Differential approaches for computing Euclidean diagonal norm balanced realizations in control theory

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Abstract
In this paper we propose dynamical systems for computing diagonal balanced realizations arising in control theory, particularly, we deal with Euclidean diagonal norm balanced realizations. The limiting solution of a differential flow appears to be a possible way of finding such realizations since no direct algebraic algorithm is known.

Keywords: Balanced realization; Gradient flow; Isodynamical methods

1. Introduction

In this paper balanced realizations arising in control theory are discussed. In particular, we consider the classical linear dynamical system:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad t > 0, \quad (1)
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) and \(x, y\) and \(u\) are the state, input and output vectors of suitable dimensions, respectively. In the case of asymptotically stable systems, i.e. for systems with matrix \(A\) having eigenvalues with negative real parts, the controllable and observable Gramian matrices are defined, respectively, by

\[
W_c(A,B) = \int_0^{\infty} e^{tA}BB^T e^{tA^T} dt, \quad W_o(A,C) = \int_0^{\infty} e^{tA}C^T C e^{tA} dt.
\]  

Instead, in the case of unstable systems, the Gramian matrices are defined instead as

\[
W_c(A,B,\tau) = \int_0^{\tau} e^{tA}BB^T e^{tA^T} dt, \quad W_o(A,C,\tau) = \int_0^{\tau} e^{tA}C^T C e^{tA} dt
\]  

with \(\tau > 0\) sufficiently large. The triple \((A, B, C)\) is said to be controllable and observable when the Gramians \(W_c(A,B)\) and \(W_o(A,C)\) (or \(W_c(A,B,\tau)\) and \(W_o(A,C,\tau)\)) are symmetric positive definite matrices.

Definition 1. A realization \((A, B, C)\) is said to be balanced when \(W_c(A,B) = W_o(A,C)\) and is said to be diagonal balanced when \(W_c(A,B) = W_o(A,C) = \Sigma\), with \(\Sigma\) diagonal matrix, with diagonal entries \(\sigma_1 \geq \cdots \geq \sigma_n > 0\).

Balanced realizations for asymptotically stable linear systems have been introduced in [7] and have quickly found widely employed in model reduction theory and approximation theory of linear systems (see also [9]). Moreover, balanced realizations (or other
related classes of realizations) play an important role in digital control and signal processing [8]. Several algebraic methods and some algorithms based on the gradient flow theory are known for computing diagonal balanced realizations (see, for instance [5,6,9,10]).

In this paper we consider the class of Euclidean norm balanced realization. This form of balancing was introduced in [4,11] and it has interesting connections with least squares matching problems arising in computer graphics [1]. It is particularly useful for systems which are not asymptotically stable and in $L^2$-sensitivity optimization problems [5].

**Definition 2.** A realization $(A, B, C)$ is said to be Euclidean norm balanced if and only if

$$AA^T + BB^T = A^T A + C^T C$$

and it is said to be Euclidean diagonal norm balanced if and only if

$$AA^T + BB^T = A^T A + C_1^T C_1 = \Sigma,$$

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with diagonal entries $\sigma_1 \geq \cdots \geq \sigma_n > 0$. The $\sigma_i$'s are called the generalized singular values of $(A, B, C)$ and $AA^T + BB^T$, $A^T A + C_1^T C_1$ are called the Euclidean norm controllability Gramian and Euclidean norm observability Gramian, respectively.

In the literature there are only few numerical algorithms for computing Euclidean norm balanced realizations. One of these achieves the balancing via the solution of a gradient flow obtained by the minimization of a suitable cost function [5]. A second one, instead, computes Euclidean balanced realization via a Newton based iterative procedure for the solution of a matrix equation [2]. However, these algorithms do not seem suitable for finding Euclidean diagonal norm balanced realizations. Here, we investigate different approaches to obtain a set of differential equations whose limiting solutions converge to an Euclidean norm diagonal balanced realizations. A different set of differential equations can be derived starting from the minimizations problem of an ad hoc design cost function, as shown in Section 4. Finally we report some numerical tests showing the behavior of our approaches.

