Exponential monotonicity of quadratic forms in ODEs and preserving methods

C. Elia*, L. Lopez
Dipartimento Interuniversitario di Matematica, Università degli Studi di Bari, via E. Orabona 4, I-70125 Bari, Italy

Abstract

In this paper we consider ODEs whose solutions satisfy exponential monotonic quadratic forms. We show that the quadratic preserving Gauss–Legendre–Runge–Kutta methods do not preserve this qualitative feature, while certain Lie group preserving methods, as Crouch–Grossman methods and methods based on projection techniques are suitable integrators for such a kind of differential problems. We also show that these differential problems may be solved via associated differential systems with quadratic invariants. Numerical tests are reported in order to confirm our theoretical results.

Keywords: Exponential monotonicity; ODEs; Crouch–Grossman methods

1. Introduction

In last years methods for ODEs preserving certain qualitative features of the solution have been widely studied. Monotonicity of quadratic forms has been studied in [8], while preserving monotonicity property of Riccati differential equations has been considered in [4]. Our concern in this paper is to study conditions under which exponential monotonicity property is maintained under discretization. We will consider differential systems of the following form:

\[ \dot{Y}(t) = A(t, Y(t))Y(t), \quad Y(0) = Y_0, \quad t \geq 0, \]  

(1)

where \( Y(t) \in \mathbb{R}^{s \times r} \), with \( 1 \leq r \leq s \), and \( A : \mathbb{R} \times \mathbb{R}^{s \times r} \rightarrow \mathbb{R}^{s \times s} \) is a continuous and locally Lipschitz matrix function. Further, we will suppose that the solution of (1) satisfies the following exponential monotonicity:

\[ Y^T(t)HY(t) = \exp(\alpha t)Y^T_0HY_0, \quad t \geq 0, \]  

(2)

where \( H \) is a constant orthogonal matrix and \( \alpha \) is a real number different from zero. Moreover, we assume that the initial value is such that \( Y_0^THY_0 \) is different from the zero matrix. We notice that the case \( \alpha = 0 \), corresponding to differential systems with quadratic invariants, has been widely studied (see for instance [3,5,6,9,13,14]). Dynamical systems with exponential monotonicity appear in connection with symmetry properties of Lyapunov exponents in [10] and in the theory of conformal Hamiltonian systems [15,16]. Further, they are also useful in differential matrix Riccati equations to obtain stable algorithms (see [17]).

We will denote by \( S_{H,\alpha}(s) \) the set of matrices which are \( H \)-skew-symmetric with respect to \( \alpha \), that is the following subset of \( \mathbb{R}^{s \times s} \):

\[ S_{H,\alpha}(s) = \{ A \in \mathbb{R}^{s \times s} | A^T H + HA = aH \}. \]  

(3)
We will denote by \( U_{\text{diss}}(s,r) \) the set of continuous matrix functions \( Y(t) \) satisfying (2), that is
\[
U_{\text{diss}}(s,r) = \{ Y \in C^1(\mathbb{R}^+, \mathbb{R}^{n \times n}) | Y(t) \text{ satisfies } (2) \}.
\]

The following result characterizes differential systems with property (2).

**Theorem 1.** Let us consider the differential system (1). If the continuous and locally Lipschitz matrix function \( A \) is such that \( A \in C^1(\mathbb{R}^+, \mathbb{R}^{n \times n}) \), then the solution \( Y(t) \) of (1) belongs to \( U_{\text{diss}}(s,r) \).

**Proof.** Differentiating the quadratic form \( V(t) = Y(t)^T H Y(t) \) and using the hypothesis on \( A \), we get
\[
\frac{d}{dt}[Y(t)^T H Y(t)] = 2[Y(t)^T A(t) Y(t)] + 2[A(t) Y(t)^T H Y(t)] + 2[A(t)^T Y(t) H A(t)] Y(t) = 2[A^T(t) Y(t)^T H Y(t)] + 2[A(t) Y(t)^T H A(t)] Y(t)
\]
from which the thesis follows.

