Variational principle for Zakharov–Shabat equations in two-dimensions

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Abstract
We study the corresponding scattering problem for Zakharov and Shabat compatible differential equations in two-dimensions, the representation for a solution of the nonlinear Schrödinger equation is formulated as a variational problem in two-dimensions. We extend the derivation to the variational principle for the Zakharov and Shabat equations in one-dimension. We also developed an approximate analytical technique for finding discrete eigenvalues of the complex spectral parameters in Zakharov and Shabat equations for a given pulse-shaped potential, which is equivalent to the physically important problem of finding the soliton content of the given initial pulse. Using a trial function in a rectangular box we find the functional integral. The general case for the two box potential can be obtained on the basis of a different ansatz where we approximate the Jost function by polynomials of order \(n\) instead of a piecewise linear function. We also demonstrated that the simplest version of the variational approximation, based on trial functions with one, two and \(n\)-free parameters respectively, and treated analytically.

1. Introduction
The nonlinear Schrödinger equation (NLSE) is one of the most important completely integrable nonlinear equations. It has various applications in nonlinear optics, hydrodynamics, plasma physics, astrophysics and quantum field theory [1–4]. After the Korteweg–deVries (KdV) equation, the NLSE was the second for which an inverse scattering transform was discovered by Zakharov and Shabat [5,6]. Among those analytical tools is the inverse scattering transform (IST), which can be viewed as a generalization of the well known Fourier transform method used for linear problems [7–11]. A key role in the solution scheme of the NLSE is played by the Zakharov–Shabat (ZS) scattering problem. The ZS scattering problem associated with the IST for solving the NLSE can be reformulated as a variational problem.

The inverse scattering method (ISM) as applied to solving nonlinear differential equations (NDEs) can be included among the important techniques of mathematical physics [12]. As it is well known, any equation amenable to solution by means of the IST admits a representation in the form of a compatibility condition for two systems of auxiliary linear equations for the so-called Jost functions. One of them has the form of a linear scattering problem. We use the variational principle as a calculation tool, with special emphasis on methods for the construction of such principles. The ZS equations have a natural variational representation, and this fact often used as a basis for the development of an analytical approximation for finding the soliton content of a given pulse. A general review of variational principles was given in Refs. [13,14]. It is well known that...
the variational representation of the linear Schrödinger equation plays an important role in quantum mechanics for finding the energy spectra of many systems to good accuracy [15]. Returning to the NLSE, it is relevant to mention that its variational (Lagrangian) representation has been successfully used for obtaining physically meaningful approximate solutions [16,17] but the integrability of the equation and, therefore, of the ZS equation were not employed in these works at all.

Kaup and Malomed [18,19] have discussed the application of the variational principle to nonlinear dissipative systems, in particular the NLSE with diffusive and damping terms. They put forth a variational principle for the ZS equations, which is the basis of the IST or a number of important nonlinear partial differential equations (PDEs). They used the variational representation of the ZS equations, to develop an approximate analytical technique for finding discrete eigenvalues of the complex spectral parameter in the ZS equations for a given pulse-shaped potential, which is equivalent to physically important problem of finding the soliton content of the given initial pulse. They also applied the technique further to several particular shapes of the pulse. They demonstrated that the simplest version of the variational approximation, based on trial functions with one or two free parameters turned out to be fully analytically tractable and yields threshold conditions for the appearance of the first soliton, or of the first pair, which are in a fairly good agreement with available numerical results [20].

In this paper, we extend the variational formulation for the ZS equations in one-dimension [18,19] to two dimensions. Studying high-dimensional nonlinear evolution equations (NLEEs) is of a prime significance in physics [21–27]. A situation of the first soliton, or of the first pair, which are in a fairly good agreement with available numerical results [20].

This paper is organized as follows: In Section 2, the variational principle for the ZS equations in two-dimensions is formulated. In Section 3, the simplest example of the application of this technique, taking the box shaped initial pulse and an ansatz based on linear Jost functions in the region of localization of the box is demonstrated in two cases. In Section 4, we give in more details the limiting case of the above-mentioned box with the phase jump equal to $\pi$, i.e., a combination of two boxes of opposite signs, the total surface of the initial pulse thus being zero. In Section 5, we approximate the Jost functions by quadratic polynomials instead of the piece-wise linear functions, which yield similar results for the two-box potential. We extended further the approximation of the Jost functions by polynomials of order $n$ instead of a piecewise linear function. We also approximated the Jost functions by quadratic polynomials instead of the piece-wise linear functions, which yield similar results for the two-box potential. We extended further the approximation of the Jost functions by polynomials of order $n$ instead of the piece-wise linear functions recovering the results of [18,19] as special cases. After substitution of the chosen ansatz into the Lagrangian and performing the integration one obtains an effective Lagrangian as a function of the free parameters. One next finds the set of values of those parameters at which the effective Lagrangian has an extremum. Finally, substituting the corresponding values back into the Lagrangian (which coincides with the eigenvalue) one obtains a variational value for the soliton’s eigenvalue. If the variation of the effective Lagrangians produces several extrema.

