A stable numerical method for solving variable coefficient advection–diffusion models

Enrique Ponsoda*, Emilio Defez, María Dolores Roselló, José Vicente Romero

Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Edificio 8G, 2º, P.O.Box 22012 Valencia, Spain

Abstract

In a recent paper [E. Defez, R. Company, E. Ponsoda, L. Jódar, Aplicación del método CE-SE a la ecuación de advección-difusión con coeficientes variables, Congreso de Métodos Numéricos en Ingeniería (SEMNI), Granada, Spain, 2005] a modified space–time conservation element and solution element scheme for solving the advection–diffusion equation with time-dependent coefficients, is proposed. This equation appears in many physical and technological models like gas flow in industrial tubes, conduction of heat in solids or the evaluation of the heating through radiation of microwaves when the properties of the media change with time. This method improves the well-established methods, like finite differences or finite elements: the integral form of the problem exploits the physical properties of conservation of flow, unlike the differential form. Also, this explicit scheme evaluates the variable and its derivative simultaneously in each knot of the partitioned domain. The modification proposed in [E. Defez, R. Company, E. Ponsoda, L. Jódar, Aplicación del método CE-SE a la ecuación de advección-difusión con coeficientes variables, Congreso de Métodos Numéricos en Ingeniería (SEMNI), Granada, Spain, 2005] with regard the original method [S.C. Chang, The method of space–time conservation element and solution element. A new approach for solving the Navier–Stokes and Euler equations, J. Comput. Phys. 119 (1995) 295–324] consists of keeping the variable character of the coefficients in the solution element, without considering the linear approximation. In this paper the stability of the proposed method is studied and a CFL condition is obtained.

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Keywords: Advection–diffusion equation; Time dependent coefficients; CE-SE numerical scheme; Amplification matrix; Stability

1. Introduction

The advection–diffusion equation with time-dependent coefficients

\[
\frac{\partial}{\partial t} u(x, t) + a(t) \frac{\partial}{\partial x} u(x, t) - b(t) \frac{\partial^2}{\partial x^2} u(x, t) = 0, \\
(x, t) \in \mathbb{R} \times [0, +\infty[; \quad b(t) \geq 0,\]
\]

This work has been partially supported by the Spanish M.C.Y.T. and FEDER grant DPI2004–08383–C03–03 and the Generalitat Valenciana grant GV/2007/009.

* Corresponding author.

E-mail addresses: eponsoda@imm.upv.es (E. Ponsoda),edefez@imm.upv.es (E. Defez), drosello@imm.upv.es (M.D. Roselló),jvromero@imm.upv.es (J.V. Romero).

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doi:10.1016/j.camwa.2008.02.004
appears frequently in many physical and technological models. In the evaluation of the heating trough radiations of microwaves when the properties of medium, like humidity or dielectric properties for example, change with the time [3,4], the advection–diffusion equation with constant coefficients is not suitable. Also, in the study of the transmission of flows in industrial tubes [5], solutions to problems of the type (1) are necessary. In problems of heat transmission in solids, it is usual that the thermal properties depend on the time but not on the position [6,7]. It is important to note the fact that in the case of variable coefficients, unlike the constant case, it is not possible to carry out a change of variable so that the problem (1) is transformed into a problem of simple diffusion [6, p. 58].

In this work, a modified space–time conservation element and solution element scheme (CE-SE) is proposed for solving Eq. (1) under the initial condition

$$u(x, 0) = f(x).$$

(2)

Usually, the CE-SE method has been applied to solve problems of conservation laws [2], since it presents significant advantages with regard to the numerical classic schemes, like finite differences or finite elements, for example. These advantages do not consist only of the level of efficiency, but also of exploiting the conservation of the flow both in time and in space. Besides, the variables and their derivatives are considered individually and calculated simultaneously in every knot of the partitioned space–time domain.

Also we can find the application of the CE-SE method to problems of advection–diffusion for the case of constant coefficients, where the resultant scheme, $a - \mu$, presents many advantages such as behavior and stability if it is compared with other numerical methods, see [8]. In spite of the fact that our problem (1) and (2) does not correspond to a conservation law, we will verify that the results obtained and the quality of response that the method offers are highly satisfactory also in case of variable coefficients. Furthermore, in this paper we support the variable character of the coefficients in each solution element without using its linear approximation as [2] in the problems of conservation laws.

In [1] it is showed that, for time-dependent coefficients advection–diffusion problems, all these advantages are kept, and an example illustrates the high quality of the results given by this scheme. Later, in [9], the stability of the CE-SE method is studied, and it is verified that under some hypotheses imposed on the data problem, the method is stable. In this work we prove that the conditions on the data and the suitable election of the partition of the domain leads to obtaining a CFL condition that guarantees spectral properties of a certain amplification matrix so that, after the election of a suitable norm, the hypotheses necessary for the stability proposed in [9] are satisfied.

This work is organized as follows. In Section 2, the CE-SE method to solve (1) and (2) is presented. We obtain an explicit expression such that the approximate values of the solution $u(x, t)$ of problem (1) and (2) can be evaluated in a set of knots of a certain partition of space–time domain. It is important to note that the obtained result reproduces the solution given by the $a - \mu$ scheme, see [8], if we consider $a(t) = a, b(t) = \mu$, constants for all $t \geq 0$. In Section 3, the stability of the numerical method is studied. A CFL condition that guarantees the stability of the scheme is obtained. Finally, in Section 4, an illustrative example is presented such that the quality on the approximate solution given by the modified CE-SE method is compared with the answer obtained by means of other standard methods.