**Example 1** (see [18], p. 184). Let us consider the following nonlinear differential system:
\[
y(t) = A(y(t)) y(t), \quad y(0) = y_0, \quad t \geq 0
\]
with \( y = (y_1, y_2)^T \in \mathbb{R}^2 \) and
\[
A(y) = \begin{pmatrix} a/2 & -y_1/a \n y_1 & a/2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}.
\]
We have \( A^T(y) + A(y) = aI \), thus \( y^T(t) y(t) = \exp(a(t))^2 \theta_0 \), that is the Euclidean norm of the solution shows an exponential monotonicity.

We now consider
\[
A(y) = \begin{pmatrix} a/4 & -y_1/a \n y_1 & a/4 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad \text{with} \quad a > 0.
\]
In this case \( A \) does not belong to \( S_p(2) \), however, since \( y^T(A^T + A) \leq a y^2 \) for all \( y \neq 0 \), the solution \( z(t) \) of the following differential system:
\[
z(t) = \begin{pmatrix} a/2 & -z(t)/a \n z(t) & a/2 \end{pmatrix} z(t), \quad t \geq 0,
\]
\[
z(0) = y_0
\]
shows an exponential monotonicity and is an upper bound of \( y(t) \), that is \( |y(t)| \leq |z(t)| \) for all \( t > 0 \).

**Remark 2.** We notice that the class of differential systems (1) includes the dissipative systems \( y = f(y) \) for which \( f(y) = A(y) y \) and \( y^T f(y) = (a/2) y^2 \) with \( a < 0 \).

**Example 2.** Let us consider the following 2n-dimensional dissipative Hamiltonian system:
\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \text{with} \quad a > 0 \quad (6)
\]
with \( q(0) = q_0 \in \mathbb{R}^n \), \( p(0) = p_0 \in \mathbb{R}^n \). Examples of such dynamical systems may be found for instance in [7,12] or in [15,16]. Here the energy function \( H(p,q) \) is a twice continuously differentiable function on \( U \) open set of \( \mathbb{R}^{2n} \). Let us consider the flow \( \phi_t : U \to \mathbb{R}^{2n} \) of the dynamical system (6), that is the mapping that advances the solution by time \( t \), i.e. \( \phi_t(q_0, p_0) = (q(t), p(t)) \). The derivative \( \partial \phi_t/\partial(q_0, p_0) \) is a solution of the variational equation \( \dot{Y}(t) = A(q(p,q), p_0,Y(t))Y(t) \)

\[
A(q, p) = \begin{pmatrix} 2H(q, p)/(\partial q) & -2H(q, p)/(\partial q) 
 -2H(q, p)/(\partial q) & (2H(q, p)/(\partial q) - aL) \end{pmatrix}
\]
and
\[
A^2(q, p)I + 2A(q, p) = -aI
\]
where
\[
I = \begin{pmatrix} 0 & L_n \n -L_n & 0 \end{pmatrix}
\]
Then \( \partial \phi_t/\partial(q_0, p_0) \) is a matrix solution of an ODE of the form (1) satisfying (2). In this case the mapping \( \partial \phi_t/\partial(q_0, p_0) \) does not preserve the oriented area of parallelograms, as in case of symplectic maps, but shows an exponential reduction of these areas.

2. The direct approach

We commence by studying the behavior of implicit Runge–Kutta methods with respect to property (2).
2.1. Runge–Kutta methods

It is well known that Gauss–Legendre–Runge–Kutta methods of order $2r$ (denoted by GL2r) are suitable to solve either dissipative systems or differential systems with quadratic first invariants. Instead, in this section we will show that they are not suitable to solve ODEs with exponential monotonic quadratic forms. In [8,16] it has been observed that symplectic methods of order 2 are not conformal integrators which is a result similar to the one we have proved in this section.

