The Blow-up Rate Estimates for a System of Heat Equations with Nonlinear Boundary Conditions

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September 29, 2012

Abstract

This paper deals with the blow-up properties of positive solutions to a system of two heat equations

\[ u_t = \Delta u, \quad v_t = \Delta v \]

in \( B_R \times (0, T) \) with Neumann boundary conditions

\[ \frac{\partial u}{\partial \eta} = e^{v^p}, \quad \frac{\partial v}{\partial \eta} = e^{u^q} \]

on \( \partial B_R \times (0, T) \), where \( p, q > 1 \), \( B_R \) is a ball in \( \mathbb{R}^n \), \( \eta \) is the outward normal. The upper bounds of blow-up rate estimates were obtained. It is also proved that the blow-up occurs only on the boundary.

1 Introduction

In this paper, we consider the system of two heat equations with coupled nonlinear Neumann boundary conditions, namely

\[
\begin{align*}
    u_t &= \Delta u, \quad v_t = \Delta v, \quad (x, t) \in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= e^{v^p}, \quad \frac{\partial v}{\partial \eta} = e^{u^q}, \quad (x, t) \in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in B_R,
\end{align*}
\]

where \( p, q > 1, \) \( B_R \) is a ball in \( \mathbb{R}^n, \) \( \eta \) is the outward normal, \( u_0, v_0 \) are smooth, radially symmetric, nonzero, nonnegative functions satisfy the condition

\[
\Delta u_0, \Delta u_0 \geq 0, \quad u_0(|x|), v_0(|x|) \geq 0, \quad x \in \overline{B}_R.
\]

The problem of system of two heat equations with nonlinear Neumann boundary conditions defined in a ball,

\[
\begin{align*}
    u_t &= \Delta u, \quad v_t = \Delta v, \quad (x, t) \in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= f(v), \quad \frac{\partial v}{\partial \eta} = g(u), \quad (x, t) \in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in B_R,
\end{align*}
\]
was introduced in [1, 2, 5, 6], for instance, in [1] it was studied the blow-up solutions to the system (1.3), where
\[ f(v) = v^p, \quad g(u) = u^q, \quad p, q > 1. \tag{1.4} \]
It was proved that for any nonzero, nonnegative initial data \((u_0, v_0)\), the finite time blow-up can only occur on the boundary, moreover, it was shown in [5] that, the blow-up rate estimates take the following form
\[
\begin{align*}
c \leq \max_{x \in \Omega} u(x, t)(T - t)^{\frac{p+1}{pq-1}} \leq C, & \quad t \in (0, T), \\
c \leq \max_{x \in \Omega} v(x, t)(T - t)^{\frac{q+1}{pq-1}} \leq C, & \quad t \in (0, T).
\end{align*}
\]
In [2, 6], it was considered the solutions of the system (1.3) with exponential Neumann boundary conditions model, namely
\[ f(v) = e^{pv}, \quad g(u) = e^{qu}, \quad p, q > 0. \tag{1.5} \]
It was proved that for any nonzero, nonnegative initial data, \((u_0, v_0)\), the solution blows up in finite time and the blow-up occurs only on the boundary, moreover, the blow-up rate estimates take the following forms
\[
\begin{align*}
C_1 \leq e^{pu(R,t)}(T - t)^{1/2} \leq C_2, & \quad C_3 \leq e^{pu(R,t)}(T - t)^{1/2} \leq C_4.
\end{align*}
\]
In this paper, we prove that the upper blow-up rate estimates for problem (1.1) take the following form
\[
\begin{align*}
\max_{\overline{B}_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), & \quad 0 < t < T, \\
\max_{\overline{B}_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), & \quad 0 < t < T,
\end{align*}
\]
where \(\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}\). Moreover, the blow-up occurs only on the boundary.

2 Preliminaries

The local existence and uniqueness of classical solutions to problem (1.1) is well known by [8]. On the other hand, every nontrivial solution blows up simultaneously in finite time, and that due to the known blow-up results of problem (1.3) with (1.4) and the comparison principle [8].

In the following lemma we study some properties of the classical solutions of problem (1.1). We denote for simplicity \(u(r, t) = u(x, t)\).

**Lemma 2.1.** Let \((u, v)\) be a classical unique solution of (1.1). Then

(i) \(u, v\) are positive, radial. Moreover, \(u_r, v_r \geq 0\) in \([0, R] \times (0, T)\).
(ii) \(u_t, v_t > 0\) in \(\overline{B}_R \times (0, T)\).
3 Rate Estimates

In order to study the upper blow-up rate estimates for problem (1.1), we need to recall some results from [3, 5].