Throughout this paper, we will denote by $\parallel \parallel$ the usual Euclidean norm in $\mathbb{R}^2$. If

$$v : \mathbb{Z} \mapsto \mathbb{R}^2/v(j) = \begin{bmatrix} v_1(j) \\ v_2(j) \end{bmatrix},$$

we will denote by $\parallel \parallel_2$ the $L^2$ discrete norm with the partition size $\Delta x$

$$\|v\|^2_2 = \Delta x \sum_{j \in \mathbb{Z}} \left[ v_1^2(j) + v_2^2(j) \right].$$

(3)

The discrete Fourier transform $\hat{v}(\theta)$ of $v(j)$ is given by

$$\hat{v}(\theta) = \sum_{j \in \mathbb{Z}} e^{-ij\theta} \begin{bmatrix} v_1(j) \\ v_2(j) \end{bmatrix}; \quad \theta \in [-\pi, \pi],$$

(4)
and its inverse transform by
\[ v(j) = \int_{-\pi}^{\pi} e^{ij\theta} \hat{v}(\theta) d\theta. \]

Note that
\[ \| \hat{v} \|^2 = \int_{-\pi}^{\pi} \| \hat{v}(\theta) \|^2 d\theta = \int_{-\pi}^{\pi} \left\| \sum_{j \in \mathbb{Z}} e^{-ij\theta} \begin{bmatrix} v_1(j) \\ v_2(j) \end{bmatrix} \right\|^2 d\theta, \]

and since \( \{ e^{-ij\theta} \}_{j \in \mathbb{Z}} \) is an orthogonal system, from the Parseval property we have
\[ \| \hat{v} \|^2 = \int_{-\pi}^{\pi} \sum_{j \in \mathbb{Z}} \left[ (v_1(j))^2 + (v_2(j))^2 \right] d\theta = \frac{2\pi}{\Delta x} \| v \|^2, \]

see [10, p. 25].

For a matrix \( A \in \mathbb{C}^{n \times n} \) and \( \delta > 0 \), there exists a norm \( \| \cdot \|_\delta \) such that
\[ \| A \|_\delta \leq \rho(A) + \delta, \]
when \( \rho(A) \) is the spectral radius of \( A \).

We remark that all matrix norms are equivalent, i.e., given \( \| \cdot \| \) and \( \| \cdot \| \) two different matrix norms, there exist constants \( k_1, k_2 > 0 \) such that
\[ k_1 \| A \| \leq \| A \| \leq k_2 \| A \|, \quad \forall A \in \mathbb{C}^{n \times n}, \]
see [10].

2. The numerical scheme

The space–time domain is partitioned in a grid such that the knots \((j, n)\) are obtained for \( n = 0, 1/2, 1, 3/2, \ldots \) and, for each \( n, j = n \pm 1/2, n \pm 3/2, n \pm 5/2, \ldots \) see [2] for details. Then, we define the solution element \( SE(j, n) \) as the space–time region enclosed inside the rhombus centered in \((x_j, t^n)\) and whose diagonals are \( \Delta t \) and \( \Delta x \), see [1] or [2] for details.

In each solution element, we define
\[ U(x, t; j, n) = U^0_j + (U_x)^n_j (x - x_j) + (U_t)^n_j (t - t^n), \quad \forall (x, t) \in SE(j, n), \]
(7)
where \( U^0_j \), \((U_x)^n_j \) and \((U_t)^n_j \) are constants to be determined into \( SE(j, n) \). Nevertheless, imposing that \( U(x, t; j, n) \), given by (7), satisfies the Eq. (1) in \((x_j, t^n) \subset SE(j, n)\), it is possible to express \((U_t)^n_j \) in terms of the other constant and substituting in (7) we obtain
\[ U(x, t; j, n) = U^0_j + (U_x)^n_j \left[ (x - x_j) - a(t^n) (t - t^n) \right], \]
and then, only two constants are necessary to be determined in each solution element.

To calculate these constants the so called elements of conservation \( CE(j, n) \) are in use, which exploit the conservation of the flow in certains space–time regions that we will describe later. It is for this reason that the integral formulation of problem (1) – whose proof can be found in [1] – turns out to be important.

**Theorem 1.** Let \( \Gamma \in \mathbb{R} \times [0, +\infty[ \) be a space–time region. If we denote by
\[ h(x, t) = (a(t)u(x, t) - b(t)u_x(x, t), u(x, t)), \]
(9)
and \( S(\Gamma) \) is the border of the domain \( \Gamma \), then Eq. (1) is the differential form of the integral equation

\[
\int_{S(\Gamma)} h(x, s) \, dS = 0, \tag{10}
\]

where \( dS = n \, ds \), \( n \) is the area of a surface element and \( n \) is the unitary normal vector of the surface element, which direction is towards the exterior one.

From (8) and (9), in the solution elements we define

\[
H(x, t; j, n) = \left( a(t)U(x, t; j, n) - b(t)(U_x)_n^j, U(x, t; j, n) \right), \quad \forall (x, t) \in SE(j, n). \tag{11}
\]

Note that in (11), coefficients \( a(t) \) and \( b(t) \) are not approximated by the constants \( a(t^n) \), \( b(t^n) \), then the variable character remains guaranteed and no precision is lost due to this assumption.

Now, we define the conservation elements \( CE_+(j, n) \) and \( CE_-(j, n) \) as the rectangular regions whose sides have a length \( \Delta x/2 \), \( \Delta t/2 \), and the knot \((x_j, t^n)\) is in the top right corner for \( CE_+(j, n) \), and the same knot is in the top left corner for \( CE_-(j, n) \), see [1] for more details.

In order to calculate the constants \( U_j^n \) and \( (U_x)_j^n \) in (8), we use the following approximation of the integral Eq. (10)

\[
F_\pm(j, n) = \int_{S(CE_\pm(j, n))} H(x, t; j, n) \, dS = 0, \tag{12}
\]

where \( CE_\pm(j, n) \) is the conservation element and \( H(x, t; j, n) \) is given by (11). Solving the integrals in (12), we obtain

\[
\frac{4}{\Delta x} F_\pm(j, n) = \pm \frac{1}{2} \left[ \left( 1 - v^n v_n^n + \zeta^n \right) (U_x)_n^j + \left( 1 - v^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \right) (U_x)_{j+\frac{1}{2}}^n \right]
\]

\[
\quad + \left( U_j^n - U_{j+\frac{1}{2}}^n \right) \left[ \frac{2}{\Delta x} \left( 1 + v_0^n \right) \right], \tag{13}
\]

where

\[
v^n = a(t^n) \frac{\Delta t}{\Delta x}, \quad k = n, n - \frac{1}{2}; \quad v_0^n = a(t_0^n) \frac{\Delta t}{\Delta x}; \quad \zeta^n = 4 \langle b \rangle_0^n \frac{\Delta t}{(\Delta x)^2};
\]