2. Geometric concepts on balancing

This section is essentially concerned with a brief recall of some geometric concepts and known results on balancing. Consider the real vector space of the triple:

$$\mathcal{L}_{\alpha,\nu,\rho}(\mathbb{R}) = \{(A, B, C) | A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}\},$$

equipped with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle (A_1, B_1, C_1), (A_2, B_2, C_2) \rangle = \text{trace}(A_1 A_2^T + B_1 B_2^T + C_1^T C_2).$$

Then the orbit of similarity action on $\mathcal{L}_{\alpha,\nu,\rho}(\mathbb{R})$

$$O(A, B, C) = \{(SAS^{-1}, SB, CS^{-1}) | S \in GL(n)\}$$

is a sub-manifold of $\mathcal{L}_{\alpha,\nu,\rho}(\mathbb{R})$. The tangent space of $O(A, B, C)$ at $(F, G, H) \in O(A, B, C)$ is given by

$$T_{F,G,H}O(A,B,C) = \{(X,F),XG,-HX) \in \mathcal{L}_{\alpha,\nu,\rho}(\mathbb{R}) | X \in \mathbb{R}^{n \times n}\}.$$

A more important Riemannian metric on $O(A, B, C)$ is the normal Riemannian metric defined as follows:

$$\langle (X_1,F),X_1G,-HX_1), (X_2,F), X_2G,-HX_2) \rangle = 2 \text{trace}(X_1^T X_2) \equiv \langle X_1, X_2 \rangle$$

for any $(X_1,F),X_1G,-HX_1), (X_2,F), X_2G,-HX_2) \in T_{F,G,H}O(A,B,C)$. As noted in [4] this is a particular convenient Riemannian metric in which the associated gradient takes a particularly simple form.

If $(A, B, C)$ is controllable and observable, $O(A, B, C)$ is the set of all realization of a transfer function $G(s) = C(sI - A)^{-1}B \in \mathbb{R}^{m \times n}$, $s \in \mathbb{R}$ (i.e. $G(s)$ is constant on $O(A, B, C)$). Hence, moving on the orbit $O(A, B, C)$ we obtain differential systems isomorphic to (1).
Using the gradient flow theory applied to a suitable cost function, the following result has been derived [5].

**Theorem 3.** Let $\langle A, B, C \rangle \in \mathcal{L}_{\mu, n, p}(\mathbb{R})$ be a controllable and observable realization and consider the following cost function $\Gamma : \mathcal{O}(A, B, C) \to \mathbb{R}$

$$
\Gamma(F, G, H) = \text{tr}(FF^T + GG^T + HH^T)
$$

(5)

for all $(F, G, H) \in \mathcal{O}(A, B, C)$. Then, the gradient flow of $\Gamma$ (i.e. $A = -\nabla \Gamma_A, B = -\nabla \Gamma_B, C = -\nabla \Gamma_C$) with respect to the normal Riemannian metric on $\mathcal{O}(A, B, C)$ is given by

$$
\dot{A} = [A, [A, A^T]] + BB^T - C^T C,
$$

$$
\dot{B} = -[A, A^T] + BB^T - C^T C B,
$$

$$
\dot{C} = C([A, A^T] + BB^T - C^T C),
$$

(6)

where $[\cdot, \cdot]$ denotes the Lie-bracket operator on matrices. Then for all initial condition $(A_0, B_0, C_0) \in \mathcal{O}(A, B, C)$, the solution $(A(t), B(t), C(t))$ belongs to $\mathcal{O}(A, B, C)$ and converges to an Euclidean norm balanced realization.