For the sake of simplicity we concentrate our attention on the vectorial case ($i.e.$ $r = 1$), noticing that the results of this section may also be shown in the general matrix case. Consider the following differential system:

$$ y = A(y)y, \quad y(0) = y_0, $$

where $y(t) \in \mathbb{R}^r$ and $A: \mathbb{R}^r \rightarrow S_{n \times n}(\mathbb{R})$.

Let us consider for the following only nonstiff problems where $\alpha$ is not too large and the derivatives of $f(y) = A(y)y$ are assumed to be bounded.

Consider the $v$-stages Runge–Kutta method defined by the Butcher array

$$ c^T A, \quad b, $$

where $c^T = (c_1, \ldots, c_v)$, $A = (a_{ij})_{i,j=1,\ldots,v}$ and $b = (b_1, \ldots, b_v)$. Denote by $y_h$ the numerical approximation of $y(t)$ at $t_h$, where $0 = t_0 < t_1 < \cdots < t_s < \cdots$ is a partition of the time interval, with $t_n = t_{n-1} + h_n$ for $n \geq 1$ and $h = \sum_{n=1}^{\infty} h_n$ is the time step. If we apply (7) to (8), we obtain

$$ y_1 = y_0 + h \sum_{i=1}^{v} b_i Y_i, \quad k_i = A(Y_i)Y_i $$

with

$$ Y_i = y_0 + h \sum_{j=1}^{i-1} a_{ij} k_j, \quad i = 1, \ldots, v, \quad k \geq 0. $$

It is well known that if the Runge–Kutta method satisfy

$$ b_0a_0 + b_1a_1 - b_1b_0 = 0, $$

then it conserves quadratic invariants. This suggests to investigate the behavior of these methods when applied to our particular case.

**Theorem 3.** Let us consider an implicit Runge–Kutta method of order $p$, whose coefficients satisfy (11). Let assume for the following that the derivatives of $f(y) = A(y)$ exist up to order $p$ and that they are bounded independently of $a$. Then, if the method is applied to (7), it follows that

$$ y^T (b) H_y (h) - y^T H_y = O(ah^{p+1}). $$

**Proof.** The relation (9) yields

$$ y^T H_y = y^T H_y + h \sum_{i=1}^{v} b_i Y_i + h \sum_{i=1}^{v} b_i Y_i H_k $$

$$ + h^2 \sum_{i=1}^{v} b_i Y_i H_k, $$

(13)

If we rewrite $y_1 = Y_1 - h \sum_{i=1}^{v} a_{ik} k_i$ and we insert it in the central term in (13) we have

$$ y^T H_y = y^T H_y + h \sum_{i=1}^{v} b_i Y_i + y^T H_k $$

$$ + h^2 \sum_{i=1}^{v} b_i Y_i H_k, $$

(14)

from which, by using (9) and (11), it follows that

$$ y^T H_y = y^T H_y + h \sum_{i=1}^{v} b_i Y_i (A(h)Y + H k), $$

(15)

Moreover, we have $Y_i (A(h)Y + H k) = a Y_i^T H Y_i$, for $i = 1, \ldots, v$, thus

$$ y^T H_y = y^T H_y + ah \sum_{i=1}^{v} b_i Y_i^T H Y_i. $$

(16)

The internal stages $Y_i$ can be developed into a Taylor series

$$ Y_i = y_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots, $$

(17)

where the coefficients $c_k$, for $k = 1, 2, \ldots$, depend on derivatives of the vector field at $y_0$ (elementary differentials) and on the coefficients of the method. Then by inserting (15) into (14) we have

$$ y^T H_y = y^T H_y + ah \sum_{i=1}^{v} b_i Y_i^T H Y_i + ah \sum_{i=1}^{v} b_i Y_i^T H Y_i, $$

(18)
where the coefficients $d_k$, for $k = 1, 2, \ldots$, are again bounded independently of $\alpha$. Together with the formula

$$y^2(h)H_2(h) = \exp(ah)g(\bar{h}H_0),$$

for the exact solution, (16) implies

$$y^2(h)H_2(h) - y^2_1H_1 = (e\epsilon h + e\epsilon h^2 + e\epsilon h^3 + \cdots)$$

(17)

with $e_k$, for $k = 1, 2, \ldots$, suitable coefficients.