and \( a(t)_1^n, a(t)^{n-\frac{1}{2}}, a(t)_0^n, b(t)_0^n \) are the mean values in \( [t^{n-\frac{1}{2}}, t^n] \)

\[
\langle a \rangle_0^n = \frac{2}{\Delta t} \int_{t^{n-\frac{1}{2}}}^{t^n} a(t) \, dt; \quad \langle b \rangle_0^n = \frac{2}{\Delta t} \int_{t^{n-\frac{1}{2}}}^{t^n} b(t) \, dt
\]

\[
\langle a \rangle_{1,2}^{n-\frac{1}{2}} = \frac{8}{(\Delta t)^2} \int_{t^{n-\frac{1}{2}}}^{t^n} a(t) \left( t - t^{n-\frac{1}{2}} \right) \, dt,
\]

\[
\langle a \rangle_1^n = \frac{8}{(\Delta t)^2} \int_{t^{n-\frac{1}{2}}}^{t^n} a(t) \left( t^n - t \right) \, dt. \tag{15}
\]

Furthermore, note that adding the last two equalities in (15), it follows that

\[
\langle a \rangle_{1,2}^{n-\frac{1}{2}} + \langle a \rangle_1^n = \frac{8}{(\Delta t)^2} \left( \frac{\Delta t}{2} \right)^2 \langle a \rangle_0^n,
\]

and, we can then write

\[
v_0^n = \frac{v_1^n + v_2^n}{2}.\]
Finally, we can obtain $U^n_j$ and $(U_x)^n_j$ and thus, for (8), it is possible to obtain the value of $U(x, t; j, n)$ in each solution element. In fact, we define

$$q(j, n) = \frac{\Delta x}{4} \left[ \begin{array}{c} U^n_j \\ (U_x)^n_j \end{array} \right].$$

(16)

From the equations $F_{\pm}(j, n) = 0$ given by (12) and (13), and expressing them in the matrix form, we have

$$\begin{bmatrix} 1 - \nu^n_0 & 1 - \nu^n v^n + \zeta^n \\ 1 + \nu^n_0 & - (1 - \nu^n v^n_1 + \zeta^n) \end{bmatrix} q(j, n) = \begin{bmatrix} 1 - \nu^n_0 & - \left( 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \right) \\ 0 & 0 \end{bmatrix} q \left( j + \frac{1}{2}, n - \frac{1}{2} \right) + \begin{bmatrix} 0 & 0 \\ 1 + \nu^n_0 & 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \end{bmatrix} q \left( j - \frac{1}{2}, n - \frac{1}{2} \right).$$

(17)

Using $\alpha^n_+ = 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n$ and with a suitable choice of the partition of the space–time domain, we can see that $|\nu^n|$ and $|v^n_1|$ are both less than one, and since $b(t) \geq 0$, it turns out that $\zeta^n$, given by (14), is bigger than zero. Then, we can assure that

$$\alpha^n_+ \neq 0; \quad \forall n = \frac{1}{2}, \frac{3}{2}, 2, \ldots.$$

In this way, system (17) has a unique solution because

$$\det \begin{bmatrix} 1 - \nu^n_0 & 1 - \nu^n v^n + \zeta^n \\ 1 + \nu^n_0 & - (1 - \nu^n v^n_1 + \zeta^n) \end{bmatrix} = -2\alpha^n_+ \neq 0.$$

Then $q(j, n)$ is given by

$$q(j, n) = Q_+(n) q \left( j - \frac{1}{2}, n - \frac{1}{2} \right) + Q_-(n) q \left( j + \frac{1}{2}, n - \frac{1}{2} \right); \quad n \geq \frac{1}{2},$$

(18)

with

$$Q_+(n) = \frac{1}{2} \begin{bmatrix} 1 + \nu^n_0 & 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \\ 1 - \nu^n v^n_1 + \zeta^n & - (1 - \nu^n_0) \left( 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \right) \end{bmatrix},$$

(19)

and

$$Q_-(n) = \frac{1}{2} \begin{bmatrix} 1 - \nu^n_0 & - \left( 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \right) \\ 1 - (\nu^n_0)^2 & - (1 + \nu^n_0) \left( 1 - \nu^{n-\frac{1}{2}} v^{n-\frac{1}{2}} - \zeta^n \right) \end{bmatrix},$$

(20)

where all factors are given by (14) and (15).

**Remark 2.** The scheme (18) is constructed from the initial conditions

$$q(j, 0) = \left[ \frac{\Delta x}{4} \frac{\partial}{\partial x} u(x, 0) \right] = \left[ \frac{\Delta x}{4} f(x_j) \right].$$

(21)

where, from (2), $u(x, 0) = f(x)$. 
3. Analysis of stability

From (18) and (21) we define the sequence

\[ q(n) \equiv \{q(j, n)\}_{j=\infty}^{j=+\infty} = \{\ldots, q(-1, n), q(0, n), q(1, n), \ldots\}. \]  

(22)

We suppose that \( q(n) \in L^2(\mathbb{R}^+), \) i.e., from (3)

\[ \|q(n)\|_2^2 = \sum_{j \in \mathbb{Z}} \|q(j, n)\|_2^2 \Delta x < \infty. \]

Applying the discrete Fourier transform (4) to (22), we obtain

\[ q(n, \theta)^* = \sum_{j \in \mathbb{Z}} q(j, n)e^{-ij\theta}; \quad -\pi < \theta < \pi, \]

(23)

and from (18)–(20) and (23) we can write

\[ q(n + 1, \theta)^* = \sum_{j \in \mathbb{Z}} q(j, n + 1)e^{-ij\theta} \]

\[ = \sum_{j \in \mathbb{Z}} \left\{ Q_+(n + 1)Q_+ \left( n + \frac{1}{2} \right) q(j - 1, n) \right. \]

\[ + \left[ Q_+(n + 1)Q_- \left( n + \frac{1}{2} \right) + Q_-(n + 1)Q_+ \left( n + \frac{1}{2} \right) \right] q(j, n) \]

\[ + \left. Q_-(n + 1)Q_- \left( n + \frac{1}{2} \right) q(j + 1, n) \right\} e^{-ij\theta}. \]  