This paper is organized as follows: In Section 2, the variational principle for the ZS equations in two-dimensions is formulated. In Section 3, the simplest example of the application of this technique, taking the box shaped initial pulse and an ansatz based on linear Jost functions in the region of localization of the box is demonstrated in two cases. In Section 4, we give in more details the limiting case of the above-mentioned box with the phase jump equal to $\pi$, i.e., a combination of two boxes of opposite signs, the total surface of the initial pulse thus being zero. In Section 5, we approximate the Jost functions by quadratic polynomials instead of the piece-wise linear functions, which yield similar results for the two-box potential. In Section 6, furthermore, we extend the approximation of the Jost functions by polynomials of order $n$ instead of the piece-wise linear functions recovering the results of [18,19] as special cases. Finally the paper ends with a conclusion in Section 7.

2. Formulation of the variational principle

The ZS system [18,19] can be extended to two-dimensions for the two component complex Jost function ($\Psi^{(1)}(x,y)$ and $\Psi^{(2)}(x,y)$) in the form

\begin{align}
  i\Psi^{(1)}_x + i\Psi^{(1)}_y + 2\Psi^{(1)} + u(x,y)\Psi^{(2)} &= 0, \\
  i\Psi^{(2)}_x + i\Psi^{(2)}_y - 2\Psi^{(2)} + u(x,y)\Psi^{(1)} &= 0,
\end{align}

where $\Psi^{(i)}_x$ and $\Psi^{(i)}_y$ are the partial derivatives, $u(x,y)$ is the potential whose soliton content must be found and $u^*(x,y)$ is its complex conjugate, $\lambda$ is the complex spectral parameter. Multiplying Eq. (1) by $\Psi^{(2)}$ and Eq. (2) by $\Psi^{(1)}$ and integrating the sum over $dxdy$, one can immediately obtain the following representation for the complex spectral parameter

\begin{equation}
  \lambda = \frac{L}{N},
\end{equation}

where

\begin{align}
  L &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{i}{2} \left( (\Psi^{(1)} + \Psi^{(1)}_y)\Psi^{(2)} - (\Psi^{(2)} + \Psi^{(2)}_y)\Psi^{(1)} \right) + \frac{1}{2} (u\Psi^{(2)} - u^*\Psi^{(1)})^2 \right] dxdy, \\
  N &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^{(1)}\Psi^{(2)} dxdy.
\end{align}

Now, one can immediately verify that varying the expression (3) with respect to the functions $\Psi^{(1)}(x,y)$ and $\Psi^{(2)}(x,y)$ produces exactly Eqs. (1) and (2). Thus Eqs. (3)–(5) represent the effective Lagrangian for the ZS equations in two-dimensions.
There are two points, which must be noted about Eqs. (4) and (5). First the normalization coefficient $N$, is not positive definite. Thus for an arbitrary trial function, $N$ could pass through zero and even go negative. Second, the kinetic term in (4) is also not positive definite. Furthermore it vanishes if $\Psi^{(1)}(x, y)$ is chosen to be proportional to $\Psi^{(2)}(x, y)$. Thus another key feature of this Lagrangian requires that trial functions for these components should be functionally different.

3. The rectangular box

We demonstrate the simplest example of the application of this technique, taking the box shaped initial pulse and an ansatz based on linear Jost functions in a single nontrivial variational parameter, we will consider the potential in the following form

$$u(x, y) = \begin{cases} 0, & \text{at } |x| > 1, |y| > 1, \\ A \exp(i\varepsilon \text{ sgn } x) \exp(i\varepsilon \text{ sgn } y), & \text{at } |x| < 1, |y| < 1. \end{cases}$$  \hspace{1cm} (6)$$