Now, by using the fact that $y_1 = y(h) + O(h^{n+1})$, it follows that

$$y^2(h)H_2(h) - y^2_1H_1 = O(h^{n+1}).$$

Comparing this result with the formula (17), we conclude that $e_1 = e_2 = \cdots = e_p = 0$, and this proves the thesis. \hfill \Box

Thus, in spite of the fact that GL2e methods are algebraically stable and preserve any quadratic forms, we have seen that they do not preserve the exponential monotonicity in (2).

2.2. Projected methods

In order to have a numerical solution satisfying (2) we may correct it by a projection technique. Assume we know the approximation $y_n$ and suppose that it satisfies property (2) at $t = t_n$, that is $y^2_nH_n = \exp(ah)y^2_nH_n$. Compute the numerical solution $\tilde{y}_{n+1}$ of (1) by a Runge–Kutta method of order $p$ for which

$$y^2_nH_{n+1} - \exp(ah)y^2_nH_{n+1} = O(h^{n+1}).$$

Then we project $\tilde{y}_{n+1}$ onto the manifold $g(y) = 0$ where

$$g(y) = y^2H_2 - \exp(ah)y^2H_0.$$ (18)

In this way we obtain $y$ and set $y_{n+1} = y$. To find $y$ we need to minimize the quantity:

$$\|y_{n+1} - \tilde{y}_{n+1}\|,$$ subject to $g(y_{n+1}) = 0$.

Define the Lagrange function:

$$L = \frac{1}{2}\|y_{n+1} - \tilde{y}_{n+1}\|^2 - \lambda g(y_{n+1}),$$

where $\lambda$ is a scalar. We want

$$\frac{\partial L}{\partial y_{n+1}} = y_{n+1} - \tilde{y}_{n+1} - \lambda g'(y_{n+1}) = 0,$$ from which

$$y_{n+1} = \tilde{y}_{n+1} + \lambda g'(y_{n+1}).$$ (19)

In order to save some evaluations the quantity $g'(y_{n+1})$ is usually replaced by $g'(y_{n+1})$, then applying the Newton method to $g(y_{n+1}) = 0$, we can find $\lambda$ by the iteration $\lambda_{n+1} = \lambda_n + \Delta\lambda_n$, with initial value $\lambda_0 = 0$, and

$$\Delta\lambda_n = -(g'(y_{n+1})^T g'(y_{n+1}))^{-1} g'(y_{n+1}) + g'(y_{n+1}) \lambda_n.$$ Substituting the value of $\lambda$ obtained in this way into Eq. (19) we obtain $y_{n+1}$. Usually one iteration of the Newton scheme is needed. By observing that when $g(y)$ is given by (18), then $g'(y) = (H + H^2)y$, and the projection scheme becomes

$$y_{n+1} = \left(1 - \frac{y^2_nH_{n+1} - \exp(ah)y^2_nH_{n+1}}{Ky_{n+1}}\right) y_{n+1},$$

(20)

where $K = H + H^T$.

The projection technique is cheap and attractive because allow us to employ classical explicit Runge–Kutta methods as basic methods. However, when $H$ is not symmetric and positive definite, the projection technique could have some computational problem; in fact the projection of a vector of $\mathbb{R}^2$ on the manifold $g(y) = 0$ could not be unique or the scalar product $y^T(H + H^T)y$ for $y \neq 0$, could vanish as in the case of $H = I$.

