A Low-Complexity Scheme for Frequency Estimation in Uplink OFDMA Systems
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Abstract—Frequency estimation in the uplink of an orthogonal frequency-division multiple-access system is a challenging task due to the presence of multiple carrier frequency offsets. Many existing solutions are too complex for practical implementation, while others are restricted to a specific subcarrier assignment strategy. Motivated by the above consideration, in this letter we propose a novel frequency estimator that allows flexible subcarrier assignment while requiring a low computational burden. Our scheme exploits the repetitive structure of the users’ training sequences, which are properly designed so as to minimize the multiple-access interference arising in the presence of frequency errors. Computer simulations are used to assess the effectiveness of the proposed method and to make comparisons with competing alternatives.

Index Terms—Orthogonal frequency-division multiple-access (OFDMA), low-complexity frequency synchronization, interference mitigation.

I. INTRODUCTION

In orthogonal frequency-division multiple-access (OFDMA), the presence of carrier frequency offsets (CFOs) between the uplink signals and the base station (BS) local reference gives rise to interchannel interference (ICI) as well as multiple-access interference (MAI), with ensuing limitations of the system performance. For this reason, the problem of frequency recovery in the OFDMA uplink has received much attention in recent years (see [1]–[9] and references therein). Existing solutions can be categorized on the basis of the specific subcarrier assignment scheme (SAS) adopted in the system, i.e., the strategy according to which subcarriers are allocated to the active users. The methods discussed in [1] and [2] are suited for a subband SAS, where groups of adjacent subcarriers are exclusively assigned to each user. This facilitates the frequency synchronization task as user separation can easily be accomplished at the BS through a bank of band-pass filters. However, the subband SAS exhibits poor performance in the presence of dispersive channels as a deep fade might hit a substantial number of users’ subcarriers. Better results are expected by adopting an interleaved SAS, where subcarriers of a given user are uniformly distributed over the available spectrum. This provides the system with full channel diversity, but complicates the CFO recovery task since in the presence of CFOs the uplink signals cannot be simply separated through a filter bank. Some prominent solutions in this context can be found in [3]–[5]. Specifically, the scheme proposed in [3] relies on the multiple signal classification algorithm and provides estimates of the CFOs in a decoupled fashion. However, it requires significant computational burden as it involves a grid-search over the uncertainty frequency range. To overcome this problem, the signal parameters via rotational invariance technique is suggested in [4] to compute CFO estimates in closed-form without the need for any peak search procedure. Alternative approaches illustrated in [5] make use of the space-alternating projection expectation-maximization algorithm. Unfortunately, all aforementioned schemes are not suited for a generalized SAS, in which the available subcarriers are opportunistically assigned to the active users on the basis of the measured channel quality.

CFO recovery for an OFDMA uplink with generalized SAS is investigated in [6]–[8]. In particular, the algorithms derived in [6] and [7] are based on the maximum likelihood (ML) principle and provide CFO estimates by exploiting a training block transmitted by each user at the beginning of the uplink frame. In spite of their effectiveness, such solutions are too computationally demanding as they require a complete search in order to locate the maximum of the likelihood function. Some computational saving is achieved in [8] by replacing the exhaustive search with a line search. This results into a scheme with reduced complexity but slower convergence rate.

From the above discussion, it follows that the main drawback of existing CFO recovery schemes is that they are either restricted to a particular SAS or require a large computational burden, which prevents their practical implementation. A possible approach to overcome these problems has recently been proposed by Zeng and Leyman in [9]. This scheme relies on a flexible SAS and computes CFO estimates in the frequency domain with affordable complexity by means of a dedicated training block. As shown later, however, the price for such advantages is a substantial degradation of the estimation accuracy.