Lemma 3.1. [5] Let $A(t)$ and $B(t)$ be positive $C^1$ functions in $[0, T)$ and satisfy

$$A'(t) \geq c \frac{B^p(t)}{\sqrt{T - t}}, \quad B'(t) \geq c \frac{A^q(t)}{\sqrt{T - t}} \text{ for } t \in [0, T),$$

where $p, q > 0, c > 0$ and $pq > 1$. Then there exists $C > 0$ such that

$$A(t) \leq C(T - t)^{-\alpha/2}, \quad B(t) \leq C(T - t)^{-\beta/2}, \quad t \in [0, T),$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Lemma 3.2. [3] Let $x \in \overline{B}_R$. If $0 \leq a < n - 1$. Then there exist $C > 0$ such that

$$\int_{S_R} \frac{ds_y}{|x - y|^a} \leq C.$$

Theorem 3.3. (Jump relation, [3]) Let $\Gamma(x, t)$ be the fundamental solution of heat equation, namely

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp\left[ -\frac{|x|^2}{4t} \right]. \quad (3.1)$$

Let $\varphi$ be a continuous function on $S_R \times [0, T]$. Then for any $x \in B_R, x^0 \in S_R, 0 < t_1 < t_2 \leq T$, for some $T > 0$, the function

$$U(x, t) = \int_{t_1}^{t_2} \int_{S_R} \Gamma(x - y, t - z)\varphi(y, z) ds_y d\tau$$

satisfies the jump relation

$$\frac{\partial}{\partial \eta} U(x, t) \to -\frac{1}{2} \varphi(x^0, t) + \frac{\partial}{\partial \eta} U(x^0, t), \quad \text{as } x \to x^0.$$

Theorem 3.4. Let $(u, v)$ be a solution of (1.1), which blows up in finite time $T$. Then there exist positive constants $C_1, C_2$ such that

$$\max_{\overline{B}_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T,$$

$$\max_{\overline{B}_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T.$$
Proof. We follow the idea of [5], define the functions $M$ and $M_b$ as follows

$$M(t) = \max_{B_R} u(x, t), \quad M_b(t) = \max_{S_R} u(x, t).$$

Similarly,

$$N(t) = \max_{B_R} v(x, t), \quad N_b(t) = \max_{S_R} v(x, t).$$

Depending on Lemma 2.1, both of $M, M_b$ are monotone increasing functions, and since $u$ is a solution of heat equation, it cannot attain interior maximum without being constant, therefore,

$$M(t) = M_b(t). \quad \text{Similarly} \quad N(t) = N_b(t).$$

Moreover, since $u, v$ blow up simultaneously, therefore, we have

$$M(t) \to +\infty, \quad N(t) \to +\infty \quad \text{as} \quad t \to T^-.$$ (3.2)

As in [4, 5], for $0 < z_1 < t < T$ and $x \in B_R$, depending on the second Green’s identity with assuming the Green function:

$$G(x, y; z_1, t) = \Gamma(x - y, t - z_1),$$

where $\Gamma$ is defined in (3.1), the integral equation to problem (1.1) with respect to $u$, can be written as follows

$$u(x, t) = \int_{B_R} \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau$$

$$- \int_{z_1}^{t} \int_{S_R} u(y, \tau) \frac{\partial \Gamma}{\partial n_y}(x - y, t - \tau) ds_y d\tau,$$

As in [4], letting $x \to S_R$ and using the jump relation (Theorem 3.3) for the third term on the right hand side of the last equation, it follows that

$$\frac{1}{2} u(x, t) = \int_{B_R} \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau$$

$$- \int_{z_1}^{t} \int_{S_R} u(y, \tau) \frac{\partial \Gamma}{\partial n_y}(x - y, t - \tau) ds_y d\tau,$$

for $x \in S_R, 0 < z_1 < t < T$.