A numerical scheme for computing Euclidean norm balanced realization can be simply obtained by the numerical solution of (6). However, for the class of Euclidean diagonal norm balanced realizations, the trivial generalization of the cost function (5) in $\Psi(F, G, H) = \text{tr}(NFF^T + GGG^T + HHH^T)$

(7)

with $N = \text{diag}(\mu_1, \ldots, \mu_k)$, $\mu_1 \geq \cdots \geq \mu_k > 0$, does not allow numerical algorithms for solving our problem, since (7) does not possess critical points which are Euclidean diagonal norm balanced realizations. In fact, the gradient flow $\dot{\Psi} = -\nabla \Psi_A, B = -\nabla \Psi_B, \dot{C} = -\nabla \Psi_C$ with respect to the normal Riemannian metric on $\mathcal{O}(A, B, C)$ is given by

$$
\dot{A} = -([N, A, A^T] + NBB^T - C^T CN, A),
$$

$$
\dot{B} = ([N, A, A^T] + NBB^T - C^T CN) B,
$$

$$
\dot{C} = C([N, A, A^T] + NBB^T - C^T CN).
$$

The equilibria of the above differential system are the triples $(A, B, C)$ such that $[N, A, A^T] + NBB^T - C^T CN = 0$. These conditions are equivalent to ask that $N(AA^T + BB^T) = A^T NA + C^T CN$, which does not lead to an Euclidean norm diagonal balancing. Hence, different approaches should be used to tackle the problem of computing Euclidean norm diagonal realizations. In particular, differential systems converging to Euclidean norm balanced triple can be derived using the correspondence between diagonal balanced realizations and diagonal balanced factorizations of a given matrix.

Given $H \in \mathbb{R}^{d \times d}$, let $X \in \mathbb{R}^{d \times n}$ and $Y \in \mathbb{R}^{d \times d}$ a factorization of $H$, i.e. $H = XY$. The factorization $(X, Y)$ is said to be balanced if $X^T X = YY^T$ holds and is said to be diagonal balanced if $X^T X = YY^T = \Sigma$ holds for a diagonal matrix $\Sigma$. Of course, when $\text{rank}(H) = n$, then the singular value decomposition (SVD) of $H$ provides a diagonal balanced factorization. Using this definition we can try to obtain Euclidean diagonal norm balanced realizations.

Let us consider the realization $(A, B, C)$ with $\text{rank}(A) = n$. Define the following matrices

$$
X = \begin{pmatrix} A \\ C \end{pmatrix} \in \mathbb{R}^{(n+p) \times n}, \quad Y = \begin{pmatrix} A & B \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)}
$$

and

$$
H = XY = \begin{pmatrix} A^T & AB \\ CA & CB \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)}
$$

Then $(X, Y)$ is a factorization of the matrix $H$, thus by using the SVD of $H$ we derive a diagonal balanced factorization of $H$. Let

$$
H = USV^T,
$$

where $U \in \mathbb{R}^{(n+p) \times (n+p)}$ and $V^T \in \mathbb{R}^{(n+m) \times (n+m)}$ are orthogonal matrices and $S \in \mathbb{R}^{(n+p) \times (n+m)}$. Being $\text{rank}(H) = n$ then

$$
S = \begin{pmatrix} \Sigma_{x,x} & 0_{x,m} \\ 0_{m,x} & 0_{m,m} \end{pmatrix}
$$

with $\Sigma \in \mathbb{R}^{x \times x}$. Thus by setting

$$
X_{\infty} = US^{1/2}l_{(n+p) \times n}, \quad Y_{\infty} = l_{(n+p) \times m}S^{1/2}V^T,
$$

where

$$
S^{1/2} = \begin{pmatrix} \Sigma^{1/2} & 0_{x,m} \\ 0_{m,x} & 0_{m,m} \end{pmatrix}
$$

and

$$
l_{(n+p) \times n} = (I_{n \times n} A^T)_{(n+p) \times n},
$$

$$
l_{(n+p) \times m} = (I_{m \times m} 0_{m \times p}).
$$
we obtain that
\[ X_\infty^T X_\infty = Y_\infty Y_\infty^T = \Sigma. \]  
(8)