In this work, a low-complexity algorithm for CFO recovery in the OFDMA uplink is proposed. The transmission is organized in frames and each uplink frame is preceded by at least two identical training blocks. The available spectrum is partitioned into disjoint subchannels, which are exclusively assigned to the active users. Any subchannel is further divided into a given number of subbands composed by a small group of adjacent subcarriers carrying known pilot symbols. The subbands can be randomly positioned within the signal bandwidth and the CFO of each user is eventually retrieved by measuring the phase shift between pilot tones transmitted over adjacent OFDMA blocks. This approach was originally proposed by Moose in the context of single-user orthogonal
frequency-division multiplexing (OFDM) transmissions [10]. When applied to a multi-user scenario, however, it provides poor performance because in the presence of multiple CFOs the uplink signals lose their mutual orthogonality and MAI arises. To alleviate this problem, the pilot tones are properly designed so as to minimize the MAI power under a constraint on the overall pilot energy. As we shall see, this leads to an estimation algorithm of affordable complexity and with an accuracy that is close to the relevant Cramer-Rao bound (CRB).

Compared to [9], our scheme has comparable flexibility in terms of subcarrier assignment but provides improved CFO estimates and requires a lower computational burden. It is fair to say that the idea of using properly modulated subcarriers to improve the performance of OFDM(A)-based systems is not novel and has recently been adopted in many different contexts. For example, it is used in [11] and [12] to limit the out-of-band power radiation of the transmitted signal, and is also employed in [13] for peak-to-average power ratio (PAPR) control. To the best of our knowledge, however, this approach has never been used before to facilitate the CFO estimation task.

The remainder of this letter is organized as follows. Next section describes the investigated system and introduces the signal model. In Section III the low-complexity CFO estimation algorithm is derived and the pilot symbols for MAI mitigation are designed. Simulation results are illustrated in Section IV while some conclusions are drawn in Section V.

II. SYSTEM DESCRIPTION AND SIGNAL MODEL

We consider the uplink of an OFDMA system employing $N$ subcarriers with frequency spacing $\Delta f$ and indices in the set $\mathcal{U} = \{0, 1, \ldots, N - 1\}$. The transmission is organized in frames and each frame is preceded by $M \geq 2$ identical training blocks where the $N$ available subcarriers are divided into $R$ subchannels. We denote by $K \leq R$ the number of simultaneously active users and assume that a single subchannel with index $k \in \{1, 2, \ldots, K\}$ is assigned to the $k$th user. Each subchannel is divided into $L$ subbands and a given subband is composed by a set of $V$ adjacent subcarriers, which is called a tile. Hence, the total number of subcarriers in each block is $N = RLV$. The subcarrier indices of the $\ell$th tile $(\ell = 1, 2, \ldots, L)$ in the $r$th block $(r = 1, 2, \ldots, R)$ subchannel are collected into a set $\mathcal{I}_{r,\ell} = \{1, 2, \ldots, V\}$. The $r$th subchannel is thus composed of subcarriers with indices taken from $\mathcal{I}_r = \bigcup_{\ell=1}^L \mathcal{I}_{r,\ell}$. The only constraint in the selection of the indices $i_{r,\ell}$ is that the tiles must be disjoint, which amounts to saying that $\mathcal{I}_{r_1,\ell_1} \cap \mathcal{I}_{r_2,\ell_2} = \emptyset$ for $\ell_1 \neq \ell_2$ or $r_1 \neq r_2$.

The waveform transmitted by the $k$th user propagates through a multipath channel characterized by a channel impulse response (CIR) $h_k = [h_k(0), h_k(1), \ldots, h_k(Q-1)]^T$ of order $Q$ (the superscript $(\cdot)^T$ denotes the transpose operation), and arrives at the BS with a timing offset $\theta_k$ (normalized by the sampling interval) and a frequency error $\xi_k$ (normalized by the subcarrier spacing). Timing errors are related to the different positions occupied by the users within the cell. In the ensuing discussion, we assume that each training block is preceded by a cyclic prefix of $N_G$ samples accommodating both the channel delay spread and the users’ timing offsets. This results into a quasi-synchronous system where no interblock-interference is present at the BS receiver. Frequency errors arise as a consequence of oscillator instabilities and/or Doppler shifts. In time-division duplex (TDD) systems, where the same frequency band is employed for both the downlink and uplink phases, frequency errors are individually estimated and corrected at the mobile terminals by exploiting a broadcast synchron channel [14]. In such a case, the uplink CFOs are expected to be adequately small as they are only induced by Doppler shifts and downlink estimation errors. On the other hand, in frequency-division duplex (FDD) systems the uplink and downlink streams are transmitted over different frequency bands and, in consequence, they are plagued by different CFOs. Since the latter may be as large as a significant percentage of the subcarrier spacing, uplink frequency recovery becomes mandatory in FDD applications to avoid a serious degradation of the system performance. Once the CFO estimates have been obtained, they are employed by the BS receiver to restore orthogonality among subcarriers. As explained in [14], one possible solution consists of returning the estimates back to the corresponding terminals via a downlink control channel so that users can properly adjust their carrier frequencies. An alternative approach is the use of advanced signal processing techniques for directly compensating the synchronization errors at the BS [15].