Here $A$ is the real amplitude, the absolute surface of the pulse is $2A$ and $2\varepsilon$ in the phase jump at the central point of the box. Next, we adopt the following ansatz for the Jost functions, which is, as a matter of fact, the simplest possible choice

$$\Psi^{(1)}(x, y) = \begin{cases} 4 \exp(-\mu(x-1)(y-1)) & \text{at } x > 1, y > 1, \\ (x+1)(y+1) & \text{at } |x| < 1, |y| < 1, \\ 0 & \text{at } x < -1, y < -1. \end{cases}$$  \hspace{1cm} (7)$$

$$\Psi^{(2)}(x, y) = \begin{cases} 0 & \text{at } x > 1, y > 1, \\ B(1-x)(1-y) & \text{at } |x| < 1, |y| < 1, \\ 4B \exp(\mu(x+1)(y+1)) & \text{at } x < -1, y < -1. \end{cases}$$  \hspace{1cm} (8)$$

Note that, for any potential with a compact support, the components $\Psi^{(1)}(x, y)$ and $\Psi^{(2)}(x, y)$ of the Jost function must be exactly equal to zero, respectively, to the left and to the right of the support, in order to comply with the standard boundary conditions for the Jost functions at infinity for bound states. We have two free parameters, $\mu$ and $B$ in Eqs. (7) and (8). However, as it immediately follows from Eqs. (4) and (5), the Lagrangian does not depend on $\mu$. In fact, any choice of a trial function in this region will work, provided it vanishes at infinity. A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (7) and (8) to be

$$L = \frac{32}{3}iB + \frac{1}{18}A\left[B^2(49 \exp(2i\varepsilon) + \exp(-2i\varepsilon)) - (49 \exp(2i\varepsilon) + \exp(-2i\varepsilon))\right],$$  \hspace{1cm} (9)$$

$$N = \frac{16}{9}A.$$  \hspace{1cm} (10)$$

Next, substitution of these expressions into Eq. (3) gives

$$\lambda = -6i - \frac{A}{32B} \left[B^2(49 \exp(2i\varepsilon) + \exp(-2i\varepsilon)) - (49 \exp(2i\varepsilon) + \exp(-2i\varepsilon))\right].$$  \hspace{1cm} (11)$$

Varying the only nontrivial parameter $B$ yields the result that the critical points exist only if $B = \pm i$. Inserting these critical values of $B$ back into Eq. (11) yields

$$\lambda = -6i \pm \frac{25}{8}iA \cos 2\varepsilon \mp 3A \sin 2\varepsilon.$$  \hspace{1cm} (12)$$

Then

$$\text{Im} \lambda = -6 \pm \frac{25}{8}A \cos 2\varepsilon, \quad \text{and} \quad \text{Re} \lambda = \mp 3A \sin 2\varepsilon,$$  \hspace{1cm} (13)$$

where the plus or minus signs correlate with that in $B = \pm i$. Since only the eigenvalue with positive imaginary part is meaningful [5], one should keep only the upper sign. Then, the imaginary part in Eq. (13) tells us that, with the increase of the surface $S = 2A$ of the pulse (6), in 2-dimensions the soliton appears at the threshold value

$$S_{thr} = \frac{96}{25 \cos 2\varepsilon}.$$  \hspace{1cm} (14)$$

Let us first consider dwell on the case $\varepsilon = 0$, the ZS equations with the potential (6) have the solution $\frac{96}{25}$. Thus, our crude approximation using the single variational parameter gives only a small error. However, this approximation fails to predict additional solitons which would appear with the increase of the surface, simply because we have only a single nontrivial variational parameter $B$ in the ansatz. Eq. (14) demonstrates that $S_{thr}$ monotonically increases with the increase of the phase jump $4\varepsilon$. This seems to be quite a reasonable trend. However, Eq. (14) also predicts that $S_{thr}$ diverges in the limit $4\varepsilon = \pi$, when the potential (6) turns into a combination of two box potentials with opposite signs.
4. The two-box potential

We consider in more details the limiting case of the above mentioned box with the phase jump equals to π i.e., a combination of two boxes of opposite sign, the total surface of the initial pulse being thus zero. We develop a variational approximation for finding the eigenvalues of this pulse, the Jost functions being approximated by a piece-wise linear ansatz, which has two variational parameters. Then we generalize this ansatz to a two-parameter ansatz, approximating the Jost functions by quadratic polynomials. We will use an improved ansatz for the two-box potential as

\[
u(x, y) = \begin{cases} 
0, & \text{at } |x| > 1, |y| > 1, \\
A, & \text{at } 0 < x < 1; 0 < y < 1, \\
-A, & \text{at } -1 < x < 0; -1 < y < 0.
\end{cases}
\]  

(15)

Since this potential is discontinuous, it is natural to try a generalization of the linear ansatz of Eqs. (7) and (8), which allows discontinuity of the first derivatives of the Jost functions.