(24)

Note that

\[ \sum_{j \in \mathbb{Z}} q(j - 1, n)e^{-ij\theta} = \sum_{j \in \mathbb{Z}} q(j - 1, n)e^{-i(j-1)\theta}e^{-i\theta} \]

\[ = e^{-i\theta}q(n, \theta)^*, \]  

(25)

and, in the same way

\[ \sum_{j \in \mathbb{Z}} q(j + 1, n)e^{-ij\theta} = \sum_{j \in \mathbb{Z}} q(j + 1, n)e^{-i(j+1)\theta}e^{i\theta} \]

\[ = e^{i\theta}q(n, \theta)^*. \]  

(26)

Substituting (23), (25) and (26) in (24), and denoting \( Q_{\pm}^n = Q_{\pm}(n) \) we obtain

\[ q(n + 1, \theta)^* = Q_{\pm}^{n+1}(\theta) Q_{\pm}^{n+\frac{1}{2}}(\theta) q(n, \theta)^*, \]  

(27)

where

\[ Q^k(\theta) = Q_+^k e^{-i\frac{k}{2} \theta} + Q_-^k e^{i\frac{k}{2} \theta}; \quad k = n + 1, n + \frac{1}{2}. \]  

(28)

The matrix \( Q_{\pm}^{n+1}(\theta) Q_{\pm}^{n+\frac{1}{2}}(\theta) \), that appears in (27), is called amplification matrix. This matrix connects the state \( q(n + 1, \theta)^* \) with \( q(n, \theta)^* \). The eigenvalues of this matrix are the so-called amplification factors.

We will next show a previous result that establishes a CFL condition after a suitable partition of the space-time domain, in which the stability is guaranteed.

**Theorem 3.** Let us consider \( t \in [0, T]; T > 0 \) fixed. If

\[ |v(t)| = |a(t)| \frac{\Delta t}{\Delta x} < 1; \quad \forall t \in [0, T], \]  

(29)
and
\[
(\zeta^n + \Delta_-) > 0; \quad \forall n > 0 \text{ half-entire},
\]
with \(n \Delta t < T\), and
\[
\Delta_- = \frac{\nu^{n-\frac{1}{2}}_1 v^{n-\frac{1}{2}}_1 - \nu^n v^n_1}{2},
\]
where \(\nu^k, v^k_1, \zeta^n, k = n, n - \frac{1}{2}\); are defined by (14), then
\[
\rho(Q^n(\theta)) \leq 1; \quad \forall \theta \in [-\pi, \pi].
\]

Furthermore
\[
\max_{\theta \in [-\pi, \pi]} \rho(Q^n(\theta)) = 1.
\]

**Proof.** From (28) we can write
\[
Q^n(\theta) = Q^n_+ e^{-i\frac{\theta}{2}} + Q^n_- e^{i\frac{\theta}{2}} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix},
\]
where
\[
q_{11} = \cos \left(\frac{\theta}{2}\right) - i v^n_0 \sin \left(\frac{\theta}{2}\right),
q_{12} = -i \alpha_- \sin \left(\frac{\theta}{2}\right),
q_{21} = i \frac{1 - (v^n_0)^2}{\alpha_+} \sin \left(\frac{\theta}{2}\right),
q_{22} = -\frac{\alpha_-}{\alpha_+} \left(\cos \left(\frac{\theta}{2}\right) + i v^n_0 \sin \left(\frac{\theta}{2}\right)\right),
\]
\[
\alpha_- = 1 - \nu^{n-\frac{1}{2}}_1 v^{n-\frac{1}{2}}_1 - \zeta^n,
\alpha_+ = 1 - \nu^n v^n_1 + \zeta^n.
\]

Note that \(\sigma(Q^n(\theta))\) are the values \(\lambda \in \mathbb{C}\) such that \(\det(Q^n(\theta) - \lambda I) = 0\), then the immediate calculation of the characteristic polynomial results
\[
\lambda_{\pm}(\theta) = \frac{1}{\alpha_+} \left(\eta \pm \sqrt{\eta^2 + (1 - \Delta_+)^2 - (\zeta^n + \Delta_-)^2}\right),
\]
where
\[
\eta = (\zeta^n + \Delta_-) \cos \left(\frac{\theta}{2}\right) - i \sin \left(\frac{\theta}{2}\right) v^n_0 (1 - \Delta_-),
\Delta_+ = \frac{\nu^n v^n_1 + \nu^{n-\frac{1}{2}}_1 v^{n-\frac{1}{2}}_1}{2}.
\]
\(\Delta_-\) is given by (31) and all the other factors that appears in (34) are given by (14) and (15).

From hypothesis (29), \(|\nu(t)| < 1\), and from (14) and (29), \(|v^{n-\frac{1}{2}}_1|\) and \(|v^n_1|\) are both less than one. Then
\[
|\Delta_+| < 1,
\]
and thus
\[
(1 - \Delta_+) > 0.
\]
Denoting
\[ \zeta \equiv \zeta^n = \frac{\zeta^n + \Delta_\pm}{1 - \Delta_\pm}, \] (36)
from (30), (35) and (36) it follows that
\[ \zeta \equiv \zeta^n > 0, \] (37)
and we can express the eigenvalues given by (34), as
\[ \lambda_{\pm}(\theta) = \frac{1}{1 + \zeta} \left( \hat{\eta} \pm \sqrt{\hat{\eta}^2 + 1 - \zeta^2} \right), \] (38)
where
\[ \hat{\eta} = \zeta \cos \left( \frac{\theta}{2} \right) - i \nu_0^n \sin \left( \frac{\theta}{2} \right). \] (39)

In the case that \( \nu_0^n = 0 \), it is easy to establish the result. In fact, if \( \nu_0^n = 0 \) we get
\[ \lambda_{\pm}(\theta) = \frac{1}{1 + \zeta} \left( \zeta \cos \left( \frac{\theta}{2} \right) \pm \sqrt{1 - \zeta^2 \sin^2 \left( \frac{\theta}{2} \right)} \right), \] and thus, if \( \zeta^2 \sin^2 \left( \frac{\theta}{2} \right) < 1 \), then
\[ |\lambda_{\pm}(\theta)|^2 \leq \frac{1}{(1 + \zeta)^2} \left( \zeta \cos \left( \frac{\theta}{2} \right) + 1 \right)^2 \leq 1. \]