Without loss of generality, we concentrate on the $j$th subchannel and denote by $Y_j(m, n)$ with $n \in \mathcal{I}_j$ the discrete Fourier transform (DFT) output over the $n$th subcarrier of the $m$th training block, with $m = 0, 1, \ldots, M - 1$. Assuming that the channel does not vary over the training period, we may write

$$Y_j(m, n) = \sum_{k=1}^{K} e^{j2\pi m c_k N_T / N} X_k(n) + W_j(m, n), \quad n \in \mathcal{I}_j$$

where $N_T = N + N_G$ is the length of the cyclically extended block and $W_j(m, n)$ accounts for background noise, which is modeled as a circularly symmetric complex Gaussian random variable with zero mean and variance $\sigma^2$. The quantity $X_k(n)$ is the signal component of the $k$th user over the $n$th subcarrier and is given by

$$X_k(n) = \sum_{\eta \in \mathcal{I}_k} u(\xi_k + \eta - n) c_k(\eta) H_k(\eta) e^{-j2\pi n \theta_k / N}$$

where $c_k(\eta)$ is the pilot symbol transmitted over the $\eta$th subcarrier, $H_k(\eta) = \sum_{\ell=0}^{Q-1} h_k(\ell) e^{-j2\pi \ell \eta / N}$ is the corresponding channel frequency response and, finally,

$$u(x) = e^{j2\pi x (N-1) / N} \frac{\sin(\pi x)}{N \sin(\pi x / N)}.$$
The resulting scheme operates in the frequency domain and provides the CFO estimate by measuring the phase shift between corresponding DFT outputs over adjacent training blocks. For illustration purposes, we begin by considering the favourable situation where only the $j$th user is active in the system. Then, from (1) we have

$$Y_j(m, n) = e^{j2\pi m\varepsilon_j/N}X_j(n) + W_j(m, n)$$  \hspace{1cm} (4)

disregarding for simplicity the noise contribution, it follows that

$$Y_j(m + 1, n)Y_j^*(m, n) = |X_j(n)|^2 e^{j2\pi m\varepsilon_j/N}$$.  \hspace{1cm} (5)

The above result indicates that an estimate of $\varepsilon_j$ can easily be obtained as

$$\hat{\varepsilon}_j = \frac{N}{2\pi N_T} \arg \left\{ \sum_{n \in \mathbb{I}_j} \sum_{m=0}^{M-2} Y_j(m + 1, n)Y_j^*(m, n) \right\}$$ \hspace{1cm} (6)

which is referred to as the ad-hoc frequency estimator (AHFE). Inspection of (6) reveals that AHFE provides ambiguous estimates unless the CFO belongs to the interval $\left[-N/(2N_T), N/(2N_T)\right]$. To make an example, assume that $N = 1024$ and $N_G = 256$. In this case, the estimation range of AHFE is approximately 40% of the subcarrier spacing.