Thus partitioning in this way
\[ X_\infty = \begin{pmatrix} A_\infty \\ C_\infty \end{pmatrix}, \quad Y_\infty = (D_\infty B_\infty) \]
from (8), it follows that
\[ A_\infty^T A_\infty + C_\infty^T C_\infty = D_\infty^T D_\infty + B_\infty^T B_\infty = \Sigma. \]  
(9)

We now observe that \( A_\infty \) is in general different by \( D_\infty \), so that (9) does not define an Euclidean norm diagonal balanced realization. Furthermore, even if \( A_\infty = D_\infty \), we do not have no guarantee that \( (A_\infty, B_\infty, C_\infty) \) is on the orbit \( O(A, B, C) \).

Let us consider the vector space \( V \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times l} \) and define the action on \( V \) in the following way:
\[ \alpha : G(n) \times V \to V, \]
\[ (T, (X, Y)) \mapsto (X T^{-1}, Y T). \]  
(10)

Since \( \alpha \) is a linear algebraic group action we can define the orbit of \( (X, Y) \) as the set of all elements in \( V \) which are \( G(n) \) equivalent to \( (X, Y) \), i.e.: \( O(X,Y) = \{(X T^{-1}, Y T) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times l} | T \in G(n)\} \),
which is a smooth sub-manifold of \( \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times l} \) and its tangent space at \((X, Y)\) is
\[ T_{(X,Y)}O(X,Y) = \{ (\dot{X}, \dot{Y}) | (X, Y) \in \mathbb{R}^{n \times n} \} \]
We recall a continuous algorithm for balanced factorizations proposed in the following theorem [5].

**Theorem 4.** Let \((X, Y) \in V\) such that \( \text{rank}(X) = \text{rank}(Y) = n \). Let us consider the cost function \( \Phi_X : \mathcal{O}(X,Y) \to \mathbb{R} \), defined by
\[ \Phi_X(F, G) = \text{trace}(N(I^T F + G G^T)) \]
for all \( (F, G) \in \mathcal{O}(X,Y) \).
(11)

Then it follows that
(i) The gradient flow \((\dot{X}, \dot{Y}) = -\nabla \Phi_X(X, Y)\) with respect to the normal Riemannian metric is
\[ \dot{X} = -X(X^2 X - NY Y^2)^2, \]
\[ \dot{Y} = (X^2 X - NY Y^2)^2 Y. \]  
(12)

(ii) For any initial condition \((X_0, Y_0)\) the solution \((X(t), Y(t))\) of (12) belongs to \( \mathcal{O}(X,Y) \) for all \( t \geq 0 \), moreover it converges to a point \((X_\infty, Y_\infty)\) such that
\[ X_\infty^T X_\infty = Y_\infty Y_\infty^T = \Sigma, \]
where \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \) with diagonal entries \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \).

3. Main results

Based on the previous result, we now derive a continuous-time flow whose limiting solutions converge to an Euclidean norm balanced realizations.

**Theorem 5.** Let \((A, B, C) \in L_{(n,n,p)}(\mathbb{R})\) be a realization such that \( \text{rank}(A) = n \). Then, for any initial condition \((X_0, B_0, C_0) \in L_{(n,n,p)}(\mathbb{R})\), the solution of the differential system
\[ \dot{X} = \frac{1}{2}(A^T A + C^T C) X - N(A A^T + B B^T), \]
\[ \dot{Y} = (A^T A + C^T C) Y - N(A A^T + B B^T) B, \]
\[ \dot{Z} = -G((A^T A + C^T C) X - N(A A^T + B B^T)) B. \]  
(13)

converges, when \( t \to \infty \), to an Euclidean norm diagonal balanced realization \((A_\infty, B_\infty, C_\infty)\).