Furthermore, for \( \theta = 0 \) we have that \( |\lambda_{-}(\theta)| = 1 \). In the case \( \zeta^2 \sin^2 \left( \frac{\theta}{2} \right) > 1 \) it follows that
\[ |\lambda_{\pm}(\theta)|^2 = \frac{1}{(1 + \zeta)^2} \left( \zeta^2 \cos^2 \left( \frac{\theta}{2} \right) + \zeta^2 \sin^2 \left( \frac{\theta}{2} \right) - 1 \right) \]
\[ = \frac{1}{(1 + \zeta)^2} (\zeta^2 - 1) = \frac{\zeta - 1}{\zeta + 1} < 1. \]

Then, for \( \nu_0^n = 0 \), the result is established. Let us consider the case \( \nu_0^n \neq 0 \). Our aim is to prove that \( |\lambda_{-}(\theta)| \) is a monotonic function that has only one relative maximum in \( \theta = 0, \forall \theta \in [-\pi, \pi] \), and \( |\lambda_{+}(0)| = 1 \). Finally we will show that
\[ \max_{\theta \in [-\pi, \pi]} |\lambda_{-}(\theta)| < 1. \]

Note that
\[ |\lambda_{\pm}(\theta)|^2 = \lambda_{\pm}(\theta) \bar{\lambda}_{\pm}(\theta) = \bar{\lambda}_{\pm}(-\theta) \lambda_{\pm}(-\theta) = |\lambda_{\pm}(-\theta)|^2, \]
and then \( |\lambda_{\pm}(\theta)| \) is an even function of \( \theta \) and we can study only the interval \([0, \pi] \).

From (38) we can write
\[ |\lambda_{\pm}(\theta)|^2 = \lambda_{\pm}(\theta) \bar{\lambda}_{\pm}(\theta) \]
\[ = \frac{1}{(1 + \zeta)^2} \left( |\hat{\eta}|^2 + |\delta|^2 \pm 2 \Re (\bar{\eta} \delta) \right), \] (40)
where
\[ \delta = \sqrt{\hat{\eta}^2 + 1 - \xi^2}, \] (41)
with \( \xi^2 = \zeta^2 > 0 \).
From (39) it follows that

\[ \delta^2 = \left( \xi^2 \cos^2 \left( \frac{\theta}{2} \right) - (v_0^n)^2 \sin^2 \left( \frac{\theta}{2} \right) + 1 - \xi^2 \right) - i \left( 2v_0^n \xi \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \right). \]  \hspace{1cm} (42)

Since the imaginary part of \( \delta^2 \) is not zero for any \( \theta \in [0, \pi] \), we have that \( \delta \neq 0, \forall \theta \in [0, \pi] \). Furthermore, if \( \theta = 0 \) then \( \delta^2 = 1 \). Thus \( \delta \neq 0, \forall \theta \in [0, \pi] \). In this way, \( |\delta|^2 \) is derivable, and from (40) and (41), \( |\lambda_\pm(\theta)|^2 \) is also derivable in \([0, \pi]\). From (38) one concludes that \( \lambda_\pm(\pi) \neq 0 \), and then, \( |\lambda_\pm(\theta)|^2 \) is derivable in \([0, \pi]\).

Then, if \( \alpha_- \), given by (33), is not zero, it follows that \( |\lambda_\pm(\theta)| \) is also derivable in \([0, \pi]\). In fact, from (38) and (41) it follows that

\[ |\lambda_\pm(\theta)| \neq 0 \iff \hat{\eta} \neq \delta \neq 0 \iff \hat{\eta}^2 \neq \delta^2 \iff 1 - \xi^2 \neq 0 \iff (1 - \xi)(1 + \xi) \neq 0, \]

but from (37) it follows that \( 1 + \xi > 0 \), then

\[ |\lambda_\pm(\theta)| \neq 0 \iff 1 - \xi \neq 0, \]

and from (36), we can write

\[ |\lambda_\pm(\theta)| \neq 0 \iff 1 - \xi^2 + \Delta_- = \frac{1 - v_1^{n-\frac{1}{2}}v_1^{n-\frac{1}{2}} - \xi^2}{1 + \Delta_+} 
eq 0, \]

in this way, if \( \alpha_- \neq 0 \), we can be sure that \( |\lambda_\pm(\theta)| \neq 0 \).

Note that in the case \( \alpha_- = 0 \), the result of the theorem is established immediately because from (32) and (33) it follows that

\[ Q^n(\theta) = \begin{bmatrix} q_{11} & 0 \\ q_{21} & 0 \end{bmatrix}, \]

and then

\[ \sigma \left( Q^n(\theta) \right) = \left\{ 0, \cos \left( \frac{\theta}{2} \right) - i v_0^n \sin \left( \frac{\theta}{2} \right) \right\}, \]

and we can conclude that

\[ \rho \left( Q^n(\theta) \right) = \sqrt{\cos^2 \left( \frac{\theta}{2} \right) + (v_0^n)^2 \sin^2 \left( \frac{\theta}{2} \right)} \leq 1, \]

because \( (v_0^n)^2 < 1 \) from hypothesis (29).

Then, let’s recover the case \( \alpha_- \neq 0 \). We are going to verify that \( |\lambda_\pm(\theta)| \) is a monotonic function in \([0, \pi]\) and it only has relative critical points at the ends of the interval.