In assessing the computational load of AHFE, we assume that the DFT outputs have already been evaluated and are available at the BS. Then, computing $\hat{\varepsilon}_j$ as in (6) approximately needs $(M - 1)LV$ complex products plus $(M - 1)LV$ complex additions. The overall complexity of AHFE for $K$ active users is summarized in the first line of Table I and corresponds to $8(M - 1)LKV$ floating point operations (flops). For comparison, in the second line of Table I we report the processing load of the iterative algorithm discussed in [6] (AAPFE), where $N_i$ is the number of employed iterations and $N_c$ the number of points at which the frequency metric is evaluated. The computational burden of the CFO estimators proposed by Na and Minn (NM) in [8] and by Zeng and Leyman (ZL) in [9] is also shown in the third and fourth lines of Table I, respectively. To fix the ideas, assume that $N = 1024$ and $N_G = 256$, with $R = K = 32$ and $M = 2$. Letting $V = 4$ and $L = 8$, from Table I it follows that 8 kfps and 104 kfps are approximately needed by AHFE and ZL, respectively. On the other hand, setting $N_i = 2$ and $N_c = 400$ we see that 6700 Mfps are involved with AAPFE, while 7000 Mfps are required by NM when $Q = 14$. The above results indicate that AHFE allows a significant complexity saving with respect to ZL, with a reduction of the number of flops by a factor of 13. As expected, the complexity of AAPFE and NM is so high to prevent their application in commercial systems. For this reason, in the sequel both AAPFE and NM will not be considered any further.

Inspection of (1) indicates that the single-user model in (4) holds true even in a multi-user setting provided that $X_k(n) = 0$ if $n \in \mathbb{I}_j$ and $k \neq j$. From (2) and (3) we see that this occurs in a perfectly synchronized scenario where $\varepsilon_k = 0$ for $k \neq j$, while it is not met whenever $\varepsilon_k \neq 0$ for at least one value of $k \neq j$. In the latter case, the DFT outputs corresponding to the $j$th subchannel will be plagued by MAI, which is a consequence of the high sidelobe power associated to each subcarrier in the frequency domain. Since the presence of MAI leads to a severe performance degradation if not properly compensated, it makes sense to look for some sidelobe reduction technique to be used in conjunction with AHFE.

### B. MAI mitigation

In order to mitigate the MAI arising in the presence of residual CFOs, the out-of-band (OOB) radiation associated to each tile has to be kept as low as possible. Existing approaches for OOB power reduction operate in either the time or frequency domain. The former includes adaptive symbol transition [16] or windowing the transmitted signal according to a suitably designed pulse shape [17]. These solutions reduce the spectrum sidelobes at the price of prolonged OFDM blocks, which inevitably affects the system throughput. Frequency-domain schemes involve weighting the subcarriers [18], expanding the signal constellation [19], using multiple choice sequences (MCS) [20] and inserting guard bands [17] or cancellation carriers [11] at the spectrum edges. All these methods exploit their degrees of freedom in an attempt of achieving the best trade-off between sidelobe reduction, data throughput and error-rate performance. On the other hand, when applied to an OFDM training block carrying no information data, all degrees of freedom can be used for the purpose of sidelobe suppression. Following this approach, in the ensuing discussion we design the users’ training sequences so as to minimize the OOB radiated power in the neighbourhood of the tile edges. Without loss of generality, we concentrate on the $k$th tile of the $k$th user and let $\mathcal{V} = \{ p_1, p_2, \ldots, p_V \}$ (with $p_v = k_v \ell + v - 1$) be the indices of the corresponding subcarriers. The latter are modulated by a set of complex-valued coefficients $c = [c(p_1), c(p_2), \ldots, c(p_V)]^T$ which will be designed according to a specified optimality criterion under an overall energy constraint:

$$||c||^2 = E_T.$$ \hspace{1cm} (7)

We begin by calling $Z(m, n)$ the interference that the considered tile produces over a generic subcarrier with index $n \in \mathcal{U} \setminus \mathcal{V}$. Then, omitting the user index for notational simplicity, from (1) and (2) it easily follows that

$$Z(m, n) = e^{j2\pi \varepsilon m N_T/N}.$$ \hspace{1cm} (8)

$$\sum_{v=1}^{V} u(\varepsilon + p_v - n) c(p_v) H(p_v) e^{-j2\pi p_v \theta/N},$$
where $\varepsilon$, $\theta$, and $H(p)$ are the CFO, the timing error and the channel response of the $k$th user, respectively. As in practical situations the tile width is much smaller than the channel coherence bandwidth, we may treat the channel response as nearly flat over a tile and replace the quantities $\{H(p)\}_{i=1}^V$ by an average frequency response

$$
\bar{H} = \frac{1}{V} \sum_{i=1}^V H(p_i).
$$

(9)