4.1. First case

We try the following Jost functions as

\[
\Psi^{(1)}(x, y) = \begin{cases} 
(1 + x^2) \exp(-\mu(x - 1)(y - 1)) & \text{at } x > 1, y > 1, \\
(1 + 2x)(1 + 2y) & \text{at } 0 < x < 1, 0 < y < 1, \\
(x + 1)(y + 1) & \text{at } -1 < x < 0, -1 < y < 0, \\
0 & \text{at } x < -1, y < -1,
\end{cases}
\]  

(16)

\[
\Psi^{(2)}(x, y) = \begin{cases} 
0, & \text{at } x > 1, y > 1, \\
(1-x)(1-y) & \text{at } 0 < x < 1, 0 < y < 1, \\
(1-2x)(1-2y) & \text{at } -1 < x < 0, -1 < y < 0, \\
(1 + x^2) \exp(\mu(x + 1)(y + 1)) & \text{at } x < -1, y < -1.
\end{cases}
\]  

(17)

This ansatz now contains one nontrivial variational parameter \(\mu\).

Substituting Eqs. (16) and (17) into Eqs. (4) and (5), one can find the values of the integrals \(L\) and \(N\) which determine the Lagrangian according to Eq. (3),

\[
N = \frac{1}{18} (3 + \mu)^2,
\]

\[
L = i \left( \frac{4}{3} \mu + \frac{1}{3} \mu^2 \right) - A \left( \frac{8}{9} + 2\mu + \frac{5}{3} \mu^2 + \frac{2}{3} \mu^3 + \frac{1}{9} \mu^4 \right).
\]

Insertion of these expressions into Eq. (3) yields

\[
\lambda = \frac{A(16 + 36x + 30x^2 + 12x^3 + 2x^4) - i(18 + 24x + 6x^2)}{(3 + \mu)^2}.
\]  

(18)

Varying the expression (18) leads to the value of \(\mu\)

\[
\mu = -4.706; \quad \mu = -1.686; \quad \mu = -1.304 - 0.833i; \quad \mu = -1.304 + 0.833i.
\]  

(19)

By substitution of the values of \(\mu\) into the expression of eigenvalue (18) produces analytical expressions

\[
\lambda = -13.0354i + 82.8923A;
\]

\[
\lambda = 3.13487i - 0.44528A;
\]

\[
\lambda = (2.7998 + 0.299752i) + (0.348521 + 0.507771i)A;
\]

\[
\lambda = (2.7998 - 0.299752i) - (0.348521 - 0.507771i)A.
\]  

(20)

4.2. Second case

We choose the trial functions in the form

\[
\Psi^{(1)}(x, y) = \begin{cases} 
2(1 + x) \exp(-\mu(x + y - 2)) & \text{at } x > 1, y > 1, \\
(2 + 2x + 2y) & \text{at } 0 < x < 1, 0 < y < 1, \\
(2 + x + y) & \text{at } -1 < x < 0, -1 < y < 0, \\
0 & \text{at } x < -1, y < -1,
\end{cases}
\]  

(21)
\( \Psi^{(2)}(x,y) = \begin{cases} 
0, & \text{at } x > 1, \ y > 1, \\
(2-x-y) & \text{at } 0 < x < 1, \ 0 < y < 1, \\
(2-ax-2y) & \text{at } -1 < x < 0, \ -1 < y < 0, \\
2(1+x) \exp \left( \mu(x+y+2) \right) & \text{at } x < -1, \ y < -1.
\end{cases} \) (22)

This ansatz now contains one nontrivial variational parameter \( \alpha \). Substituting Eqs. (21) and (22) into Eqs. (4) and (5), one can find the values of the integrals \( L \) and \( N \) which determine the Lagrangian according to Eq. (3).

\[
N = 4 + 5\alpha^2,
\]

\[
L = i(4 + 4\alpha) - A \left( \frac{17}{6} + 4\alpha + \frac{7}{6} \alpha^2 \right).
\]

Insertion of these expressions into Eq. (3) yields

\[
\lambda = \frac{A(17 + 24\alpha + 7\alpha^2) - i(24 + 24\alpha)}{24 + 10\alpha}. \quad (23)
\]

Varying the expression (23) leads to the value of \( \alpha \)

\[
\alpha = -2.4 - 0.1999i; \quad \alpha = -2.4 + 0.1999i. \quad (24)
\]

By substitution of the values of \( \alpha \) into the expression of eigenvalue (23) produces analytical expressions

\[
\lambda = (-16.8 - 2.4i) - (0.96 + 0.28i)A; \\
\lambda = (16.8 - 2.4i) - (0.96 - 0.28i)A. \quad (25)
\]

5. Quadratic polynomials

Qualitatively similar results for the two-box potential can be obtained on the basis of a different ansatz, where we approximate the Jost functions by quadratic polynomials instead of the piecewise linear functions in Eqs. (16), (17), (21) and (22). For \(|x| < 1 \) and \(|y| < 1 \).