**Proof.** Let us consider the realization \((A, B, C)\) and
\[ X = \begin{pmatrix} A \\ C \end{pmatrix} \in \mathbb{R}^{(n+p) \times n}, \quad Y = \begin{pmatrix} A & B \end{pmatrix} \in \mathbb{R}^{n \times (n+m)}. \]
Since \( \text{rank}(A) = n \) then \( \text{rank}(X) = \text{rank}(Y) = n \). Thus the matrices \((X, Y)\) satisfy the hypothesis of Theorem 4 and
\[ H = XY = \begin{pmatrix} A^2 & AB \\ CA & CB \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)}. \]
The solution \((X(t), Y(t))\) of the differential system (12) may be written as
\[ X(t) = \begin{pmatrix} A(t) \\ C(t) \end{pmatrix}, \quad Y(t) = (D(t) B(t)) \]
with \((A(t), B(t), C(t)) \in L_{(n,n,p)}(\mathbb{R})\) and \( D(t) \in \mathbb{R}^{n \times l} \). The differential system (12) becomes equivalent to
\[ \dot{X} = -A((A^T A + C^T C) X - N(A A^T + B B^T)), \]  
(14)
\[ C = -\mathcal{C}(A^T A + C^T C) N - N(A A^T + B B^T), \]
\[ \mathcal{D} = ((C^T D + C^T C) N - N(D D^T + B B^T)) D, \]
\[ B = ((C^T D + C^T C) N - N(D B B + B D^T)) B, \]
where for any initial condition and for \( t \to \infty, (X(t), Y(t)) \) converges to a limit point \( X_\infty = \left( \begin{array}{c} A_\infty \\ C_\infty \end{array} \right) \). Therefore, \( A_\infty = D_\infty A_\infty \) and \( B_\infty = D_\infty B_\infty \), such that \( X_\infty T \). Thus now \( A_\infty = D_\infty \) and by adding (14) and (16) it follows that \((A(t), B(t), C(t))\) satisfy the set of differential equation (13). We now observe that the equilibria points of (13) are the triples (A, B, C) such that
\[ (A^T A + C^T C) N = N(A A^T + B B^T), \]
i.e.: \[ N^{-1}(A^T A + C^T C) N = A A^T + B B^T. \]
Then, by using the symmetry of \( A A^T + B B^T \) and \( A^T A + C^T C \), we have
\[ N^{-1}(A^T A + C^T C) N = N(A A^T + C^T C) N^{-1} \]
and
\[ (A^T A + C^T C) A = N^2 (A^T A + C^T C). \]
Hence
\[ (a_i^2 - \mu_i^2) (A^T A + C^T C) = 0 \]
for all \( i, j = 1, \ldots, n, \ i \neq j \).
Therefore, \( A^T A + C^T C \) and \( A A^T + B B^T \) must be diagonal and equal.

**Remark 6.** Observe that the cost function (11) has compact sub-level, therefore since (13) is a gradient flow, the solutions converge to the set of equilibria of (13).

Note that the solution of the previous differential system converges to an Euclidean diagonal norm balanced realization but does not move on the similarity orbit \( \mathcal{O}(A, B, C) \). Therefore, along the solution the transfer function is not preserved.

Instead, the following result overcomes this problem.

**Theorem 7.** Let \( (A, B, C) \in \mathcal{L}_{(n, m, p)}(\mathbb{R}) \) be a realization such that \( \text{rank}(A) = n \). Then, for any initial condition \( (A_0, B_0, C_0) \in \mathcal{L}_{(n, m, p)}(\mathbb{R}), \) the solution of the differential system
\[ A = [(A^T A + C^T C) N - N(A A^T + B B^T), A], \]
\[ B = [(A^T A + C^T C) N - N(A A^T + B B^T)] B, \]
\[ C = -\mathcal{C}(A^T A + C^T C) N - N(A A^T + B B^T)) \]
belongs to \( \mathcal{O}(A, B, C) \) for all \( t \) in the existence interval. Moreover, the equilibria point of (18) are Euclidean norm diagonal balanced realization \( (A_\infty, B_\infty, C_\infty) \).