Also, we neglect the phase shift induced by $\theta$ since in practice the timing error is only a small fraction of the block duration, so that $\theta/N \ll 1$. Under the above assumptions, equation (8) reduces to $Z(m, n) = H e^{-j2\pi p_i \theta/N} e^{j2\pi m N_T/N} S(n)$, where we have defined

$$
S(n) = \sum_{i=1}^V u(\varepsilon + p_i - n) c(p_i).
$$

(10)

In principle, the complete elimination of the MAI at the DFT output can be accomplished by designing the pilot vector $c$ such that the quantities $S(n)$ in (10) are zero for any $n \in U \setminus V$. This would lead to a set of $N - V$ linear homogeneous equations in $V$ unknowns, which is only solved by $c = [0, \ldots, 0]^T$ in practical situations where $V$ is less than $N - V$. For this reason, we adopt a different strategy that aims at minimizing the power of the quantities $S(n)$ rather than at nulling them. This amounts to looking for the minimum of the cost function

$$
\mathcal{Y}(c) = \sum_{n \in U \setminus V} |S(n)|^2
$$

(11)

over the set of vectors $c$ having energy $E_T$. After substituting (3) into (10), we observe that $|S(n)|^2$ becomes vanishingly small whenever $|p_i - n|$ is sufficiently large. This means that the MAI power over subcarriers located at a large distance from the considered tile can reasonably be neglected. Accordingly, we may apply the optimization problem (11) to only a small number of DFT outputs, say $\Lambda$, located at each side of the considered tile. Following this approach, we replace $\mathcal{Y}(c)$ in (11) by

$$
\mathcal{Y}_2(c) = \sum_{n \in \Lambda} |S(n)|^2
$$

(12)

where $\Lambda$ is the set of integers in the intervals $[p_1 - \Lambda; p_1 - 1] \cup [p_V + 1; p_V + \Lambda]$, with $\Lambda$ being a design parameter.

Unfortunately, from (10) it turns out that $\mathcal{Y}_2(c)$ depends on the CFO $\varepsilon$, which is unknown. A possible way out consists of modeling $\varepsilon$ as a random variable with a uniform distribution over the interval $[-1/2; 1/2]$ and replacing $\mathcal{Y}_2(c)$ by its expectation with respect to $\varepsilon$, i.e.,

$$
\mathcal{Y}_3(c) = \sum_{n \in \Lambda} \int_{-1/2}^{1/2} |S(n)|^2 d\varepsilon.
$$

(13)

As a further simplification, we suggest to approximate the integral in (13) with a summation over a number of hypothesized CFO values $\{\varepsilon(i) = i/I; -I/2 \leq i \leq I/2 - 1\}$, with $I$ being a suitably designed even integer. Hence, bearing in mind (10), we obtain the objective function

$$
\Omega(c) = \sum_{n \in \Lambda} \sum_{i=1}^{I/2-1} \left|\sum_{\nu=1}^V u(\varepsilon(i) + p_i - n) c(p_i)\right|^2
$$

(14)

or, equivalently,

$$
\Omega(c) = \sum_{m \in M} \left|\sum_{\nu=1}^V u(p\nu - m/I) c(p\nu)\right|^2
$$

(15)

with $M$ being a set that collects the $2IA$ integers belonging to the intervals $[p_1 - \Lambda - 1/2; p_1 - 1/2] \cup [p_V + 1/2; p_V + \Lambda + 1/2]$. For notational convenience, we denote by $i_m$ the elements of $M$ for $1 \leq m \leq |M|$, where $|M| = 2IA$ is the cardinality of $M$. Then, we rewrite (15) in matrix notation as

$$
\Omega(c) = \|U^T c\|^2 - \lambda \left\|c\right\|^2 - E_T
$$

(16)

where $U \in \mathbb{C}^{|M| \times V}$ is a matrix with entries

$$
[U]_{m,p} = u(p\nu - i_m/I) \quad 1 \leq m \leq |M|, \quad 1 \leq v \leq V.
$$

(17)