We already know that

\[ |\lambda_\pm(\theta)| \neq 0, \quad \forall \theta \in [0, \pi]. \]

Deriving this expression

\[ \frac{d}{d\theta} |\lambda_\pm(\theta)| = \frac{1}{2} \frac{1}{|\lambda_\pm(\theta)|^2} \frac{d}{d\theta} |\lambda_\pm(\theta)|^2, \]  \hspace{1cm} (43)

by (40) we can write

\[ \frac{d}{d\theta} |\lambda_\pm(\theta)|^2 = \frac{1}{(1 + \xi)^2} \frac{d}{d\theta} \left( \hat{\eta} \hat{\eta} + \delta \delta \pm \hat{\eta} \delta \pm \hat{\eta} \delta \right). \]  \hspace{1cm} (44)

From (41) we can deduce that

\[ \delta' = \frac{d\delta}{d\theta} = \frac{\hat{\eta} \hat{\eta}'}{\delta}; \quad \delta' = \frac{\hat{\eta} \hat{\eta}'}{\delta}. \]
and calculating in (44), it follows that

\[
\frac{d}{d\theta} |\lambda_\pm(\theta)|^2 = \pm \frac{|\lambda_\pm(\theta)|^2}{|\delta|^2} 2 \operatorname{Re} \left( \eta' \delta \right),
\]

and from (43) one concludes that

\[
\frac{d}{d\theta} (|\lambda_\pm(\theta)|) = \pm \frac{|\lambda_\pm(\theta)|}{|\delta|^2} \operatorname{Re} \left( \eta' \delta \right).
\]  \hspace{1cm} (45)

If we write \( \delta \), given in (41), in the form

\[
\delta = \operatorname{Re}(\delta) + i \operatorname{Im}(\delta),
\]

then

\[
\delta^2 = \left( \operatorname{Re}(\delta)^2 - \operatorname{Im}(\delta)^2 \right) + 2i \operatorname{Re}(\delta) \operatorname{Im}(\delta),
\]

and, identifying in (42), one concludes that

\[
\operatorname{Re}(\delta) \operatorname{Im}(\delta) = -\nu_0^n \xi \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right)
\]

\[
\operatorname{Re}(\delta)^2 - \operatorname{Im}(\delta)^2 = 1 - \left( \xi^2 + (\nu_0^n)^2 \right) \sin^2 \left( \frac{\theta}{2} \right).
\]

Solving the system and applying that \( \delta = 1 \) if \( \theta = 0 \), it follows that

\[
\operatorname{Re}(\delta)^2 = \frac{n + \sqrt{m}}{2}, \quad \operatorname{Im}(\delta)^2 = -\frac{n + \sqrt{m}}{2}
\]  \hspace{1cm} (46)

where

\[
m = 1 - 2 \left( \xi^2 + (\nu_0^n)^2 \right) \sin^2 \left( \frac{\theta}{2} \right) + \left( \xi^2 - (\nu_0^n)^2 \right)^2 \sin^4 \left( \frac{\theta}{2} \right)
\]

\[
n = 1 - \left( \xi^2 + (\nu_0^n)^2 \right) \sin^2 \left( \frac{\theta}{2} \right)
\]  \hspace{1cm} (47)

Thus

\[
\operatorname{Re} \left( \eta' \delta \right) = \frac{1}{2} \operatorname{Re} \left[ \left( -\xi \sin \left( \frac{\theta}{2} \right) + i (\nu_0^n)^2 \sin \left( \frac{\theta}{2} \right) \right) \left( \operatorname{Re}(\delta) + i \operatorname{Im}(\delta) \right) \right]
\]

\[
= \frac{1}{2} \left[ |\nu_0^n| \cos \left( \frac{\theta}{2} \right) \sqrt{\operatorname{Im}^2(\delta) - \xi^2 \sin \left( \frac{\theta}{2} \right) \sqrt{\operatorname{Re}^2(\delta)}} \right].
\]

Note that from (46) and (47), for \( \theta = 0 \) we have \( \operatorname{Re} \left( \eta' \delta \right) = 0 \), and from (45) we note that \( \theta = 0 \) is a critical point of \( |\lambda_\pm(\theta)| \) in \([-\pi, \pi]\) because it is an even function and its derivative is zero in \( \theta = 0 \).

Furthermore, it is easy to show that in \([0, \pi]\) it does not exist any other relative critical point. In fact, if we suppose that

\[
\exists \theta \in [0, \pi] / \ |\nu_0^n| \cos \left( \frac{\theta}{2} \right) \sqrt{\operatorname{Im}^2(\delta) - \xi^2 \sin \left( \frac{\theta}{2} \right) \sqrt{\operatorname{Re}^2(\delta)}} = \xi \sin \left( \frac{\theta}{2} \right) \sqrt{\operatorname{Re}^2(\delta)},
\]

we then have

\[
(\nu_0^n)^2 \cos^2 \left( \frac{\theta}{2} \right) \operatorname{Im}(\delta)^2 = \xi^2 \sin^2 \left( \frac{\theta}{2} \right) \operatorname{Re}(\delta)^2,
\]

and from (46) and (47)