**Proof.** We observe that if we define the matrix function
\[ A_N(A, B, C) = (A^T A + C^T C) N - N(A A^T + B B^T) \]
then, the differential system (18) becomes
\[ \dot{X}(t) = S(t) A_N(S(t)) \dot{S}(t), \quad t > 0, \]
\[ B(t) = S(t) B_0, \]
\[ C(t) = C_0 S(t)^{-1}, \]
where \( S(t) \in GL(n, \mathbb{R}) \) is the solution of the following differential system:
\[ \dot{S}(t) = A_S(S(t) A_S S^{-1}(t), S(t) B_0, C_0 S^{-1}(t)) S(t), \quad t > 0, \]
\[ S(0) = I. \]
From (20) it follows that the solution of (18) evolves on the orbit \( O(A, B, C) \) and has equilibria points \((A, B, C)\) such that \(A^T A + C^T C\) and \(AA^T + BB^T\) must be diagonal and equal.

Then a numerical algorithm may be derived by using isodynamical numerical methods for solving (20), i.e. by using a scheme of the following form:

\[
\begin{align*}
A_{k+1} &= S_{k+1}A_{k}S_{k+1}^{-1}, & B_{k+1} &= S_{k+1}B_{k}, & C_{k+1} &= C_{k}S_{k+1}^{-1},
\end{align*}
\]

(22)

where \((A_0, B_0, C_0)\) is the initial realization and \(S_k\) may be obtained, for instance by the Euler method applied to (21):

\[
S_{k+1} = I + hA_k(A_k, B_k, C_k)
\]

with \(h > 0\) the time step [3].

**Remark 8.** Since \(GL(n, \mathbb{R})\) is an open space, the solution \(S(t)\) of (22) could approach a singular matrix in a finite time, although this occurrence never happened in our numerical experiments.

The convergence of the flow (19) to its equilibria points seems to be difficult to prove theoretically, however, our numerical tests show that the solution of (19) converges asymptotically to an Euclidean diagonal norm balanced realization.

### 4. The least square approach

In this section, in order to derive a differential flow whose solutions converge for all \(t > 0\) to an Euclidean norm diagonal balanced realization, we propose a cost function in the least squares sense rather than using the cost function derived by the balanced factorization problem.

Particularly, let \(N\) be a diagonal matrix as before, and consider the function \(\psi : O(A, B, C) \to \mathbb{R}\) defined as follows:

\[
\psi(F, G, H) = (F^T F + H^T H)N - N(FF^T + GG^T)\|X\|^2.
\]

(23)

where \(\|\cdot\|\) is the Frobenius norm on matrices. In order to determine the gradient flow of \(\psi(F, G, H)\) we first calculate its derivative.

**Lemma 9.** Let \(\psi : O(A, B, C) \to \mathbb{R}\) be the cost function defined by (23) for all \((F, G, H) \in O(A, B, C)\). Then the derivative of \(\psi\) at \((F, G, H)\) is the linear map

\[
D\psi_{(F,G,H)}(X, F, XG, −HX) = 2\langle \Delta_1, X \rangle,
\]

(24)

where

\[
\Delta_1(F, G, H) = \begin{cases} 
(F^T F + H^T H)N\|X\|^2 & + gN - (Ng^T + g^T N)(FF^T + GG^T) \\
F(g^T + gN)F^T + F^T(g^T N + Ng^T)F & + gN - (Ng^T + g^T N)(FF^T + GG^T)
\end{cases}
\]

(25)

and

\[
g = g(F, G, H) = (F^T F + H^T H)N - N(FF^T + GG^T).
\]