Our goal is to find a vector $c$ that minimizes $\Omega(c)$ under the energy constraint (7). The solution to this problem is obtained by looking for the minimum of the augmented cost function

$$
\Omega_2(c, \lambda) = \|U^T c\|^2 - \lambda \left\|c\right\|^2 - E_T
$$

(18)

with $\lambda$ being the Lagrange multiplier. Computing the gradient of $\Omega_2(c, \lambda)$ with respect to $c$ and setting its entries to zero produces

$$
U^H U c^{(IRT)} = \lambda c^{(IRT)}
$$

(19)

where the superscript $(\cdot)^H$ denotes the Hermitian transpose operation and IRT stands for “interference reduction tones”. The above equation indicates that $c^{(IRT)}$ is an eigenvector of $U^H U$ with $\lambda$ being the corresponding eigenvalue. On the other hand, substituting (19) into (16) and bearing in mind (7) yields $\Omega(c^{(IRT)}) = \lambda E_T$, which achieves a minimum if $\lambda$ is the smallest eigenvalue of $U^H U$, say $\lambda_{\min}$. In summary, our optimization problem is solved by choosing $c^{(IRT)}$ as the eigenvector of $U^H U$ associated to $\lambda_{\min}$ and normalized such that $\|c^{(IRT)}\|^2 = E_T$. It is worth noting that the computation of $c^{(IRT)}$ does not require any on-time operation since $U$ is a known matrix and, accordingly, the eigendecomposition of $U^H U$ can be precomputed off-line. We also point out that $c^{(IRT)}$ is univocally specified apart from an irrelevant phase shift that does not affect neither the value of $\Omega(c^{(IRT)})$ nor the energy of $c^{(IRT)}$.

At this stage, we recall that one possible approach for reducing the OOB radiation consists of turning off some subcarriers at the edges of each tile so as to provide a sufficiently large guard band between tiles of different users. As shown later, however, the use of such virtual carriers (VCs) in place of the optimum pilots $c^{(IRT)}$ reduces the accuracy of the frequency estimates. An alternative criterion for pilot design has been proposed by Zhao and Haggman (ZH) in [21] and relies on the idea of mitigating the disturbance term produced by a couple
of adjacent pilots over subcarriers located at a relatively large distance from the considered tile. This is achieved by using pilot symbols with alternating polarity of the form

\[ c(p_{2
u}) = -c(p_{2
u-1}) \quad \text{for} \quad \nu = 1, 2, \ldots, V/2. \]  

(20)

However, while the pilot symbols \( c^{(\text{IRT})} \) are derived by means of a rigorous optimization procedure, the ZH scheme is based on heuristic arguments and does not obey to any optimality criterion.

IV. SIMULATION RESULTS

Computer simulations have been run to assess the performance of the proposed IRT-based CFO estimator in a scenario inspired by the IEEE 802.16 family of standards for wireless MANs [22]. The DFT size is \( N = 1024 \) with a sampling period \( T_s = 1/(N\Delta f) = 87.5 \) ns. We assume that the maximum number of active users is \( K = 32 \). In order to keep the system overhead to a tolerable level, the number of training blocks is fixed to \( M = 2 \). The channel taps are generated as independent and circularly symmetric Gaussian random variables with zero mean (Rayleigh fading) and power delay profile as specified in the ITU IMT-2000 Vehicular-B channel model [23]. Channels of different users are assumed to be statistically independent of each other. We consider a cell radius of 1.5 km, corresponding to a maximum propagation delay of \( \theta_{\max} = 114 \) sampling periods. The training blocks are preceded by a cyclic prefix of length \( N_G = 256 \) to avoid interblock interference. The normalized CFOs are uniformly distributed over the interval \([-\varepsilon_{\max}, \varepsilon_{\max}]\) and vary at each run. As discussed previously, parameter \( \varepsilon_{\max} \) depends on the duplexing mode adopted in the system, with larger values of \( \varepsilon_{\max} \) being expected in an FDD scenario. In our simulations we let \( \varepsilon_{\max} \leq 0.3 \), which amounts to considering frequency offsets as large as 30% of the subcarrier spacing. In order to obtain an accurate approximation of the integral in (13), we choose \( I = 8 \). Parameter \( \Lambda \) is set equal to two while the energy of each tile is \( E_T = V \). The accuracy of the frequency estimates is measured in terms of the total mean square estimation error, which is defined as

\[ \text{MSE} = \frac{1}{K} \sum_{k=1}^{K} E\{(\hat{\varepsilon}_k - \varepsilon_k)^2\}. \]  