\[
(\nu_0^n)^2 \cos^2 \left( \frac{\theta}{2} \right) \left( \frac{\sqrt{m} - n}{2} \right) = \xi^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \frac{\sqrt{m} + n}{2} \right).
\]
Reordering and using
\[ m - n^2 = 4 \left( v_0^n \right)^2 \xi^2 \sin^2 \left( \frac{\theta}{2} \right) \cos^2 \left( \frac{\theta}{2} \right) , \] (48)
then
\[ 4 \left( v_0^n \right)^4 \cos^4 \left( \frac{\theta}{2} \right) = 4 \xi^2 \sin^2 \left( \frac{\theta}{2} \right) \Re (\delta)^4 , \]
and since \( \theta \neq 0 \)
\[ \Re (\delta)^4 = \left( \left( v_0^n \right)^2 \cos^2 \left( \frac{\theta}{2} \right) \right)^2 . \]
From (46) and (48) it follows that
\[ 4 \left( v_0^n \right)^2 \cos^2 \left( \frac{\theta}{2} \right) \left( \left( v_0^n \right)^2 \cos^2 \left( \frac{\theta}{2} \right) - n \right) = 4 \left( v_0^n \right)^2 \xi^2 \cos^2 \left( \frac{\theta}{2} \right) \sin^2 \left( \frac{\theta}{2} \right) . \]
From \( \theta \neq \pi \) and substituting \( n \) by its expression given by (47), one concludes that
\[ \left( v_0^n \right)^2 \cos^2 \left( \frac{\theta}{2} \right) - 1 + \left( \xi^2 + \left( v_0^n \right)^2 \right) \sin^2 \left( \frac{\theta}{2} \right) = \xi^2 \sin^2 \left( \frac{\theta}{2} \right) . \]
i.e.,
\[ \left( v_0^n \right)^2 - 1 = 0 , \]
which contradicts the hypotheses. Hereby, \( \theta = 0 \) is the only one relative critical point of \( |\lambda_+ (\theta)| \) in \([-\pi, \pi] \). Taking into account that \( |\lambda_+ (\theta)| \) is an even function, we can write
\[ \max_{\theta \in [-\pi, \pi]} |\lambda_+ (\theta)| = \max \{|\lambda_+ (0)| , |\lambda_+ (\pi)| \} . \]
But from (38) and (39) we have that
\[ \lambda_+ (0) = \frac{1}{1 + \xi} \left( \xi + \sqrt{\xi^2 + 1 - \xi^2} \right) = 1 , \]
and
\[ \lambda_+ (\pi) = \frac{1}{1 + \xi} \left( -iv_0^n + \sqrt{-(v_0^n)^2 + 1 - \xi^2} \right) . \]
If the radicand is positive, then
\[ |\lambda_+ (\pi)| = \frac{1}{1 + \xi} \left( (v_0^n)^2 - (v_0^n)^2 + 1 - \xi^2 \right)^{1/2} = \sqrt{1 - \xi} < 1 , \]
but if the radicand is negative and under hypothesis (29), then
\[ |\lambda_+ (\pi)| = \left| \frac{-v_0^n + \sqrt{(v_0^n)^2 + \xi^2 - 1}}{1 + \xi} \right| \leq \frac{|v_0^n| + \xi}{1 + \xi} < 1 . \]
Hence we conclude that
\[ \max_{\theta \in [-\pi, \pi]} |\lambda_+ (\theta)| = |\lambda_+ (0)| = 1 . \]
Finally, let’s analyze \( |\lambda_- (\theta)| \). From (38) one concludes immediately that
\[ \lambda_+ (\theta) \lambda_- (\theta) = \frac{\xi - 1}{\xi + 1} \in \mathbb{R} , \]
independently of $\theta$. From this, we can write
\[
|\lambda_-(0)| = \frac{1}{|\lambda_+(0)|} \left| \frac{\xi - 1}{\xi + 1} \right| < \frac{\xi + 1}{\xi + 1} = 1,
\]
and
\[
|\lambda_-(\pi)| = \frac{1}{1 - \xi} \left| -iv_0^n - \sqrt{(-v_0^n)^2 + 1 - \xi^2} \right|.
\]
If the radicand is positive, then
\[
|\lambda_-(\pi)| = \sqrt{\frac{1 - \xi}{1 + \xi} < 1},
\]
but if the radicand is negative, from (29) it follows that
\[
|\lambda_-(\pi)| \leq |v| + \xi < 1.
\]
Then
\[
\rho(Q^n(\theta)) \leq 1; \quad \max_{\theta \in [-\pi, \pi]} \rho(Q^n(\theta)) = 1,
\]
and thus, the result is established. \(\square\)

On the other hand, we can prove that under (30), if $|v(t)| > 1$, then
\[
\max_{\theta \in [-\pi, \pi]} \rho(Q^n(\theta)) > 1.
\]
Under conditions of Theorem 3, the next result, presented in [9], is established.

**Theorem 4.** The numerical scheme (18), under conditions (29) and (30) of Theorem 3, is stable.

**Proof.** From (5) we can write
\[
\|q(n+1)\|_2^2 = \frac{\Delta x}{2\pi} \|q^*(n+1)\|_2^2 = \frac{\Delta x}{2\pi} \int_{-\pi}^{\pi} \|q^*(n+1, \theta)\|^2 d\theta.
\]
Applying (27)
\[
\|q(n+1)\|_2^2 = \frac{\Delta x}{2\pi} \int_{-\pi}^{\pi} \|Q^{n+1}(\theta)Q^{n-\frac{1}{2}}(\theta)q^*(n+1, \theta)\|^2 d\theta \leq \frac{\Delta x}{2\pi} \int_{-\pi}^{\pi} \|Q^{n+1}(\theta)\|_2^2 \|Q^{n-\frac{1}{2}}(\theta)\|_2^2 \|q^*(n+1, \theta)\|^2 d\theta,
\]
where $\|\|_2$ is the 2–norm for a matrix. In the next step we evaluate $\|Q^k(\theta)\|_2$, $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots, n + 1$, with $(n+1)\Delta t \leq T$.

From the norm given by (6), for $Q^k(\theta)$ and $\delta \Delta t/2 > 0$, there exists a norm, denoted by $\|\|_{k-\theta}$, such that
\[
\|Q^k(\theta)\|_{k-\theta} \leq \rho(Q^k(\theta)) + \frac{\delta \Delta t}{2}.
\]
From Theorem 3, under hypotheses (29) and (30) we have that
\[
\|Q^k(\theta)\|_{k-\theta} \leq 1 + \frac{\delta \Delta t}{2},
\]
and by the equivalence between norms, $\exists M(k, \theta) > 0$ such that
\[
\|Q^k(\theta)\|_2 \leq M(k, \theta) \left( 1 + \frac{\delta \Delta t}{2} \right).
\]
Substituting (50) in (49) it follows that
\[
\|q(n+1)\|_2^2 \leq \frac{\Delta x}{2\pi} \int_{-\pi}^{\pi} M^2(n+1, \theta)M^2(n, \theta) \left(1 + \frac{\delta \Delta t}{2}\right)^4 \|q^*(n+1, \theta)\|^2 d\theta.
\]
If we take
\[
M(k) = \max_{\theta \in [-\pi, \pi]} \max_{\theta \in [0, \pi]} M(k, \theta),
\]
then
\[
\|q(n+1)\|_2^2 \leq \frac{\Delta x}{2\pi} M^2(n+1)M^2(n) \left(1 + \frac{\delta \Delta t}{2}\right)^4 \int_{-\pi}^{\pi} \|q^*(n+1, \theta)\|^2 d\theta,
\]
and, from (5),
\[
\|q(n+1)\|_2^2 \leq M^2(n+1)M^2(n) \left(1 + \frac{\delta \Delta t}{2}\right)^4 \|q(n)\|_2^2.
\]
Doing the iteration from \(q(0)\) to \(q(n)\), with \(n \Delta t \leq T\), we obtain that
\[
\|q(n+1)\|_2 \leq C(T) \left(1 + \frac{\delta \Delta t}{2}\right)^{2n} \|q(0)\|_2,
\]
where
\[
C(T) = M(n)M(0) \left(\prod_{i=1}^{n-1} M^2(i)\right).
\]
From
\[
\left[\left(1 + \frac{\delta \Delta t}{2}\right)^n\right]^2 \leq \left(e^{\frac{\delta \Delta t}{2}}\right)^2,
\]
one concludes that
\[
\|q(n+1)\|_2 \leq C(T)e^{\frac{\delta \Delta t}{2}} \|q(0)\|_2.
\]  \(\square\)