**Proof.** Consider the smooth map \(\sigma : GL(n, \mathbb{R}) \to O(A, B, C)\) defined by \(\sigma(S) = (SFS^{-1}, SG, HS^{-1})\) for any \(S \in GL(n, \mathbb{R})\) and the composed map of \(\psi = \psi \circ \sigma\). Observe that by the chain rule we have \(D\psi_{\sigma(S)} = D\sigma_{\sigma(S)}\left(\nabla g(\sigma(S))\right)\). After some calculations it is easy to verify that

\[
D\psi_{\sigma(S)}(X, X\sigma(S)) = 2\langle \Delta_1, X \rangle
\]

with \(\Delta_1\) defined by (25). □

**Theorem 10.** Let \(\psi : O(A, B, C) \to \mathbb{R}\) be the cost function defined as in (23). Then

(i) The gradient flow \((\tilde{A}(t), \tilde{B}(t), \tilde{C}(t))\) with respect to the normal Riemannian metric on \(O(A, B, C)\) is given by

\[
\dot{A} = -\Delta_1(A, B, C), A, \\
\dot{B} = -\Delta_1(A, B, C), B, \\
\dot{C} = \Delta_1(A, B, C)
\]

(26)

with \(\Delta_1\) given by (25).

(ii) For any initial conditions \((A_0, B_0, C_0)\) on the orbit \(O(A, B, C)\) the solution \((\tilde{A}(t), \tilde{B}(t), \tilde{C}(t))\) of (26) exists for all \(t \geq 0\) and (26) is an isodynamical flow on the set of controllable and observable triples \((A, B, C)\).

(iii) For any initial conditions \((A_0, B_0, C_0)\) the solution \((\tilde{A}(t), \tilde{B}(t), \tilde{C}(t))\) converges to an Euclidean diagonal norm realization.
Proof. The proof of (i) follows immediately from the expression of the gradient vector field of \( \psi \) with respect to the normal Riemannian metric on \( \mathcal{O}(A, B, C) \).

In fact, let \( \text{grad} \psi = (\nabla \phi_A, \nabla \phi_B, \nabla \phi_C) \) be the components of the gradient with respect to the normal Riemannian metric.

Since \( \text{grad} \psi(F, G, H) \in T_F \mathcal{O}(A, B, C) \), by the characterization of the tangent space of the similarity orbit \( \mathcal{O}(A, B, C) \), it results

\[
\text{grad} \psi(F, G, H)(X, F, XG, -HX) \quad \text{for} \quad X \in \mathbb{R}^{n \times n}.
\]

Moreover, it is known that

\[
\text{D} \phi(F, G, H)(X, F, XG, -HX) = ((\nabla \phi_A, \nabla \phi_B, \nabla \phi_C), (X, F, XG, -HX)).
\]

Thus, the previous Lemma and the uniqueness of the matrix \( X_T \) prove (i).

Observe that from the closure of \( \mathcal{O}(A, B, C) \) in \( \mathbb{E}_{n \times n}(\mathbb{R}) \) it follows immediately that \( \psi \) has compact sub-level set. Thus, the gradient flow (26) is complete, i.e. the solution \((A(t), B(t), C(t))\) exists for every \( t > 0 \).

To prove (iii) note that the equilibria of (26) are the critical points of \( \phi(A, B, C) \), i.e. the triple \((A_0, B_0, C_0)\) such that

\[
N (A_0, B_0, C_0) + (g(A_0, B_0, C_0)) T N = 0,
\]

\[
N (A_0, B_0, C_0) + p(A_0, B_0, C_0) N = 0.
\]

Substituting the expression of \( g(A, B, C) \), we get

\[
2 N (A_0, B_0, C_0) = N (A_0, B_0, C_0) + (A_0^T A_0 + B_0^T B_0 + C_0^T C_0) N = 0.
\]

Using the symmetry of the matrices \((A_0^T A_0 + B_0^T B_0 + C_0^T C_0)\), we get

\[
N (A_0, B_0, C_0) = 0.
\]

Remark 11. Note that the equilibria of (26) are global minima of \( \psi \).