5.1. First case

We choose the following Jost functions as

\[
\Psi^{(1)}(x,y) = \begin{cases} 
(4 + 16\alpha) \exp \left( -\mu(x-1)(y-1) \right) & \text{at } x > 1, \ y > 1, \\
(x+1)(y+1) + \alpha(x+1)^2(y+1)^2 & \text{at } |x| < 1, \ |y| < 1, \\
0 & \text{at } x < -1, \ y < -1.
\end{cases} \quad (26)
\]

\[
\Psi^{(2)}(x,y) = \begin{cases} 
0, & \text{at } x > 1, \ y > 1, \\
(1-x)(1-y) + \alpha(1-x)^2(1-y)^2 & \text{at } |x| < 1, \ |y| < 1, \\
(4 + 16\alpha) \exp \left( \mu(x+1)(y+1) \right) & \text{at } x < -1, \ y < -1.
\end{cases} \quad (27)
\]

A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (26) and (27) to be

\[
N = \frac{16}{9} + \frac{32}{9} \alpha + \frac{256}{225} \alpha^2,
\]

\[
L = \frac{4}{1575} \left[ 2i(875 + 4\alpha(7 + 4\alpha)(105 + 32\alpha)) - 105A(20 + 3\alpha(35 + 48\alpha)) \right].
\]

Then we have

\[
\lambda = -\frac{2i(875 + 4\alpha(7 + 4\alpha)(105 + 32\alpha)) + 105A(20 + 3\alpha(35 + 48\alpha))}{28(5 + 2\alpha)(5 + 8\alpha)}. \quad (28)
\]

Subsequently varying the eigenvalue (28) leads to a equation for \( \alpha \)

\[
\frac{d\lambda}{d\alpha} = \frac{1}{84} \left[ -192i + \frac{5(-260i + 5523A)}{(5 + 2\alpha)^2} - \frac{20(52i + 357A)}{(5 + 8\alpha)^2} \right] = 0.
\]
the roots of this equation are
\[ \lambda = -0.836469; \quad \lambda = -0.351936; \quad \lambda = -2.49 - 1.29723i; \]
\[ \lambda = -2.49 + 1.29723i; \quad \lambda = -0.6349 - 0.23989i; \quad \lambda = -0.6349 + 0.23989i. \] (29)

By substitution of the values of \( \lambda \) into the expression of eigenvalue (28) produces analytical expressions
\[ \lambda = -1.04309i + 0.352589A; \]
\[ \lambda = -1.03421i - 21.9359A; \]
\[ \lambda = (1.61584 - 0.89848i) - (9.82756 - 0.049i)A; \]
\[ \lambda = (-1.61584 - 0.89848i) - (9.82756 + 0.049i)A; \]
\[ \lambda = (5.9961 + 1.28737i) + (32.7859 + 63.0185i)A; \]
\[ \lambda = (-5.9961 + 1.28737i) + (32.7859 - 63.0185i)A. \] (30)

Finally, the results for the imaginary and real parts of the eigenvalues (30) from that expressions prove to be quite similar to those displayed in the previous first cases for the piece-wise linear ansatz: at the threshold there appears only a symmetric pair of solitons, and no more pairs appear with the subsequent increase of the amplitude A.

5.2. Second case

We choose the following Jost functions in the form
\[
\Psi^{(1)}(x, y) = \begin{cases} 
(4 + 8x) \exp \left(-\mu(x - 1)(y - 1)\right) & \text{at } x > 1, \quad y > 1, \\
(2 + x + y) + \alpha(x + 1)^2 + \alpha(y + 1)^2 & \text{at } |x| < 1, \quad |y| < 1, \\
0 & \text{at } x < -1, \quad y < -1,
\end{cases}
\] (31)

\[
\Psi^{(2)}(x, y) = \begin{cases} 
0, & \text{at } x > 1, \quad y > 1, \\
(2 - x - y) + \alpha(1 - x)^2 + \alpha(1 - y)^2 & \text{at } |x| < 1, \quad |y| < 1, \\
(4 + 8x) \exp \left(\mu(x + 1)(y + 1)\right) & \text{at } x < -1, \quad y < -1.
\end{cases}
\] (32)