2.3. Exponential integrators

The computational problems of the projection techniques disappear when we use exponential matrix based methods. In this section we will see how certain exponential integrators are suitable to solve the class of differential problems considered. In particular we will consider Magnus and Crouch–Grossmann methods belonging to the class of Lie group methods (see [13]). The most simple methods in the class of Magnus methods are the first-order Lie Euler method:

$$y_{n+1} = \exp(hA_n)y_n,$$ (21)

where $A_n = A(y_n)$, and the second-order implicit midpoint method:

$$y_{n+1} = \exp(hA_{n+1/2})y_n.$$ (22)
Lemma 4. Let $A \in S_{d, \alpha}(s)$. Then we have
\[ \exp(hA^n) \exp(hA) = \exp(hA^n)H \exp(hA) \]
with $h > 0$.

Proof. Using Taylor expansion of the exponential matrix we have
\[
\exp(hA^n)H \exp(hA) = (I + hA^n + \frac{1}{2!}h^2(A^n)^2 + \cdots)H(I + hA + \frac{1}{2!}h^2A^2 + \frac{1}{3!}h^3A^3 + \cdots)
\]
\[= H + h(A^nH + HA) + \frac{1}{2!}h^2((A^n)^2H + 2A^nHA + HA^2) + \cdots.
\]
Notice that
\[ (A^n)^2H + 2A^nHA + HA^2 = A^n(aH^2 + 2A^nHA + (A^n)^2H) = a^2H^2 + 2aA^2H + aA^3H = a^2H^2 + aA^2H + aA^3H = a^2A^2H = a^2H.\]

The class of Crouch–Grossman methods preserves property (2).

Proof. If we consider the class of Crouch–Grossman methods then it follows that
\[ \gamma_{v+1}^T H_{\gamma_{v+1}} = \gamma_v^T \exp(hb_1A_1) \cdots \exp(hb_vA_v)H \]
and recursively applying Lemma 4 we obtain
\[ \gamma_{v+1}^T H_{\gamma_{v+1}} = \exp \left( ab \sum_{i=1}^{v} h_i^2 H \right) = \exp(ab\gamma_v^2 H), \]
for the Lie midpoint scheme we can use similar arguments. While it is easy to see that higher-order Magnus methods do not preserve exponential monotonicity.

To find higher-order methods preserving (2) we can consider the class of Crouch–Grossman methods (see [2]) based on the $v$-stage explicit Runge–Kutta method defined by the Butcher table in (8), that is
\[ y_{\gamma_{v+1}} = \exp(hb_{\gamma_{v+1}}) \cdots \exp(hb_1A_1)y_0, \]
\[ A_i = A(Y_i), \]
\[ Y_i = \exp(ha_{\gamma_{v+1}}(A_{i-1}) \cdots \exp(ha_{\gamma_{v+1}}A_1), \ i = 1, \ldots, v. \]

Theorem 5. The class of Crouch–Grossman methods applied to the differential system (7) preserves property (2).

Proof. Here we need to evaluate matrix exponentials and this may be too expensive for matrices of large size. However, if $A$ belongs to $S_{d, \alpha}(s)$, then $A = B + (a/2)I$, where $B$ belongs to the Lie algebra $S_{d, \alpha}(s)$ associated to the quadratic group $\mathcal{O}(H) = \{ X \in \mathbb{R}^{n \times n} | X^2H = H \}$. Thus, we can write
\[ \exp(A) = \exp \left( \frac{1}{2} a \right) H \exp(B) \]
and, in order to save some computations, we can evaluate $\exp(B)$ by using splitting techniques for the approximation of the matrix exponential in a Lie-algebraic setting (see for instance [1,19]). We have to observe that Crouch–Grossman methods automatically preserve the exponential monotonicity property, in particular the knowledge of $H$ and $a$ is not needed. Instead when a splitting technique is applied, then $H$ and $a$ are required.
3. The use of quadratic preserving methods

We observe that the behavior of differential system (1) may be studied via the one of an associated differential system whose solution satisfies a quadratic form. The relationship between a dynamical system with exponential monotonicity (2) and a dynamical system evolving on a quadratic group has been observed in [10]. Here it is used to derive numerical methods preserving property (2). The following result provides the corresponding differential system and the relation between the solutions of the associated systems.