Depending on Lemma 2.1 we notice that $u, v$ are positive and radial. Thus

$$\int_{B_R} \Gamma(x - y, t - z_1) u(y, z_1) dy > 0,$$

$$\int_{z_1}^{t} \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau = \int_{z_1}^{t} e^{\nu(R, \tau)} \int_{S_R} \Gamma(x - y, t - \tau) ds_y d\tau.$$
This leads to
\[
\frac{1}{2} M(t) \geq \int_{z_1}^{t} e^{N_p(\tau)} \left[ \int_{S_R} \Gamma(x-y, t-\tau) ds_y \right] d\tau \\
- \int_{z_1}^{t} M(\tau) \left[ \int_{S_R} \frac{\partial \Gamma}{\partial \eta_y} (x-y, t-\tau) ds_y \right] d\tau, \quad x \in S_R, 0 < z_1 < t < T.
\]

It is known that (see [3]) there exist \( C_0 > 0 \), such that \( \Gamma \) satisfies
\[
|\frac{\partial \Gamma}{\partial \eta_y} (x-y, t-\tau)| \leq \frac{C_0}{(t-\tau)^\mu} \cdot \frac{1}{|x-y|^{(n+1-2\mu-\sigma)}}, \quad x, y \in S_R, \sigma \in (0,1).
\]

Choose \( 1 - \frac{\sigma}{2} < \mu < 1 \), from Lemma 3.2, there exist \( C^* > 0 \) such that
\[
\int_{S_R} \frac{ds_y}{|x-y|^{(n+1-2\mu-\sigma)}} < C^*.
\]

Moreover, for \( 0 < t_1 < t_2 \) and \( t_1 \) is closed to \( t_2 \), there exists \( c > 0 \), such that
\[
\int_{S_R} \Gamma(x-y, t_2-t_1) ds_y \geq \frac{c}{\sqrt{t_2-t_1}}.
\]

Thus
\[
\frac{1}{2} M(t) \geq c \int_{z_1}^{t} e^{N_p(\tau)} \frac{1}{\sqrt{T-\tau}} d\tau - C \int_{z_1}^{t} M(\tau) |t-\tau|^{\mu} d\tau.
\]

Since for \( 0 < z_1 < t_0 < t < T \), it follows that \( M(t_0) \leq M(t) \), thus the last equation becomes
\[
\frac{1}{2} M(t) \geq c \int_{z_1}^{t} e^{N_p(\tau)} \frac{1}{\sqrt{T-\tau}} d\tau - C^*_1 M(t)|T-z_1|^{1-\mu}.
\]

Similarly, for \( 0 < z_2 < t < T \), we have
\[
\frac{1}{2} N(t) \geq c \int_{z_2}^{t} e^{M_q(\tau)} \frac{1}{\sqrt{T-\tau}} d\tau - C^*_2 N(t)|T-z_2|^{1-\mu}.
\]

Taking \( z_1, z_2 \) so that
\[
C^*_1|T-z_1|^{1-\mu} \leq 1/2, \quad C^*_2|T-z_2|^{1-\mu} \leq 1/2,
\]

it follows
\[
M(t) \geq c \int_{z_1}^{t} e^{N_p(\tau)} \frac{1}{\sqrt{T-\tau}} d\tau, \quad N(t) \geq c \int_{z_2}^{t} e^{M_q(\tau)} \frac{1}{\sqrt{T-\tau}} d\tau. \tag{3.3}
\]

Since both of \( M, N \) increasing functions and from (3.2), we can find \( T^* \) in \( (0,T) \) such that
\[
M(t) \geq q^{\frac{1}{(p-1)}}, \quad N(t) \geq p^{\frac{1}{(p-1)}}, \quad \text{for } T^* \leq t < T.
\]
Thus
\[ e^{M(t)} \geq e^{M(t)}, \quad e^{N(t)} \geq e^{N(t)}, \quad T^* \leq t < T. \]

Therefore, if we choose \( z_1, z_2 \) in \((T^*, T)\), then (3.3) becomes
\[ e^{M(t)} \geq c \int_{z_1}^{t} \frac{e^{pN(\tau)}}{\sqrt{T - \tau}} d\tau \equiv I_1(t), \quad e^{N(t)} \geq c \int_{z_2}^{t} \frac{e^{qM(\tau)}}{\sqrt{T - \tau}} d\tau \equiv I_2(t). \]

Clearly,
\[ I_1'(t) = c \frac{e^{pN(t)}}{\sqrt{T - t}} \geq c \frac{I_1^2}{\sqrt{T - t}}, \quad I_2'(t) = c \frac{e^{qM(t)}}{\sqrt{T - t}} \geq c \frac{I_2}{\sqrt{T - t}}. \]

By Lemma 3.1, it follows that
\[
I_1(t) \leq \frac{C}{(T - t)^{\frac{2}{p}}}, \quad I_2(t) \leq \frac{C}{(T - t)^{\frac{2}{q}}}, \quad t \in \{\max\{z_1, z_2\}, T\}. \tag{3.4}
\]