A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (31) and (32) to be
\[
N = \frac{40}{3} + 32x + \frac{832}{45} x^2,
\]
\[
L = \frac{2}{45} i(1 + 2x)(480 + 15iA(12 + 17x) + 4\alpha(225 + 104\alpha)).
\]

Then we have
\[
\lambda = \frac{(1 + 2x)(15A(12 + 17x) - 4i(120 + \alpha(225 + 104\alpha)))}{300 + 16\alpha(45 + 26x)}. \] (33)

Subsequently varying the eigenvalue (33) leads to a equation for \( \alpha \)
\[
\frac{d\alpha}{d\alpha} = \frac{(1 + 2x)(255A - 4i(225 + 208\alpha))}{300 + 16\alpha(45 + 26x)} + \frac{2(15A(12 + 17x) - 4i(120 + \alpha(225 + 104\alpha))}{300 + 16\alpha(45 + 26x)} \times \frac{15A(915 + 4\alpha(651 + 464\alpha)) - 4i(13275 + 4\alpha(14535 + 4\alpha(6135 + 104\alpha(45 + 13\alpha)))}{(300 + 16\alpha(45 + 26x))^2},
\]
the roots of this equation are
\[ \alpha = -0.702 - 0.0297i; \quad \alpha = -0.702 + 0.0297i; \quad \alpha = -1.015 - 0.117i; \]
\[ \alpha = -1.015 + 0.117i; \quad \alpha = -0.7156 - 0.275i; \quad \alpha = -0.7156 - 0.275i. \] (34)
By substitution of the values of $\lambda$ into the expression of eigenvalue (33) produces analytical expressions

$$\lambda = (-5.2699 - 0.79698i) - (0.675 + 0.3444i)A;$$
$$\lambda = (5.2699 - 0.79698i) - (0.675 - 0.3444i)A;$$
$$\lambda = (-0.7636 - 0.2148i) + (0.4733 - 5.359)iA;$$
$$\lambda = (0.7636 - 0.2148i) + (0.4733 + 5.359)iA;$$
$$\lambda = (-1.183 - 0.389i) + (0.0777 - 1.0178)iA;$$
$$\lambda = (1.183 - 0.389i) + (0.0777 + 1.0178)iA. \quad (35)$$

Finally, the results for the imaginary and real parts of the eigenvalues (35) from that expressions prove to be quite similar to those displayed in the previous first cases for the piece-wise linear ansatz: at the threshold there appears only a symmetric pair of solitons, and no more pairs appear with the subsequent increase of the amplitude $A$.

6. General case

Qualitatively similar results for the two-box potential can be obtained on the basis of a different ansatz, where we approximate the Jost functions by $n$-dimensional polynomials instead of the piece-wise linear functions in Eqs. (7), (8), (16), (17), (26) and (27); (16)-(17) and (26)-(27) as

$$\Psi^{(1)}(x,y) = \begin{cases} \sum_{m=1}^{N} C_m 2^{2m} \exp(-\mu(x-1)(y-1)) & \text{at } x > 1, \ y > 1, \\ \sum_{m=1}^{N} C_m (1 + x)^m (1 + y)^m & \text{at } |x| < 1, \ |y| < 1, \\ 0 & \text{at } x < -1, \ y < -1, \end{cases} \quad (36)$$

$$\Psi^{(2)}(x,y) = \begin{cases} \sum_{m=1}^{N} C_m (1 - x)^m (1 - y)^m & \text{at } |x| < 1, \ |y| < 1, \\ \sum_{m=1}^{N} C_m 2^{2m} \exp(\mu(x+1)(y+1)) & \text{at } x < -1, \ y < -1, \end{cases} \quad (37)$$

where $C_1 = 1$, $C_m (m = 1, 2, \ldots, N)$ are the variational parameters. A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (36) and (37) to be

$$N = \sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{(\Gamma(1 + k))^2 (\Gamma(1 + m))^2}{(\Gamma(m + k + 2))^2} \ 2^{2k+2m+2}, \quad (38)$$

$$L = i \sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{(\Gamma(1 + k))^2 (\Gamma(1 + m))^2}{(\Gamma(1 + k + m + 2))^2} \ 2^{2k+2m+2} - A \sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{2^{k+m+2} (2^{k+m} - 1)}{(k+m+1)^2}. \quad (39)$$