Lemma 6. Let \( Y(t) \) be the solution of (1) and let \( A : \mathbb{R} \times \mathbb{R}^{r} \to \mathbb{R}^{r} \). Then the matrix \( B(t, Y(t)) = A(t, Y(t)) - (a/2)I \), belongs to the Lie algebra \( \mathfrak{so}(0) \) associated to the quadratic group \( \mathbb{C}(H) \) for all \( t \geq 0 \). Moreover, the solution of the differential system:

\[
\dot{X}(t) = B(t, Y(t))X(t), \quad X(0) = X_0 \in \mathbb{R}^{r/r}
\]

preserves the quadratic form

\[
X^T(t)B(t, X(t))X(t) = Y^T(t)HY(t)
\]

for all \( t > 0 \) \((25)\)

and if \( X_0 \in \mathbb{C}(H) \), then \( X(t) \) belongs to \( \mathbb{C}(H) \) for all \( t \geq 0 \). Furthermore, we have

\[
Y(t) = \exp\left(\frac{a}{2}t\right)X(t), \quad t \geq 0.
\]

Proof. The only thing to prove is the relation (26).

We define

\[
Z(t) = \exp\left(\frac{a}{2}t\right)X(t), \quad t \geq 0.
\]

By differentiating we have

\[
Z(t) = \frac{a}{2} \exp\left(\frac{a}{2}t\right)X(t) + \exp\left(\frac{a}{2}t\right)X(t)
\]

or

\[
Z(t) = \frac{a}{2} \exp\left(\frac{a}{2}t\right)X(t) + \exp\left(\frac{a}{2}t\right)B(t, Y(t))X(t)
\]

\[
= \frac{a}{2} \exp\left(\frac{a}{2}t\right)X(t) + \exp\left(\frac{a}{2}t\right)A(t, Y(t))X(t)
\]

\[
= A(t, Y(t))Z(t).
\]

Thus, since \( Z(0) = X_0 \) we get \( Y(t) = Z(t) \). \( \square \)

The relation (26) also provides a computational way to preserve the exponential monotonicity in (2). In fact, any quadratic preserving numerical method, as Gauss–Legendre–Runge–Kutta methods (see [5]), Lie group methods (see [13]), Cayley based methods (see [6,14]), applied to

\[
\dot{X}(t) = B(t, \exp\left(\frac{a}{2}t\right)X(t))X(t),
\]

\[
X(0) = X_0 \in \mathbb{R}^{r/r}
\]

(27)

provides a numerical solution \( X_n \) which preserves the quadratic form in (25), i.e. \( X_n^T B_n X_n = Y_n^T H Y_n \), from which

\[
Y_n = \exp\left(\frac{a}{2}t_n\right)X_n
\]

satisfies (2).

We observe that this approach is equivalent to apply a splitting technique to the original differential system. In fact, by observing that \( A(t, Y(t)) = B(t, Y(t)) + (a/2)I \), with \( B \) matrix function on the Lie algebra \( \mathfrak{so}(0) \), the original system may be written as

\[
\dot{Y}(t) = B(t, Y(t))Y(t) + \frac{a}{2}Y(t), \quad t > 0.
\]

Hence we can solve separately the two differential systems

\[
\dot{Z}(t) = \frac{a}{2}Z(t), \quad Z(0) = I,
\]

\[
\dot{X}(t) = B(t, Z(t))X(t)X(t), \quad X(0) = X_0
\]

the first of which is explicitly integrable in closed form. Then we may give the solution by

\[
Y(t) = Z(t)X(t), \quad t > 0.
\]

Thus, if we solve the first differential system by a quadratic preserving method the exponential monotonicity property is preserved.