On the other hand, for \( t^* = 2t - T \) (Assuming that \( t \) is close to \( T \)).
\[
I_1(t) \geq c \int_{t^*}^{t} \frac{e^{pN(\tau)}}{\sqrt{T - \tau}} d\tau \geq ce^{pN(t^*)} \int_{2t - T}^{t} \frac{1}{\sqrt{T - \tau}} d\tau = 2c(\sqrt{2} - 1)\sqrt{T - t}e^{pN(t^*)}.
\]

Combining the last inequality with (3.4) yields
\[
e^{N(t^*)} \leq \frac{C}{2c(\sqrt{2} - 1)(T - t)^{\frac{p+1}{2pq - p}} + \frac{1}{2}} = \frac{2^{\frac{q+1}{2pq - p}}C}{2c(\sqrt{2} - 1)(T - t)^{\frac{q+1}{2pq - p}}}.
\]

Thus, there exists a constant \( c_1 > 0 \) such that
\[
e^{N(t^*)}(T - t)^{\frac{q+1}{2pq - p}} \leq c_1.
\]

In the same way we can show
\[
e^{M(t^*)}(T - t)^{\frac{p+1}{2pq - p}} \leq c_2.
\]

This leads to, there exists \( C_1, C_2 > 0 \) such that
\[
\max_{\mathcal{B}_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T, \tag{3.5}
\]
\[
\max_{\mathcal{B}_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T. \tag{3.6}
\]
4 Blow-up Set

In order to show that the blow-up to problem (1.1) occurs only on the boundary, we need to recall the following lemma from [6].

**Lemma 4.1.** Let \( w \) is a continuous function on the domain \( B_R \times [0,T) \) and satisfies

\[
\begin{align*}
  w_t &= \Delta w, & (x,t) \in B_R \times (0,T), \\
  w(x,t) &\le \frac{C}{(T-t)^m}, & (x,t) \in S_R \times (0,T), & m > 0.
\end{align*}
\]

Then for any \( 0 < a < R \)

\[
\sup\{w(x,t) : 0 \le |x| \le a, \ 0 \le t < T\} < \infty.
\]

**Proof.** Set

\[
h(x) = (R^2 - r^2)^2, \ r = |x|,
\]

\[
z(x,t) = \frac{C_1}{[h(x) + C_2(T-t)]^m}.
\]

We can show that:

\[
\Delta h - \frac{(m+1)|\nabla h|^2}{h} = 8r^2 - 4n(R^2 - r^2) - (m+1)16r^2
\]

\[
\ge -4nR^2 - 16R^2(m+1),
\]

\[
z_t - \Delta z = \frac{C_1m}{[h(x) + C_2(T-t)]^{m+1}}(C_2 + \Delta h - \frac{(m+1)|\nabla h|^2}{h + C_2(T-t)})
\]

\[
\ge \frac{C_1m}{[h(x) + C_2(T-t)]^{m+1}}(C_2 - 4nR^2 - 16R^2(m+1)).
\]

Let

\[
C_2 = 4nR^2 + 16R^2(m+1) + 1
\]

and take \( C_1 \) to be large such that

\[
z(x,0) \ge w(x,0), \ x \in B_R.
\]

Let \( C_1 \ge C(C_2)^m \), which implies that

\[
z(x,t) \ge w(x,t) \text{ on } S_R \times [0,T).
\]

Then from the maximum principle [7], it follows that

\[
z(x,t) \ge w(x,t), \ (x,t) \in \overline{B}_R \times (0,T)
\]

and hence

\[
\sup\{w(x,t) : 0 \le |x| \le a, 0 \le t < T\} \le C_1(R^2 - a^2)^{-2m} < \infty, \ 0 \le a < R.
\]
**Theorem 4.2.** Let the assumptions of Theorem 3.4 be in force. Then \((u,v)\) blows up only on the boundary.

**Proof.** Using equations (3.5), (3.6)

\[
 u(R, t) \leq \frac{c_1}{(T-t)^{\frac{\alpha}{2}}}, \quad v(R, t) \leq \frac{c_2}{(T-t)^{\frac{\beta}{2}}}, \quad t \in (0, T).
\]

From Lemma 4.1, it follows that

\[
 \sup \{ u(x, t) : (x, t) \in B_a \times [0, T) \} \leq C_1 (R^2 - a^2)^{-\alpha} < \infty, \\
 \sup \{ v(x, t) : (x, t) \in B_a \times [0, T) \} \leq C_1 (R^2 - a^2)^{-\beta} < \infty,
\]

for \(a < R\).

Therefore, \(u, v\) blow up simultaneously and the blow-up occurs only on the boundary. \(\square\)

**References**