(21)

Comparisons are made with the pilot-based scheme suggested by Huang and Letaief (HL) in [15] and with the ZL and ZH algorithms illustrated in [9] and [21], respectively. When using HL, the pilot symbols \( c(p_v) \) are the same in each tile and no MAI mitigation is attempted. With ZL, only one subcarrier is active in a tile while the other \( V - 1 \) subcarriers are left unmodulated. Comparisons are made under a common simulation set-up, which includes the same number of subchannels and tiles as well as the same transmitted energy \( E_T \) per tile. As mentioned previously, iterative methods like AAPFE and NM are not considered as they are too complex.

Fig. 1 illustrates the frequency MSE of the investigated estimators as a function of the signal-to-noise ratio (SNR) \( 1/\sigma^2 \) expressed in dB. In this experiment, a tile is composed by \( V = 4 \) subcarriers and \( \varepsilon_{\max} \) is set to 0.15. Any subchannel comprises \( L = N/(KV) = 8 \) tiles, which are uniformly distributed across the signal spectrum at a distance of \( KV = 128 \) subcarriers. The VC curve refers to a system employing the pilot sequence \( c = [0, \sqrt{2}, \sqrt{2}, 0]^T \), which amounts to inserting one virtual carrier at the tile edges for MAI mitigation, while the CRB represents the Cramer-Rao bound and serves as a benchmark. The bound is computed from (1) under the optimistic assumption that only one user is active in the uplink, and is given by

\[ \text{CRB}(\varepsilon) = \frac{3N^2\sigma^2}{2\pi^2 N_G^3 VLM(M^2 - 1)}. \]  

(22)

We see that IRT attains the CRB at SNR values of practical interest and outperforms the other schemes, which are plagued by an irreducible floor. This indicates that they are not able to completely suppress the MAI induced by frequency errors. The impact of MAI is barely visible at low SNR values due to the masking effect of thermal noise, but it becomes evident as the SNR grows large.

In Fig. 2 the frequency MSE is shown as a function of \( \varepsilon_{\max} \) for SNR = 30 dB and \( V = 4 \). While the accuracy of HL and VC rapidly deteriorates with \( \varepsilon_{\max} \) as a consequence of the increased MAI power, both IRT and ZH are close to the CRB even at CFO values as large as 0.15. This reveals a remarkable capability of these schemes to keep the OOB radiation to a much lower level than HL and VC. As for ZL, it performs poorly at all investigated \( \varepsilon_{\max} \) values, thereby indicating that the MAI power is not the primary source of degradation for this estimator.

The results of Fig. 2 are replicated in Fig. 3, with the only difference that now the tiles are composed of \( V = 8 \) subcarriers. It follows that the number of tiles in any subchannel is reduced to \( L = 4 \), while the distance between tiles in the same subchannel is increased to 256 subcarrier spacings. Six unmodulated subcarriers are used by the VC scheme for MAI mitigation, which amounts to setting \( c = [0, 0, 0, 2, 2, 0, 0, 0]^T \). Compared to Fig. 2, the accuracy of
the investigated estimators is somewhat improved. The reason is that increasing $V$ provides the system with more degrees of freedom to be exploited for MAI mitigation. Again, we see that the IRT attains the CRB at all investigated CFO values, while the other schemes depart from the bound as $\varepsilon_{\text{max}}$ increases. A possible explanation is that these methods are based on heuristic arguments and do not exploit the available degrees of freedom as efficiently as IRT.