4. Numerical examples

In this section we will perform some numerical tests by using Magnus, Crouch–Grossmann and projected methods denoted, respectively, by MAGp, Table 1

<table>
<thead>
<tr>
<th>Method</th>
<th>Error</th>
<th>Method</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAG1</td>
<td>5.551E−16</td>
<td>QRO04</td>
<td>3.3307E−16</td>
</tr>
<tr>
<td>MAG2</td>
<td>6.1062E−16</td>
<td>QGL2</td>
<td>1.2601E−16</td>
</tr>
<tr>
<td>CR03</td>
<td>1.6658E−16</td>
<td>PRO02</td>
<td>2.2204E−16</td>
</tr>
<tr>
<td>CR04</td>
<td>8.810E−16</td>
<td>PRO04</td>
<td>2.2204E−16</td>
</tr>
<tr>
<td>QPRO02</td>
<td>3.3307E−16</td>
<td>GL2</td>
<td>2.2242E−04</td>
</tr>
</tbody>
</table>
CG$p$, PROJ$p$ and quadratic preserving methods applied to the transformed problem. For these methods we solve the associated quadratic system by projection methods (QPROJ$p$) or by GL methods (QGL$p$).

The integer $p$ after the label denotes the order of the considered method. The exponential monotonicity of the numerical solution $y_n$ is measured at each step by $\|y_n^T H y_n - \exp(\alpha n) y_0^T H y_0\|$, where $\|\|$ is the Euclidean norm.
We consider some numerical tests for Example 1 described in Section 1. The maximum exponential error for every method is given in Table 1. As we can see all methods preserve the exponential monotonicity of the solution except for GL2. To check the order of every method, we evaluate the solution at time $T = 10$ for different stepsizes starting by $h = 0.5$. The other stepsizes are given by $h_i = h/2^i$ for $i = 1, 2, 3, 4$. Figs. 1 and 2 show how second- and fourth-order methods, respectively, behave when we vary the stepsize. We plot the quantities $p_i$ versus the stepsize, where $p_i = \log(e_i - 1)/\log(h_{i+1}/h_i)$, and $e_i$ is the global error at stepsize $h_i$. As we can see all methods reach the desired order.

In Fig. 3 we plot the norm of the solution (exponentially increasing when $\alpha$ is positive, decreasing when $\alpha$ is negative) for four different values of $\alpha = \pm 2, \pm 0.1$. The norm of the approximation given by all methods is the same (plotted by a solid line) except for the one given by the Gauss–Legendre–Runge–Kutta (plotted by a dotted line). As we can see when $\alpha$ is positive the Gauss–Legendre–Runge–Kutta methods does not perform well.

5. Conclusions

Differential systems with exponential monotonicity appear in several situations. Our end was to study numerical methods preserving property (2). The key reason for this work is that classical methods, such as Gauss–Legendre–Runge–Kutta schemes, do not preserve exponential monotonicity.

We have seen that certain exponential integrators and Crouch–Grossman methods, usually used as Lie group preserving methods, automatically maintain the desired exponential monotonicity property. Projected techniques may be also applied although these procedures may show computational problems when $H$ is different from the identity matrix. Furthermore we have seen that a differential system whose solution satisfies a quadratic form may be associated to our differential system so that quadratic preserving methods may be considered.

Acknowledgements

We thank Ernst Hairer for his useful comments on the proof of Theorem 3 and for alerting us on the existence of the references [15,16].
References


C. Elia is a PhD student in mathematics at University of Bari, Italy. She received her Laurea degree in mathematics from the same university in 2000. Recently, she is a visiting student at Georgia Tech, Atlanta, USA. Her research interest is mainly in numerical methods for ordinary differential equations and Lyapunov exponents. She is member of Italian National Group for the Scientific Computation (GNCS).

L. Lopez was born on 22 July 1955 in Bari, Italy. He received his Laurea degree in mathematics from the University of Bari in 1980. In 1995 he became full professor of numerical analysis at University of Bari where he is currently employed. Initially his research activity was in the field of numerical methods for partial differential equations arising in biological and population models. Then his interests moved to the field of numerical methods for stiff and differential algebraic equations. Recently, his research concerns the area of numerical solution of ordinary differential equations evolving on manifolds, particularly numerical methods for isospectral and orthogonal flows. He is author of more than 40 papers that have been published on international journals.