5. Numerical test

In this section we report some numerical tests in order to illustrate the behavior of the proposed approaches. All the numerical results have been obtained by Matlab 6.1 codes implemented on Pentium III 1 GHz with 256 MbRAM.

We have compared the approaches in term of error on the Euclidean norm realization, \( E_{\text{norm}} = \|A_k A_k^T + B_k B_k^T - A_k^T A_k - C_k^T C_k\|\) and error on the extra diagonal elements of the matrices \( A A_k^T + B B_k^T \) and \( A^T A + C C_k^T \) evaluated as \( E_{\text{extra}} = \|\text{off}(A_k A_k^T + B_k B_k^T) - \text{off}(A_k^T A_k + C_k^T C_k)\|\) (where \( \text{off}(M) = M - \text{diag}(M) \), for all \( M \in \mathbb{R}^{n \times n} \)).

As first example we have solved both systems (18) (DF approach) and (26) (LS approach) by the forward Euler method with constant step-size \( h = 0.0001 \).

The initial matrices are

\[
A_0 = \begin{bmatrix}
-3 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}, \quad C_0 = \begin{bmatrix}
2 & 1 & 3
\end{bmatrix}.
\]

and diagonal matrix \( N = \text{diag}(3, 2, 1) \). Table 1 summarizes the result at the end of the integration interval [0, 20], the results obtained by the two approaches are quite similar; the reduced accuracy of the results obtained by the solution of system (26) are due to the number of matrix multiplications in the expression of the vector field.

The diagonal matrices (corresponding to the solutions obtained by DF and DL approach, respectively) at which the Euclidean norm controllability Gramian and Euclidean norm observability Gramian converge are reported below:

\[
D_{\text{GF}} = \text{diag}([1.9818, 5.8420, 16.3196])
\]

\[
D_{\text{LS}} = \text{diag}([15.9352, 5.8137, 2.0119]).
\]

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Example 1 (comparisons between the two differential approaches at ( T = 20 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approach</td>
<td>( E_{\text{norm}} )</td>
</tr>
<tr>
<td>DF approach</td>
<td>1.6594E−12</td>
</tr>
<tr>
<td>LS approach</td>
<td>4.8188E−10</td>
</tr>
</tbody>
</table>
Fig. 1 plots the error on the Euclidean norm realization on the time interval [0 20], while Fig. 2 gives the behavior of the error on extra diagonal elements. Both figures refer to the numerical solution of differential equation (18), however similar results can be obtained by the solution of (26). Furthermore, we observe due to the use of a general purpose integrator the transfer function of the system is not preserved.

To overcome this problem the isodynamical forward Euler method should be used (see [3] for more details). As second example we have reported the results obtained solving both systems (18) and (26).
Table 2
Example 2: comparisons between the two differential approaches at $T = 10$

<table>
<thead>
<tr>
<th>Approach</th>
<th>$E_{\text{norm}}$</th>
<th>$E_{\text{extra}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DF approach</td>
<td>5.2342E−5</td>
<td>5.2142E−5</td>
</tr>
<tr>
<td>LS approach</td>
<td>3.8521E−4</td>
<td>3.6732E−4</td>
</tr>
</tbody>
</table>

with random generated initial condition matrices $A_0 \in \mathbb{R}^{5 \times 5}$, $B_0 \in \mathbb{R}^{5 \times 2}$ and $C_0 \in \mathbb{R}^{1 \times 5}$, by the isodynamical forward Euler method with constant step-size $h = 0.0001$. Table 2 summarizes these results. Note that in this case the transfer function of the system is preserved.

6. Conclusion

Ordinary differential equations evolving on the orbit of similarity of a given triple $(A, B, C)$ are achieved which converge to Euclidean norm diagonal realizations. These differential systems are isodynamical flows, hence to preserve completely the right behavior of the solution isodynamical schemes are needed.

References


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