It is interesting to compare the investigated CFO estimators in terms of PAPR level, which is measured as [24]

$$
\text{PAPR} = \max_{0 \leq m \leq N-1} \frac{\left| s_k(m) \right|^2}{\frac{1}{N} \sum_{m=0}^{N-1} \left| s_k(m) \right|^2}
$$

where $s_k(m) = \frac{1}{\sqrt{N}} \sum_{\eta \in \mathcal{I}_k} c_k(\eta) e^{j2\pi m\eta/N}, \quad 0 \leq m \leq N - 1$ are the time-domain samples at the input of the D/A converter. The PAPR values corresponding to $V = 8$ are listed in the first column of Table II. These results indicate that all schemes, except for ZL, are characterized by very large PAPR values that cannot be handled in practice. This problem can be mitigated by means of some PAPR reduction technique. Among many available solutions, here we adopt the scheme proposed in [25] as it provides a reasonable tradeoff between complexity and PAPR reduction capability. This method relies on the simple idea of multiplying all pilot symbols in the $\ell$th tile by the complex exponential $e^{j\phi(\ell)}$, yielding

$$
c_k(\eta) = c_k(\eta)e^{j\phi(\ell)} \quad \text{for} \quad \eta \in \mathcal{I}_{k,\ell}
$$

(25)

where $\phi(\ell)$ is a phase shift that can arbitrarily vary from tile to tile without compromising the MAI reduction capability of the frequency recovery schemes. Although in principle one may look for the optimal phase shifts $\{\phi(\ell); \ell = 1, 2, \ldots, L\}$ that minimize the PAPR, in practice the complexity associated with the optimization problem is greatly reduced if we constraint $\phi(\ell)$ to vary within a finite set of $\Phi$ elements $A_\phi = \{2\pi(i - 1)/\Phi; i = 1, 2, \ldots, \Phi\}$. Since we may arbitrarily set $\phi(1) = 0$ without incurring any performance penalty, the sequence $\{\phi(\ell)\}$ is found through a complete search over $\Phi^{L-1}$ possible candidates. In our simulations we let $\Phi = 4$ to keep the complexity of the search to a tolerable level. In such a case, recalling that $V = 8$ and $L = 4$, the sequence $\{\phi(\ell)\}$ is chosen among 64 candidates. As shown in the second column of Table II, this approach reduces the PAPR of all investigated schemes by a factor of 4. Interestingly, ZL is now characterized by unitary PAPR, meaning that the corresponding training sequence has constant amplitude in the time domain. This advantage, however, is achieved at the price of a significant degradation of the estimation accuracy as it is seen from the results of Figs. 1-3.

We now compare the investigated schemes in terms of bit-error-rate (BER) performance. In doing so, we assume that the CFO estimates computed at the BS are returned back to the mobile terminals via a downlink control channel and exploited by the users to adjust their carrier frequency. After the synchronization procedure has been completed, however, the $k$th uplink signal will still be plagued by a residual frequency error $\Delta \varepsilon_k = \varepsilon_k - \hat{\varepsilon}_k$. The latter produces an accumulated phase error that grows linearly in time, thereby preventing coherent data detection if not properly compensated. For this

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Without PAPR control</th>
<th>With PAPR control</th>
</tr>
</thead>
<tbody>
<tr>
<td>HL</td>
<td>32</td>
<td>8</td>
</tr>
<tr>
<td>VC</td>
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<td>2</td>
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<tr>
<td>ZH</td>
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</tr>
<tr>
<td>ZL</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>IRT</td>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 2. Accuracy of the investigated schemes vs. $\varepsilon_{\text{max}}$ with $V = 4$ and SNR = 30 dB.

Fig. 3. Accuracy of the investigated schemes vs. $\varepsilon_{\text{max}}$ with $V = 8$ and SNR = 30 dB.
and $\text{SNR} = 30 \text{ dB}$.

Numerical simulations indicate that the accuracy of the proposed scheme attains the Cramer-Rao bound in many practical situations. Compared to some alternative approaches with similar complexity, it exhibits improved performance while maintaining a tolerable PAPR level at the remote terminals. The increased accuracy of the frequency estimates results into smaller accumulated phase errors over the payload section of the frame. This advantage can be exploited to improve the BER performance or, alternatively, to reduce the rate at which channel estimates must be updated, with a corresponding saving in terms of system overhead.

**REFERENCES**


