \ldots, m,
\]

\[
\prod_{j \in J} d_r^{\lambda_r} \times \prod_{j \in J} y_j^{\lambda_j} \geq \beta^*_r d_r y_r, \quad r = 1, \ldots, s.
\]

Since \( \lambda \in \Lambda \), by dividing both sides of the \( i \)th input constraint by \( c_i, i = 1, \ldots, m \), and the \( r \)th output constraint by \( d_r, r = 1, \ldots, s \), we retrieve the constraints of model (18), so the proof is complete.

**Proof of Theorem 3.2.3.** Let \( (x'_j, y'_j), j \in J \), be a transformed input-output vector of DMU \( j \) given by:

\[
x'_j = x'_j, \quad y'_j = y'_j; \quad i = 1, \ldots, m, \quad r = 1, \ldots, s,
\]

where the \( c_i \) and \( d_r \) are any collection of positive constants. Then, we have

\[
\log x'_j = \log x'_j = c_i \log x_j, \quad \log y'_j = \log y'_j = d_r \log y_j; \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.
\]
for any \( j \in J \). According to the assumption of the theorem, we have
\[
g_i' = c_i g_i', \quad d_i g_i' = d_i g_i'; \quad i = 1, \ldots, m, \quad r = 1, \ldots, s,
\]
where \( g' \) is the new direction corresponding to the transformed data. Now, substitution of (A12), (A13) and (A14) in the constraints of NLP (18) yields:
\[
\prod_{j \in J} x_{i_j j}^{c_i h_j} \leq \alpha_i^{-c_i h_j} x_{i_0}^{c_i}, \quad i = 1, \ldots, m,
\]
\[
\prod_{j \in J} y_{d_j j}^{d_i h_j} \geq \beta_i^{d_i h_j} y_{d_0}^{d_i}, \quad r = 1, \ldots, s,
\]
\[\lambda \in \Lambda.\]

These constraints can be equivalently expressed as
\[
\left( \prod_{j \in J} x_{i_j j}^{\lambda_i} \right)^{c_i} \leq \left( \alpha_i^{-c_i h_j} x_{i_0} \right)^{c_i}, \quad i = 1, \ldots, m,
\]
\[
\left( \prod_{j \in J} y_{d_j j}^{\lambda_i} \right)^{d_i} \geq \left( \beta_i^{d_i h_j} y_{d_0} \right)^{d_i}, \quad r = 1, \ldots, s,
\]
\[\lambda \in \Lambda.\]

Since \( c_i, \ i = 1, \ldots, m \), and \( d_r, \ r = 1, \ldots, s \), are all positive constants, the above constraints are mathematically equivalent to the constraints of (18), so the proof is complete. \(\blacksquare\)