4. Example

In order to test the proposed scheme, we consider the equation
\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) + a(t) \frac{\partial}{\partial x} u(x, t) - b(t) \frac{\partial^2}{\partial x^2} u(x, t) &= 0, \\
x \in [0, 1], \quad t \in [0, \pi[,
\end{align*}
\]  \(\tag{52}\)
with
\[
a(t) = \cos(t), \quad b(t) = 1 + \cos(t), \quad u(x, 0) = e^x, \quad u(0, t) = e^t, \quad u(1, t) = e^{1+t}.
\]  \(\tag{53}\)
The exact solution of this problem is
\[
u(x, t) = e^{x+t}.
\]  \(\tag{54}\)
In this section we compare the numerical method presented in this article with the original CE-SE method and with the central finite-difference approximation
\[
\begin{align*}
q(j, n+1) &= q(j, n) - a(t^j) \frac{\Delta t}{2\Delta x} (q(j+1, n) - q(j-1, n)) \\
&\quad + b(t^j) \frac{\Delta t}{\Delta x^2} (q(j+1, n) - 2q(j, n) + q(j-1, n)).
\end{align*}
\]  \(\tag{55}\)
Table 1
Relative errors with CE-SE method

<table>
<thead>
<tr>
<th>x</th>
<th>T = 0.628319</th>
<th>T = 1.88496</th>
<th>T = 3.14159</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.00058</td>
<td>0.037</td>
<td>0.018</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0070</td>
<td>0.052</td>
<td>0.011</td>
</tr>
<tr>
<td>0.6</td>
<td>0.010</td>
<td>0.048</td>
<td>0.0042</td>
</tr>
<tr>
<td>0.8</td>
<td>0.011</td>
<td>0.033</td>
<td>0.0040</td>
</tr>
<tr>
<td>Maximum error</td>
<td>0.011</td>
<td>0.052</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Table 2
Relative errors with the modified CE-SE method

<table>
<thead>
<tr>
<th>x</th>
<th>T = 0.628319</th>
<th>T = 1.88496</th>
<th>T = 3.14159</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.0060</td>
<td>0.036</td>
<td>0.010</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0077</td>
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<td>0.0046</td>
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<tr>
<td>0.6</td>
<td>0.0072</td>
<td>0.038</td>
<td>0.0011</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0048</td>
<td>0.021</td>
<td>0.000013</td>
</tr>
<tr>
<td>Maximum error</td>
<td>0.0077</td>
<td>0.045</td>
<td>0.010</td>
</tr>
</tbody>
</table>

Table 3
Relative errors with a central finite-difference method

<table>
<thead>
<tr>
<th>x</th>
<th>T = 0.628319</th>
<th>T = 1.88496</th>
<th>T = 3.14159</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.000052</td>
<td>0.000024</td>
<td>0.00020</td>
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<tr>
<td>0.4</td>
<td>0.000070</td>
<td>0.000030</td>
<td>0.00022</td>
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<tr>
<td>0.6</td>
<td>0.000064</td>
<td>0.000026</td>
<td>0.00016</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000038</td>
<td>0.000014</td>
<td>0.000089</td>
</tr>
<tr>
<td>Maximum error</td>
<td>0.000070</td>
<td>0.000030</td>
<td>0.00043</td>
</tr>
</tbody>
</table>

Table 4
Relative errors with the modified CE-SE method

<table>
<thead>
<tr>
<th>x</th>
<th>T = 0.628319</th>
<th>T = 1.88496</th>
<th>T = 3.14159</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.000014</td>
<td>0.000022</td>
<td>0.00024</td>
</tr>
<tr>
<td>0.4</td>
<td>0.000019</td>
<td>0.000028</td>
<td>0.00022</td>
</tr>
<tr>
<td>0.6</td>
<td>0.000017</td>
<td>0.000024</td>
<td>0.00016</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000010</td>
<td>0.000013</td>
<td>0.000082</td>
</tr>
<tr>
<td>Maximum error</td>
<td>0.000019</td>
<td>0.000028</td>
<td>0.00024</td>
</tr>
</tbody>
</table>

This scheme is stable [12] if \( C_r < P_e/2 \) and \( P_e < 2 \), where \( C_r = a(t) \Delta t / \Delta x \) and \( P_e = a(t) \Delta x / b(t) \).

In the proposed method, the values of \( U^n_j \) for \( n = 0 \) and \( n = N \) are given by the boundary conditions of the problem, and the corresponding values for \( (U_x)^n_j \) are deduced from \( F_+(0, n) \) and \( F_-(N, n) \), being \( 2N + 1 \) the number of spatial points.

First we compare the errors of CE-SE method (Table 1) and the modified CE-SE method (Table 2) using \( \Delta x = 0.1 \) and \( \Delta t = \pi/50 \). We can observe that the global error of the modified CE-SE method improves the one obtained with CE-SE method.

Now we compare the central finite-difference method (Table 3) with the modified CE-SE method (Table 4). To do this we must reduce the time step in order to make the finite-difference stable. So we use \( \Delta x = 0.1 \) and \( \Delta t = \pi/1300 \). We can observe that finite-difference errors are bigger than the error provided by means of the modified CE-SE method.

References