Then we have

$$\lambda = \frac{A \sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{2^{k+m+2} (2^{k+m-1})}{(k+m+1)^2}}{\sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{(\Gamma(1+k))^2 (\Gamma(1+m))^2}{(\Gamma(1+m+k))^2} 2^{2k+2m+2}} \quad \frac{i \sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{(\Gamma(1+k))^2 (\Gamma(1+m))^2}{(\Gamma(1+m+k))^2} 2^{2k+2m+2}}{\sum_{m=1}^{N} \sum_{k=1}^{N} C_m C_k \frac{(\Gamma(1+k))^2 (\Gamma(1+m))^2}{(\Gamma(1+m+k))^2} 2^{2k+2m+2}} \quad (40)$$

Subsequently varying the Lagrangian (40) with respect to $C_m$ with $L$ and $N$ taken as Eqs. (36) and (37), which allows $n$-equations in $n$-unknowns. By solving these equations we get the values of $C_m$. Lastly, substitution of these values of $C_m$ into the expression for the eigenvalue of $\lambda$ produces a rather cumbersome analytical expression.

6.1. Example

We choose the trial function in the form

$$\Psi^{(1)}(x,y) = \begin{cases} \sum_{m=1}^{N} C_m 2^{2m} \exp(-\mu(x-1)(y-1)) & \text{at } x > 1, \ y > 1, \\ \sum_{m=1}^{N} C_m (1 + x)^m (1 + y)^m & \text{at } |x| < 1, \ |y| < 1, \\ 0 & \text{at } x < -1, \ y < -1, \end{cases} \quad (41)$$
\[ \psi^{(2)}(x, y) = \begin{cases} 0, & \text{at } x > 1, \ y > 1, \\ \sum_{n=1}^{3} c_m (1 - x^n)(1 - y^n) & \text{at } |x| < 1, \ |y| < 1, \\ \sum_{n=1}^{3} c_m 2^n \exp(\mu(x + 1)(y + 1)) & \text{at } x < -1, \ y < -1, \end{cases} \]  

(42)

where \( C_1 = 1 \) and \( C_m (m = 1, 2, 3) \) are the variational parameters. A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (41) and (42) to be

\[
N = \frac{16}{9} + \frac{32}{9} C_2 + \frac{128}{25} C_3 + \frac{512}{225} C_2 C_3 + \frac{256}{225} C_2^2 + \frac{1024}{1225} C_3^2,
\]

\[
L = \frac{16i}{1575} \left( 525 + 280(5C_2 + 9C_3) + 16 \left( 35C_2^2 + 84C_2C_3 + 36C_3^2 \right) \right)
\]

\[ - \frac{4A}{315} \left( 420 + 63(35C_2 + 96C_3) + 16 \left( 189C_2^2 + 1085C_2C_3 + 1620C_3^2 \right) \right). \]

Then we have

\[
\lambda = \frac{4A \left( 420 + 63(35C_2 + 96C_3) + 16 \left( 189C_2^2 + 1085C_2C_3 + 1620C_3^2 \right) \right)}{315 \left( \frac{16}{9} + \frac{32}{9} C_2 + \frac{128}{25} C_3 + \frac{512}{225} C_2 C_3 + \frac{256}{225} C_2^2 + \frac{1024}{1225} C_3^2 \right)}
\]

\[ - \frac{16i \left( 525 + 280(5C_2 + 9C_3) + 16 \left( 35C_2^2 + 84C_2C_3 + 36C_3^2 \right) \right)}{1575 \left( \frac{16}{9} + \frac{32}{9} C_2 + \frac{128}{25} C_3 + \frac{512}{225} C_2 C_3 + \frac{256}{225} C_2^2 + \frac{1024}{1225} C_3^2 \right)}. \]  

(43)

Subsequently varying the eigenvalue (43) with respect to \( C_m (m = 1, 2, 3, \ C_1 = 1) \). By solving these equations we get the values of \( C_m \) as:

\[
C_2 = -5.5574, \quad C_3 = 1.686;
\]

\[
C_2 = -1.9397, \quad C_1 = 0.534;
\]

\[
C_2 = -1.0477, \quad C_3 = 0.332;
\]

\[
C_2 = -2.952, \quad C_3 = 1.686;
\]

\[
C_2 = -1.3597, \quad C_3 = 0.332;
\]

\[
C_2 = -0.691, \quad C_3 = 0.534;
\]

\[
C_2 = -1.198, \quad C_3 = 0.275;
\]

\[
C_2 = -1.198, \quad C_3 = 0.3645;
\]

\[
C_2 = -2.956 + 0.6885i, \quad C_3 = 0.86 - 0.219i;
\]

\[
C_2 = -2.956 - 0.6885i, \quad C_3 = 0.86 + 0.219i;
\]

\[
C_2 = -2.956 + 0.6885i, \quad C_3 = 2.212 - 0.0597i;
\]

\[
C_2 = -2.956 - 0.6885i, \quad C_3 = 0.86 + 0.219i.
\]

Fig. 1a. The jost function of the Eq. (41) with the parameters \( C_1 = 1, \ C_2 = -5.557, \ C_3 = 1.686 \) in the interval \([-1.5, 1.5] \).
By substitution of the values of $C_2$ and $C_3$ into the expression of eigenvalue (43) produces analytical expressions

\[ \lambda = -0.146116(33.8526i - 35.2782A); \]
\[ \lambda = -1.13637(4.76691i - 225.403A); \]
\[ \lambda = 5.27201(-0.51141i - 0.521163A); \]
\( \lambda = 4.49014(0.347332i - 2.03118A); \)
\( \lambda = -0.50103(8.51855i - 57.8136A); \)
\( \lambda = -3.32836(1.06525i - 3.21724A); \)
\( \lambda = 7.8114(-0.558997i - 0.288761A); \)
\( \lambda = -7.42121(0.606113i - 2.36152A); \)
Fig. 4b. The jost function of the Eq. (42) with the parameters $C_1 = 1, \ C_2 = -2.952, \ C_3 = 1.686$ in the interval $[-1,1]$.

Fig. 5a. The jost function of the Eq. (41) with the parameters $C_1 = 1, \ C_2 = -1.3597, \ C_3 = 0.332$ in the interval $[-3,3]$.

Fig. 5b. The jost function of the Eq. (42) with the parameters $C_1 = 1, \ C_2 = -1.3597, \ C_3 = 0.332$ in the interval $[-3,3]$.

\[
\lambda = (3.04887 - 6.12103i) + (7.35151 + 6.12193i)A;
\]
\[
\lambda = (-3.04887 - 6.12103i) + (7.35151 - 6.12193i)A;
\]
\[
\lambda = (0.287599 - 5.52171i) + (307.644 - 161.949i)A;
\]
\[
\lambda = (-0.287599 - 5.52171i) + (307.644 + 161.949i)A.
\]
Finally, the results for the imaginary and real parts of the eigenvalues (45) from that expressions prove to be quite similar to those displayed in the previous first cases for the piece-wise linear ansatz.

Figs. 1a,1b,2a,2b,3a,3b,4a,4b,5a,5b,6a,6b,7a,7b,8a,8b shows an examples of Eqs.(41) and (42) with the parameters $C_1 = 1$, $C_2 = -0.691$, $C_3 = 0.534$ in the interval $[-1,2]$.

Fig. 6a. The jost function of the Eq. (41) with the parameters $C_1 = 1$, $C_2 = -0.691$, $C_3 = 0.534$ in the interval $[-1,2]$.

Fig. 6b. The jost function of the Eq. (42) with the parameters $C_1 = 1$, $C_2 = -0.691$, $C_3 = 0.534$ in the interval $[-1,2]$.

Fig. 7a. The jost function of the Eq. (41) with the parameters $C_1 = 1$, $C_2 = -1.198$, $C_3 = 0.275$ in the interval $[-1,1]$.

Fig. 7b. The jost function of the Eq. (42) with the parameters $C_1 = 1$, $C_2 = -1.198$, $C_3 = 0.275$ in the interval $[-1,1]$.

Finally, the results for the imaginary and real parts of the eigenvalues (45) from that expressions prove to be quite similar to those displayed in the previous first cases for the piece-wise linear ansatz.

Figs. 1a,1b,2a,2b,3a,3b,4a,4b,5a,5b,6a,6b,7a,7b,8a,8b shows an examples of Eqs. (41) and (42) with the parameters $C_2, C_3$ from Eq. (44) in the interval $-1 < x < 1$ and $-1 < y < 1$. A similar argument as before applies quite well for figures which corresponds to stable state.
7. Conclusion

The variational principle for the ZS equation in two-dimensions is formulated. The simplest example of the application of this technique, taking the box shaped initial pulse and an ansatz based on linear Jost functions in the region of localization of the box in two cases are demonstrated. We give in more details the limiting case of the above-mentioned box with the phase jump equal to \( \pi \), i.e., a combination of two boxes of opposite signs, the total surface of the initial pulse being thus zero. Using a trial function in a rectangular box we found a functional integral corresponding to ZS equations, the trial functions above
We also approximated the Jost functions by quadratic polynomials instead of the piece-wise linear functions, which yield similar results for the two-box potential. We extended further the approximation of the Jost functions by polynomials of order $n$ instead of the piece-wise linear functions recovering the result of [18,19] as special case.

References